Two-state model of excitable systems with time delayed feedback: renewal theory approach

Andrey Pototsky and Natalia Janson
Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom

We present a two-state model of an excitable system with time delayed feedback control. The two-state stochastic process $s(t) = \pm 1$ can be interpreted as a renewal process with history-dependent residence time distributions (RTDs). We assume that the durations of the excited and the refractory phases τ are equally long and not affected by the noise. This reduces the problem to the only unknown RTD of the activation time $\psi_-(t)$.

Two qualitatively different situations are considered. (i) For small delay times ($\tau_1 \leq 3\tau$) the history is non-variable, i.e. all the activation times are identically distributed with density which depends on the delay time τ_1. The history dependence of the transition rate from the non-excited to the excited state λ_- is assumed to be known. It is given by λ if τ_1 seconds ago the system was in the state $s = -1$ and by $\lambda + p$ if τ_1 seconds ago the system was in the state $s = +1$. The RTD $\psi_-(t)$ as function of τ_1 is computed straight forward. In the case of non-variable history the results of the renewal theory [1, 2] can be applied directly to yield an analytic expression for the power spectrum of the noise-induced oscillations.

(ii) For large delay times ($\tau_1 > 3\tau$) the history becomes variable and the renewal theory is no longer applicable. To overcome this problem the equilibrium RTDs $\psi_-^{\text{eq}}(t)$ in the sense of the averaging over all possible histories U are introduced. Based on this concept we derive an equation for the equilibrium $\psi_-^{\text{eq}}(t)$ which is valid for an arbitrary delay time τ_1

$$\psi_-^{\text{eq}}(t) = \int_{u \in U} \mathcal{P}[\psi_-^{\text{eq}}(u)] \psi_-^u(t)\, du.$$

Here $\psi_-(u)$ is the known RTD at given history u and $\mathcal{P}$ is the probability density of the history u which has to be determined separately for any fixed delay time τ_1. According to the definition of the equilibrium RTD, the probability density $\mathcal{P}$ depends on u solely through $\psi_-^{\text{eq}}(t)$.

For the delay times τ_1 in the interval $[3\tau; 4\tau]$ the set U is restricted to $[0; \tau_1 - 3\tau]$ and the equation for $\psi_-^{\text{eq}}(t)$ can be written explicitly

$$\begin{aligned}\psi_-^{\text{eq}}(t) \;=\; & \lambda e^{-\lambda t} \int_{\tau_1 - 3\tau}^{\infty} \psi_-^{\text{eq}}(\chi)\, d\chi + (\lambda + p) e^{-(\lambda + p)t} \int_0^{\tau_1 - 3\tau - t} \psi_-^{\text{eq}}(\chi)\, d\chi + \\ & \lambda e^{-\lambda t} e^{-p(\tau_1 - 3\tau)} \int_{\tau_1 - 3\tau - t}^{\tau_1 - 3\tau} e^{p\chi} \psi_-^{\text{eq}}(\chi)\, d\chi.\end{aligned}$$

Analytic solution of the last equation on the interval $t \in [0; \tau_1 - 3\tau]$ is given by

$$\psi_-^{\text{eq}}(t) = \frac{\lambda(\lambda + p) e^{-(\lambda + p)t}}{\lambda + p\, e^{-(\lambda + p)(\tau_1 - 3\tau)}}.$$

Using the combination of the renewal theory power spectrum and the derived analytic expression for the equilibrium RTD, the piece-wise linear dependence of the main period of the noise-induced oscillations on the delay time is demonstrated. This confirms the results obtained numerically [3] for the noisy FitzHugh-Nagumo system in the excitable regime with time-delayed feedback.

To compare the analytic results with the corresponding numerical results calculated for a real excitable system we design a bistable system with two time delays following the idea of the delay-induced excitability [4]. The first delay is used to model the excitability, the second delay is assigned to the controlling force. All the parameters of the two-state model are matched to the parameters of the bistable system via the Kramers formula for the transition rates (see for instance [5]).

Using the coupling between the transition rate λ_- and the noise strength we show the delay-induced onset and enhancement of the coherence resonance for positive feedback strength. The degree of the coherence measured by the correlation time is maximal when the delay time is equal to the duration of the excited and the refractory phases taken together.

[1] D. R. Cox, *Renewal theory* (Methuen and Co., 1970).
[2] I. Goychuk and P. Hänggi, Phys. Rev. E **69**, 021104 (2004).
[3] N. B. Janson, A. G. Balanov, and E. Schoell, Phys. Rev. Lett. **93**, 010601 (2004).
[4] T. Piwonski, J. Houliham, T. Busch, and G. Huyet, Phys. Rev. Lett. **95**, 040601 (2005).
[5] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, 2004).