A COMPUTATIONAL EFFICIENT APPROACH TO THE DYNAMIC MODELING OF 6-DOF PARALLEL MANIPULATORS

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Abstract
Due to their closed-loop structure and kinematic constraints, dynamic modeling of parallel manipulators presents an inherent complexity.
In this paper an approach based on the manipulator generalized momentum is proposed. This approach is used to obtain the dynamic model of a six degrees-of-freedom parallel manipulator. The computational effort is evaluated and compared with the one involved within the classic Lagrange's formulation. It is showed the proposed approach presents a much lower computational burden.

Key words
Robotics, Parallel manipulator, Dynamics, Generalized momentum.

1 Introduction

The dynamic model of a parallel manipulator operated in free space can be mathematically represented, in the Cartesian space, by a system of nonlinear differential equations that may be written in matrix form as:

$$\mathbf{I}(\mathbf{x})\cdot\ddot{\mathbf{x}}+\mathbf{V}(\mathbf{x},\dot{\mathbf{x}})\cdot\dot{\mathbf{x}}+\mathbf{G}(\mathbf{x})=\mathbf{f} \qquad (1)$$

$\mathbf{I}(\mathbf{x})$ being the inertia matrix, $\mathbf{V}(\mathbf{x},\dot{\mathbf{x}})$ the Coriolis and centripetal terms matrix, $\mathbf{G}(\mathbf{x})$ a vector of gravitational generalized forces, $\mathbf{x}$ the generalized position of the mobile platform (end-effector) and $\mathbf{f}$ the controlled generalized force applied on the end-effector. Thus,

$$\mathbf{f}=\mathbf{J}^T(\mathbf{x})\cdot\boldsymbol{\tau} \qquad (2)$$

where $\boldsymbol{\tau}$ is the generalized force developed by the actuators and $\mathbf{J}(\mathbf{x})$ is a jacobian matrix.

The dynamic model of a parallel manipulator is usually developed using one of two approaches: the Newton-Euler or the Lagrange methods. The Newton-Euler approach uses the free body diagrams of the rigid bodies. [Do and Yang, 1988] and [Reboulet and Berthomieu, 1991] use this method on the dynamic modeling of a Stewart platform. [Ji, 1994] presents a study on the influence of leg inertia on the dynamic model of a Stewart platform. [Dasgupta and Mruthyunjaya, 1998] used the Newton-Euler approach to develop a closed-form dynamic model of the Stewart platform. This method was also used by [Khalil and Ibrahim, 2007; Riebe and Ulbrich, 2003; Guo and Li, 2006], among others.

The Lagrange method describes the dynamics of a mechanical system from the concepts of work and energy. [Nguyen and Pooran, 1989] use this method to model a Stewart platform, modeling the legs as point masses. [Lebret et al., 1993] follow an approach similar to the one used by [Nguyen and Pooran, 1989]. Lagrange's method was also used by [Gregório and Parenti-Castelli, 2004] and [Caccavale et al., 2003], for example.

Unfortunately the dynamic models obtained from these classical approaches usually present high computational loads. Therefore, alternative methods have been searched, namely the ones based on the principle of virtual work [Staicu et al., 2007; Tsai, 2000], and screw theory [Gallardo et al., 2003].

In this paper the author presents a new approach to the dynamic modeling of a six degrees-of-freedom (dof) parallel manipulator: the use of the generalized momentum concept.

2 Manipulator Kinematic Structure

Manipulator kinematic structure comprises a fixed (base) platform and a moving (payload) platform, linked together by six independent, identical, open kinematic chains (Figure 1). Each chain comprises two links: the first link (linear actuator) is always normal to the base and has a variable length, l_i, with one of its ends fixed to the base and the other one attached, by a universal joint, to the second link; the second link (fixed-length link) has a fixed length, L, and is attached to the mobile platform by a spherical joint. Points B_i and P_i are the connecting points to the base and mobile platforms.

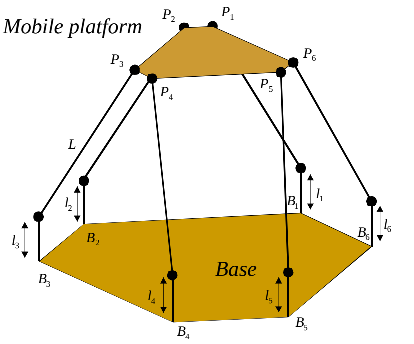


Figure 1. Manipulator kinematic structure.

