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ENUMERATION OF LOCALLY BRUNOVSKY LINEAR SYSTEMS OVER $\mathcal{C}(\mathbb{S}^1)$-MODULES. A PROCEDURE

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Abstract
In this paper we describe a procedure to visit all feedback classes of locally Brunovsky linear system over fixed $R = \mathcal{C}(\mathbb{S}^1)$ the ring of real continuos functions defined on the unit circle. Furthermore, we give the exact number of such classes throughout partitions of integers, binary strings and colored Ferrers diagrams.

Key words
Brunovsky linear system, feedback classification, Ferrers diagrams, partition of integers, projective module.

1 Introduction

Let R be a commutative ring with unit element $1 \neq 0$. A linear system over R is given by a linear rule (or right hand side) on the form $x^+ = Ax + Bu$ where $x \in X$ are states, $u \in U$ are inputs, and x^+ is the time-derivative or time-shift in the sequential case. Sets of states X and of inputs U are R-modules while maps A and B are R-linear maps. In this way, we say that a linear system Σ (see figure 1) and other analogous linear system Σ' are said to be Feedback Equivalent if we can bring one of them into the another by a finite composition of the following Basic Feedback Actions: Isomorphisms $Q : U \to U'$ in the input module, isomorphisms $P : X \to X'$ in the state module and feedback actions $F : X \to U$ which transforms (A, B) to system $(P(A + BF)P^{-1}, PBQ)$. In general, the theory of linear control systems over a commutative ring R goes back to the models of [Morse, 1976] for delay models. See [Brewer, Bunce and VanVleck, 1986], [Carriegos and Sánchez-Giralda, 2001] and [Hermida-Alonso, López-Cabeceira and Trobajo, 2005] to do general reading about equivalent linear systems over commutative rings.

On the other hand, it is a known that partial reachability linear map given by

$$\varphi_i^{\Sigma} = \begin{pmatrix} B \; AB \cdots A^{i-1}B \end{pmatrix} : U^{\oplus i} \longrightarrow X \qquad (1)$$

Σ :

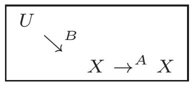


Figure 1. $\Sigma(A, B)$ Linear System

is a feedback invariant, up to equivalence, associated to Σ (see [Carriegos, 2003] and [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996]). So, we have a main set of feedback invariants, up to up to isomorphism, associated to system Σ, it is, quotient modules $N_{i+1}^{\Sigma}/N_i^{\Sigma} =$

$$\text{Im}(B, AB, ..., A^iB)/\text{Im}(B, AB, ..., A^{i-1}B). \qquad (2)$$

Furthermore, in the case of reachable linear systems over a field, or in the more general framework of projective-free rings, we known that if all R-modules $N_{i+1}^{\Sigma}/N_i^{\Sigma}$ are free, then there is a complete set of invariants verifying

$$X = N_1^{\Sigma} \oplus N_2^{\Sigma}/N_1^{\Sigma} \oplus \cdots \oplus N_s^{\Sigma}/N_{s-1}^{\Sigma}. \qquad (3)$$

Thus, once we have fixed a projective-free ring R and the dimensions $m = \dim U$ and $n = \dim X$, all feedback classes of m-input n-dimensional linear systems are in one to one correspondence with the set of partitions of integer n in decreasing sequences, equivalently, all the Ferrers diagrams of integer n (see [Knuth, 2004] to get a complete reading about partitions of integer subject).

This paper is organized as it follows: In section 2, our study is focused over continuos real functions $R = \mathcal{C}(K)$ defined in a topological space K (see [Brunovsky, 1970]), in particular is given necessary and sufficient conditions for classifying linear systems over $R = \mathcal{C}(\mathbb{S}^1)$ by Ferrers diagrams (see [Carriegos and Sánchez-Giralda, 2001] and [Ferrer, García-Planas and Puerta, 1997] to do a previous reading). In section 3, we obtain the enumeration and the number of all feedback classes of reachable linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank n. In section 4, we design a procedure to obtain and enumerate all such classes. In section 5, we extend feedback classes study of reachable linear systems over $R = \mathcal{C}(\mathbb{S}^1)$-modules.

2 The Unit Circle and Colored Ferrer's Diagrams

In this section, we strongly use next result (see [Carriegos and Sánchez-Giralda, 2001]): *The classification problem (in the case of projective invariants) is actually equivalent to the problem of characterization of all possible decompositions of finitely generated R-modules U and X on the form*

$$\begin{aligned} U &= P_0 \oplus P_1 \\ X &= P_1 \oplus P_2 \oplus \cdots \oplus P_s \end{aligned} \tag{4}$$

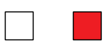
where P_0 represents a solution for $\mathrm{Ker}(B)$ and P_i represents a solution for $N_i^{\Sigma}/N_{i-1}^{\Sigma}$. Thus, the only restriction to solve the system of equations is that P_{i+1} must be a direct summand of P_i for all i.

In order to above theorem, to give the complete classification of locally Brunovsky systems is needed to know exactly the monoid $(\mathrm{Proj}(R), \oplus)$ of isomorphism classes of finitely generated R-modules with the direct sum as internal operation. The full description of the monoid $(\mathrm{Proj}(R), \oplus)$ is a great task. Of course, if finitely generated projective are free, then $(\mathrm{Proj}(R), \oplus)$ is isomorphic to $(\mathbb{N} \cup \{0\}, +)$, but in general this is not the case. If $R = \mathcal{C}(K)$ is the ring of continuos functions defined on a compact topological space K, then $(\mathrm{Proj}(R), \oplus) \equiv (\mathrm{Vect}(K), \oplus)$ depend, of course, on the topology of K (see [Swan, 1962]).

Our paper is devoted to study of $K = \mathbb{S}^1$ the real unit circumference. In this case, $(Proj(R = \mathcal{C}(\mathbb{S}^1)), \oplus)$ is the commutative monoid generated by the symbols R (representing trivial vector bundles) and P (representing the Möbius Strip) modulo the relation

$$P \oplus P = R \oplus R = R^2. \tag{5}$$

Consequently, there is only two isomorphism classes of rank r projective R-modules: R^r (the free one) and $R^{r-1} \oplus P$. Thus, we may characterize the feedback class of a locally Brunovsky linear system over R by a colored Ferrer's diagram: Because $(Proj(R), \oplus)$ is the commutative monoid generated by the symbols R and P, then every building block is a rank 1 projective module, and there are two classes depicted by



for R and P respectively. Observe that we have the rule (5), it is figure 2.

So, locally Brunovsky linear systems over the finitely generated module X of rank n would be describe by a colored Ferrer's diagram with exactly n building blocks (white or grey) where the following four restrictions apply:

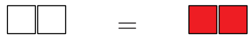


Figure 2. Equation 5

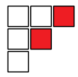


Figure 3. Example 2.1

i) There is at most one grey block on each row (by equation (5)).
ii) Parity condition: if $X = R^n$, then there are an even number of grey blocks in the whole diagram.
iii) The ith row is at most as long as the $(i-1)$th row (by decreasing ranks in the sequence (4)).
iv) If two rows have the same length r, then they are equal (by R^r is not a direct summand of $R^{r-1} \oplus P$ nor the converse).

Example 2.1. *In figure 3, we can see the feedback class of locally Brunovsky linear systems over $X = (R \oplus R \oplus P) \oplus (R \oplus P) \oplus (R)$ free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank $n = 6$.*

3 Number of Locally Brunovsky Linear Systems Over the Free $R = \mathcal{C}(\mathbb{S}^1)$-module of Rank n

Let's denote by $p_R(n)$ the number of non-isomorphic decompositions of R^n, while $\tilde{p}_R(n)$ denotes the number of non-isomorphic decompositions $R^n \cong P_1 \oplus \cdots \oplus P_s$ with P_{i+1} direct summand of P_i. Note that, if R is projectively trivial, then $\tilde{p}_R(n) = p_R(n) = p(n)$ is the number of partitions of integer n, but in general $\tilde{p}_R(n) \leq p_R(n)$.