For kinematic modeling purposes, two frames, {P} and {B}, are attached to the mobile and base platforms, respectively. The generalized position of frame {P} relative to frame {B} may be represented by the vector:

$$\begin{aligned} {}^{B}\mathbf{x}_{P}\,|_{B|E} &= \begin{bmatrix} x_P & y_P & z_P & \psi_P & \theta_P & \varphi_P \end{bmatrix}^T \\ &= \begin{bmatrix} {}^{B}\mathbf{x}^{T}_{P(pos)}\,|_{B} & {}^{B}\mathbf{x}^{T}_{P(o)}\,|_{E} \end{bmatrix}^T \end{aligned} \tag{3}$$

where ${}^{B}\mathbf{x}_{P(pos)}\,|_{B} = \begin{bmatrix} x_P & y_P & z_P \end{bmatrix}^T$ is the position of the origin of frame {P} relative to frame {B}, and ${}^{B}\mathbf{x}_{P(o)}\,|_{E} = \begin{bmatrix} \psi_P & \theta_P & \varphi_P \end{bmatrix}^T$ defines an Euler angle system representing orientation of frame {P} relative to {B}. The used Euler angle system corresponds to the basic rotations [Vukobratovic and Kircanski, 1986]: ψ_P about $\mathbf{z}_P$; θ_P about the rotated axis $\mathbf{y}_{P'}$; and φ_P about the rotated axis $\mathbf{x}_{P''}$. The rotation matrix is given by:

$${}^{B}\mathbf{R}_{P} = \begin{bmatrix} C\psi_P C\theta_P & C\psi_P S\theta_P S\varphi_P - S\psi_P C\varphi_P & C\psi_P S\theta_P C\varphi_P + S\psi_P S\varphi_P \\ S\psi_P C\theta_P & S\psi_P S\theta_P S\varphi_P + C\psi_P C\varphi_P & S\psi_P S\theta_P C\varphi_P - C\psi_P S\varphi_P \\ -S\theta_P & C\theta_P S\varphi_P & C\theta_P C\varphi_P \end{bmatrix} \tag{4}$$

$S(\cdot)$ and $C(\cdot)$ correspond to the sine and cosine functions, respectively.

The manipulator position and velocity kinematic models are known [Merlet and Gosselin, 1991], being obtainable from the geometrical analysis of the kinematics chains. The velocity kinematics is represented by the Euler angles jacobian matrix, $\mathbf{J}_E$, or the kinematics jacobian, $\mathbf{J}_C$. These jacobians relate the velocities of the active joints (actuators) with the generalized velocity of the mobile platform:

$$\dot{\mathbf{l}} = \mathbf{J}_E \cdot {}^{B}\dot{\mathbf{x}}_{P}\,|_{B|E} = \mathbf{J}_E \cdot \begin{bmatrix} {}^{B}\dot{\mathbf{x}}_{P(pos)}\,|_{B} \\ {}^{B}\dot{\mathbf{x}}_{P(o)}\,|_{E} \end{bmatrix} \tag{5}$$

$$\dot{\mathbf{l}} = \mathbf{J}_C \cdot {}^{B}\dot{\mathbf{x}}_{P}\,|_{B} = \mathbf{J}_C \cdot \begin{bmatrix} {}^{B}\dot{\mathbf{x}}_{P(pos)}\,|_{B} \\ {}^{B}\boldsymbol{\omega}_{P}\,|_{B} \end{bmatrix} \tag{6}$$

with

$$\dot{\mathbf{l}} = \begin{bmatrix} \dot{l}_1 & \dot{l}_2 & \cdots & \dot{l}_6 \end{bmatrix}^T \tag{7}$$

$${}^{B}\boldsymbol{\omega}_{P}\,|_{B} = \mathbf{J}_A \cdot {}^{B}\dot{\mathbf{x}}_{P(o)}\,|_{E} \tag{8}$$

and [Vukobratovic and Kircanski, 1986]

$$\mathbf{J}_A = \begin{bmatrix} 0 & -S\psi_P & C\theta_P C\psi_P \\ 0 & C\psi_P & C\theta_P S\psi_P \\ 1 & 0 & -S\theta_P \end{bmatrix} \tag{9}$$

Vectors ${}^{B}\dot{\mathbf{x}}_{P(pos)}\,|_{B} \equiv {}^{B}\mathbf{v}_{P}\,|_{B}$ and ${}^{B}\boldsymbol{\omega}_{P}\,|_{B}$ represent the linear and angular velocity of the mobile platform relative to {B}, and ${}^{B}\dot{\mathbf{x}}_{P(o)}\,|_{E}$ represents the Euler angles time derivative.

3 Dynamic Modeling Using the Generalized Momentum Approach

The generalized momentum of a rigid body, $\mathbf{q}_c$, may be obtained using the following general expression:

$$\mathbf{q}_c = \mathbf{I}_c \cdot \mathbf{u}_c \tag{10}$$

Vector $\mathbf{u}_c$ represents the generalized velocity (linear and angular) of the body and $\mathbf{I}_c$ is its inertia matrix. Vectors $\mathbf{q}_c$ and $\mathbf{u}_c$, and inertia matrix $\mathbf{I}_c$ must be expressed in the same referential.

Equation (10) may also be written as:

$$\mathbf{q}_c = \begin{bmatrix} \mathbf{Q}_c \\ \mathbf{H}_c \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{c(tra)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{c(rot)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v}_c \\ \boldsymbol{\omega}_c \end{bmatrix} \tag{11}$$

where $\mathbf{Q}_c$ is the linear momentum vector due to rigid body translation, and $\mathbf{H}_c$ is the angular momentum vector due to body rotation. $\mathbf{I}_{c(tra)}$ is the translational inertia matrix, and $\mathbf{I}_{c(rot)}$ the rotational inertia matrix. $\mathbf{v}_c$ and $\boldsymbol{\omega}_c$ are the body linear and angular velocities.