So, the number of feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank n is $\tilde{p}_R(n)$. Thus, in order to give all feedback classes equivalence we have to visit all partitions x of n and to determinate that colored Ferrer's diagrams verifying the above four conditions on partition x. If we denote by $\tilde{p}_R(x)$ the number of such diagrams, then

$$\tilde{p}_R(n) = \sum_x \tilde{p}_R(x). \tag{6}$$

It is known a procedure to obtain all partitions of a given integer n in inverse lexicographic order. Thus, let x be a fixed partition of n. First, condition *i*) is direct because inverse lexicographic order. Second, in order to control condition *iv*) we write $x = x_1^{c_1}, x_2^{c_2}, \ldots, x_h^{c_h}$ where $n = c_1x_1 + c_2x_2 + \ldots + c_hx_h$ and $x_i > x_{i+1}$ for all i. In this way, by condition *iv*), we can define the ith-row-block, of the Ferrers diagram, as the block of c_i rows (each row of length x_i) associated to $x_i^{c_i}$

The key is denote the colored Ferrer's diagrams, associated to a given sequence (4) with partition x of n, as a binary string $w = w_1 w_2 \dots w_h$, where $w_i = 0$ if and only if the ith-row-block associated to $x_i^{c_i}$ is white.
Finally, by condition *i*), the ith-row-block associated to $x_i^{c_i}$ affect condition *ii*) if and only if c_i is an odd exponent. Let c_r be first odd exponent in the way $x_r \geq x_j$ for all x_j such that c_j is an odd number, then following computing sentences are equivalent to condition *ii*):

1. (sum,w[i]c[i],i,1,h) = 0 mod(2)
2. if (c[i] odd number) then
   (sum,w[i]c[i],i,1,h) = 0 mod(2)
3. if (i not r and c[i] odd number) then
   (sum,w[i]c[i],i,1,h) = 1 mod(2)
4. if (i not r and c[i] odd number) then
   (sum,w[i],i,1,h) = 1 mod(2)

In this way, in order to verify above four conditions, observe that c_i exponents are free for all $i \neq r$ and only c_r control the parity condition *ii*), i.e. parity is controlled by the bit w_r throughout the equality

$$w_r + \sum_{i \neq r} w_i = 0 \,\mathrm{mod} 2, \qquad (7)$$

where w_i bits are free for all $i \neq r$. So, we have next result:

Theorem 3.1. *Let $R = \mathcal{C}(\mathbb{S}_1)$ be the ring of real continuos functions defined on the unit circle. The number of all feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank n is given by*

$$\tilde{p}_R(n) = \sum_x \tilde{p}_R(x) = \sum_x 2^k,$$

where, if x is a partition denoted by $x = x_1^{c_1}, x_2^{c_2}, \cdots, x_h^{c_h}$, then $k = h$ if not exists an odd exponent c_i in partition x, and $k = h - 1$ in other case.

Proof. It is known that the set of all feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank n is the disjoint union, in partitions of n, of sets of all feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank n throughout a given partition x. Thus, $\tilde{p}_R(n) = \sum_x \tilde{p}_R(x)$.
On the other hand, in particular case $x = x_1^{c_1}$ with c_1 odd number we have only a colored Ferrers diagram (all building blocks are white) and it verifies $\tilde{p}_R(x) = 2^{h-1} = 1$. In other cases, by sentences 1, 2, 3, 4 and equation (7), it is clear that all feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank n throughout a given partition $x = x_1^{c_1}, x_2^{c_2}, \cdots, x_h^{c_h}$ are in one-to-one correspondence with the set of all binary strings of length k, where $k = h$ if not exists an odd exponent c_i in partition x, and $k = h - 1$ in other case. So, $\tilde{p}_R(x) = 2^k$.
□

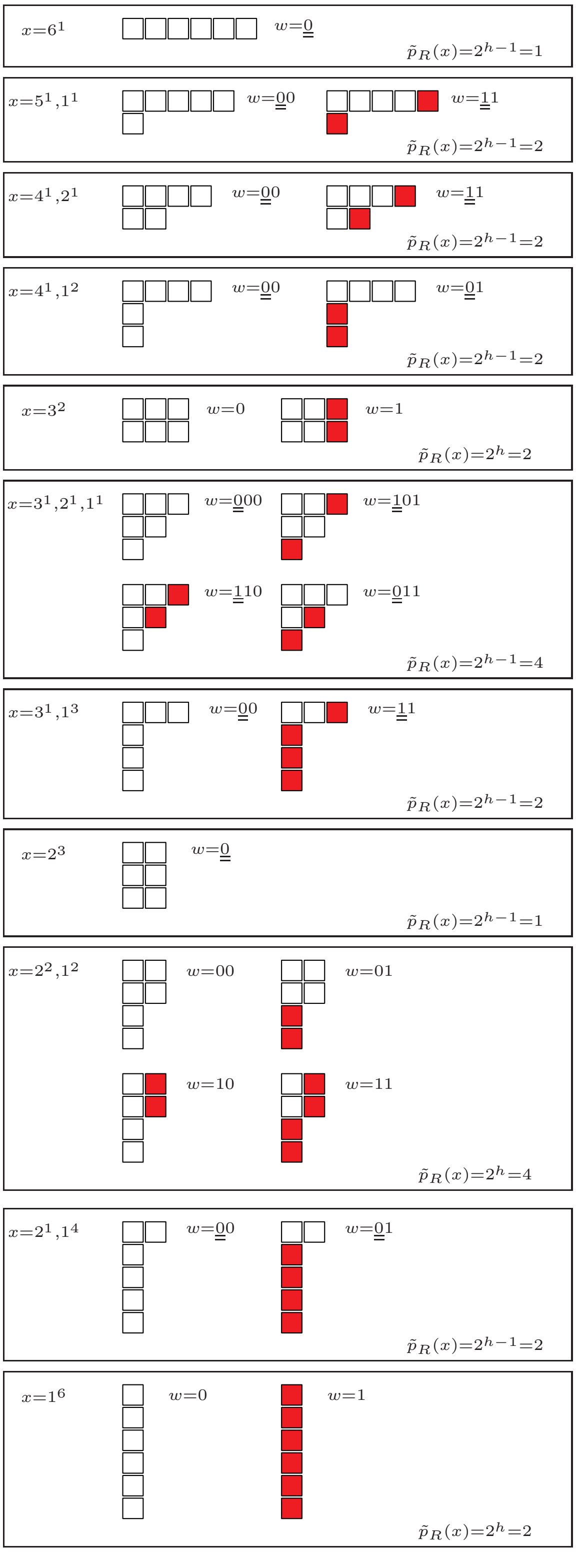


Figure 4. Feedback classes of Example 3.2

Example 3.2. *All $\tilde{p}_R(n)$ feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank $n = 6$, are listed in figure 4. We con-*

clude that there exist $\tilde{p}_R(n) = \sum_x \tilde{p}_R(x) = 24$ feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank $n = 6$. Observe that, inside each binary string w of each partition x we have mark (if possible) parity control bit w_r with double underline.

4 Procedure

It is known an algorithm to obtain all partitions of a given integer n (see [Knuth, 2004]). We include it for a complete study of our subject:

```
input(n){
m=1, h=1, x[1]=n;
for i=2 to n do x[i]=1;
output(x[1]);
while(x[1] not 1)do{
  if(x[h]==2)then{
    m=m+1, x[h]=1, x[m]=1, h=h-1
  }else{
    r=x[h]+1, t=m-h+1, x[h]=r,
    while(t>=r)do{h=h+1, x[h]=r, t=t-r}
    if(t==0)then{m=h
    }else{m=h+1, if(t>1)then{ h=h+1, x[h]=t}}
  }output(x[m])}}
```

Next, we give our procedure to give all colored Ferrers diagrams D, associated to a partition $x = x_1^{c_1}, x_2^{c_2}, \cdots, x_h^{c_h}$, verifying conditions *i*), *ii*), *iii*) and *iv*).

```
input(x[1],x[2],···,x[h],c[1],c[2],···,c[h]){
r=0, k=h, boolean cont=true;
for i=1 to h do if(c[i]mod2==1∧cont)then{
  r=i, k=h-1, cont=false}
for i=0 to pow(2,k)-1 do{
  aux=integer[i].toBinaryString.ofLength[k]
  for j=1 to h do
    if(j<r)then w[j]=aux[j]
    elseif(j>r) then w[j]=aux[j-1]
  w[r]=sum(aux[j]c[j],j,1,h,j not r)mod2
  output(w[1],w[2],···,w[h])}}
input(w[1],w[2],···,w[h]){
D = ∅;
for i=1 to h do{
  if(w[i]=0)then {
  add to D a building white block with c[i]
  rows and x[i] columns
  }else{
  add to D a building grey block with c[i]
  rows and x[i] columns }
}output(D)}
```

Example 4.1. *Enumerate all $\tilde{p}_R(n)$ feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank $n = 50$ associated to partition $x = 9^2, 6^3, 3^4, 2^1$ of n. We have $h = 4$ row-blocks in each diagram D, $r = 2$ position of control bit and $k = h - 1 = 3$ free bits, then there exist 2^3 feedback classes of locally Brunovsky linear systems over the free $R = \mathcal{C}(\mathbb{S}^1)$-module of rank $n = 380$ associated to partition $x = 9^2, 6^3, 3^4, 2^1$ of n:*

		0	1	2	3	4	5	6	7	
9^2	w_1	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	
6^3	w_2	0	1	1	0	1	0	0	1	*pcb*
3^4	w_3	**0**	**0**	**1**	**1**	**0**	**0**	**1**	**1**	
2^1	w_4	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	

pcb:parity control bit

with colored Ferrers diagrams of figure 5

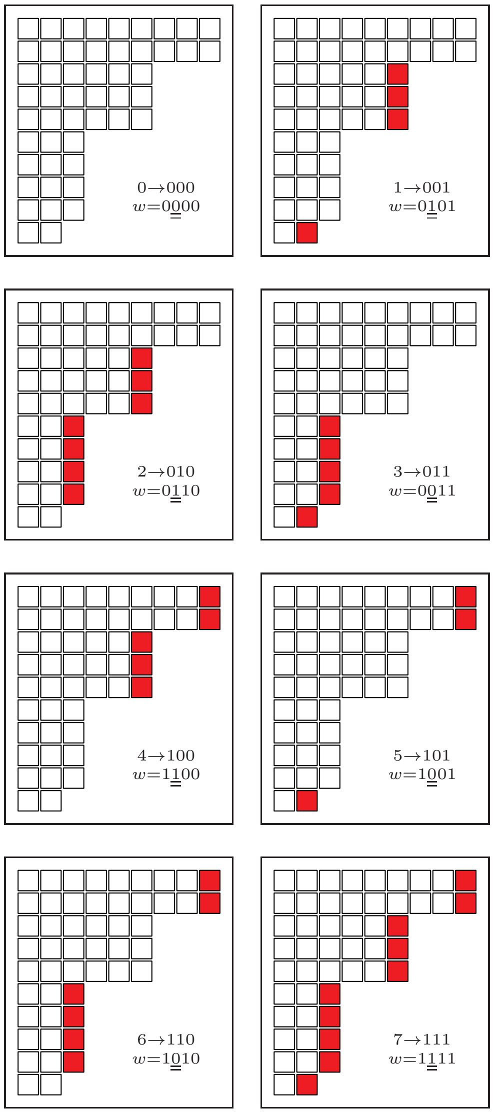


Figure 5. Feedback classes of Example 4.1

5 Number of Locally Brunovsky Linear Systems Over $X = R^{n-1} \oplus P$ an $R = \mathcal{C}(\mathbb{S}^1)$-module

Analogously to section 3, locally Brunovsky linear systems over the finitely generated module $X = R^{n-1} \oplus P$ would be describe by a colored Ferrer's diagram with exactly n building blocks (white or grey)

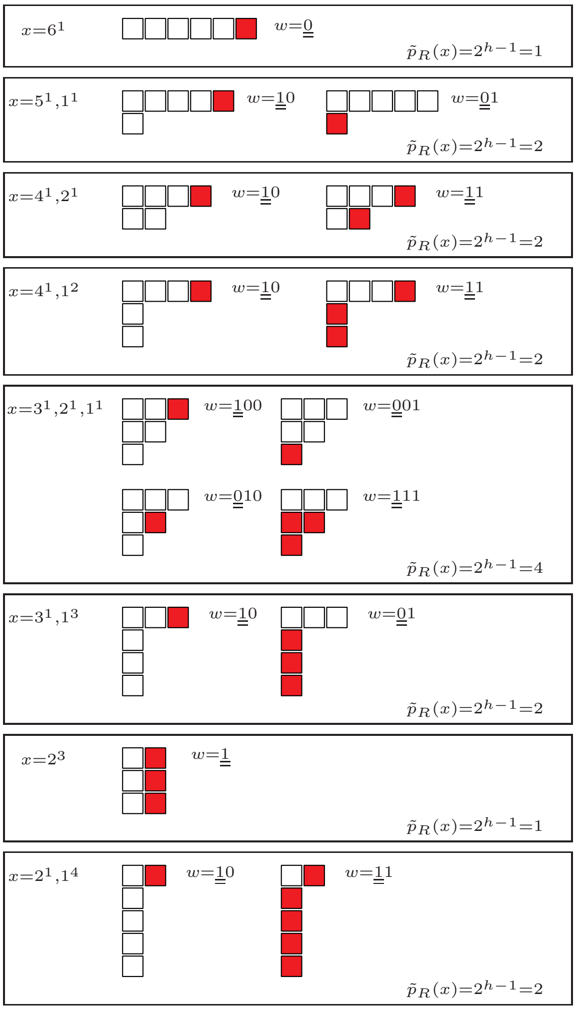


Figure 6. Feedback classes of Example 3.2

where it changes parity condition: *ii*) Parity condition: if $X = R^{n-1} \oplus P$, then there are an odd number of grey blocks in the whole diagram. So, in this case we have if (i not r and c[i] odd number) then (sum,w[i],i,1,h) = 0 mod(2). Observe that, in order to write a procedure we have to replace the computing line

w[r]=sum(aux[j]c[j],j,1,h,j not r)+1 mod2.

Theorem 5.1. *Let $R = \mathcal{C}(\mathbb{S}_1)$ be the ring of real continuos functions defined on the unit circle. The number $\tilde{p}_R(n)$ of all feedback classes of locally Brunovsky linear systems over $X = R^{n-1} \oplus P$ a $R = \mathcal{C}(\mathbb{S}^1)$-module is given by sum of $\tilde{p}_R(x)$ on x, where, if x is a partition denoted by $x = x_1^{c_1}, x_2^{c_2}, \cdots, x_h^{c_h}$, then $\tilde{p}_R(x) = 0$ if not exists an odd exponent c_i in partition x, and $\tilde{p}_R(x) = 2^{h-1}$ in other case.*

Example 5.2. *All $\tilde{p}_R(n)$ feedback classes of locally Brunovsky linear systems over $X = R^5 \oplus P$ an $R = \mathcal{C}(\mathbb{S}^1)$-module, are listed in figure 6. Note that partitions $x = 3^2$, $x = 2^2, 1^2$ and $x = 1^6$ verify $\tilde{p}_R(x) = 0$, so this partitions do not math feedback classes.*

6 Conclusion

In this paper, we design computing procedure for obtaining feedback equivalent classes of linear systems under determined conditions. In this way, our blow-up relation from integers partitions to feedback classes lead to suppose that it is possible to design computing procedures over other similar rings.

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