The kinetic component of the generalized force acting on the body can be computed from the time derivative of equation (10):

$$\mathbf{f}_{c(kin)} = \dot{\mathbf{q}}_c = \dot{\mathbf{I}}_c \cdot \mathbf{u}_c + \mathbf{I}_c \cdot \dot{\mathbf{u}}_c \tag{12}$$

with force and momentum expressed in the same frame.

3.1 Mobile Platform Modeling

The linear momentum of the mobile platform, written in frame {B}, may be obtained from the following expression:

$$\mathbf{Q}_P\,|_{B} = m_P \cdot {}^{B}\mathbf{v}_{P}\,|_{B} = \mathbf{I}_{P(tra)} \cdot {}^{B}\mathbf{v}_{P}\,|_{B} \tag{13}$$

$\mathbf{I}_{P(tra)}$ is the translational inertia matrix of the mobile platform,

$$\mathbf{I}_{P(tra)} = \mathrm{diag}([m_P \quad m_P \quad m_P]) \tag{14}$$

m_P being its mass.

The angular momentum, also written in frame {B}, is:

$$\mathbf{H}_P\,|_B = \mathbf{I}_{P(rot)}\,|_B \cdot {}^B\boldsymbol{\omega}_P\,|_B \tag{15}$$

$\mathbf{I}_{P(rot)}\,|_B$ represents the rotational inertia matrix of the mobile platform, expressed in the base frame {B}.

The inertia matrix of a rigid body is constant when expressed in a frame that is fixed relative to that body. Furthermore if the frame axes coincide with the principal directions of inertia of the body, then all inertia products are zero and the inertia matrix is diagonal. Therefore, the rotational inertia matrix of the mobile platform, when expressed in frame {P}, may be written as:

$$\mathbf{I}_{P(rot)}\,|_P = \mathrm{diag}([I_{P_{xx}} \quad I_{P_{yy}} \quad I_{P_{zz}}]) \tag{16}$$

This inertia matrix can be written in frame {B} using the following transformation [Torby, 1984]:

$$\mathbf{I}_{P(rot)}\,|_B = {}^B\mathbf{R}_P \cdot \mathbf{I}_{P(rot)}\,|_P \cdot {}^B\mathbf{R}_P^T \tag{17}$$

The generalized momentum of the mobile platform, expressed in frame {B}, can be obtained from the simultaneous use of equations (13) and (15):

$$\mathbf{q}_P\,|_B = \begin{bmatrix} \mathbf{I}_{P(tra)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{P(rot)}\,|_B \end{bmatrix} \cdot \begin{bmatrix} {}^B\mathbf{v}_P\,|_B \\ {}^B\boldsymbol{\omega}_P\,|_B \end{bmatrix} \tag{18}$$

where

$$\mathbf{I}_P\,|_B = \begin{bmatrix} \mathbf{I}_{P(tra)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{P(rot)}\,|_B \end{bmatrix} \tag{19}$$

is the mobile platform inertia matrix written in the base frame {B}.

The combination of equations (8) and (15) results into:

$$\mathbf{H}_P\,|_B = \mathbf{I}_{P(rot)}\,|_B \cdot \mathbf{J}_A \cdot {}^B\dot{\mathbf{x}}_{P(o)}\,|_E \tag{20}$$

Accordingly, equation (18) may be rewritten as:

$$\mathbf{q}_P\,|_B = \begin{bmatrix} \mathbf{I}_{P(tra)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{P(rot)}\,|_B \end{bmatrix} \cdot \begin{bmatrix} \Im & 0 \\ 0 & \mathbf{J}_A \end{bmatrix} \cdot \begin{bmatrix} {}^B\mathbf{v}_P\,|_B \\ {}^B\dot{\mathbf{x}}_{P(o)}\,|_E \end{bmatrix} \tag{21}$$

$$\mathbf{q}_P\,|_B = \mathbf{I}_P\,|_B \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P\,|_{B|E} \tag{22}$$

T being a matrix transformation defined by:

$$\mathbf{T} = \begin{bmatrix} \Im & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_A \end{bmatrix} \tag{23}$$

The time derivative of equation (22) results into:

$$\begin{aligned} {}^P\mathbf{f}_{P(kin)}\,|_B &= \dot{\mathbf{q}}_P\,|_B \\ &= \frac{d}{dt}\left(\mathbf{I}_P\,|_B \cdot \mathbf{T}\right) {}^B\dot{\mathbf{x}}_P\,|_{B|E} + \mathbf{I}_P\,|_B \cdot \mathbf{T} \cdot {}^B\ddot{\mathbf{x}}_P\,|_{B|E} \end{aligned} \tag{24}$$

${}^P\mathbf{f}_{P(kin)}\,|_B$ is the kinetic component of the generalized force acting on {P} due to the mobile platform motion, expressed in frame {B}. The corresponding actuating forces, $\boldsymbol{\tau}_{P(kin)}$, may be computed from the following relation:

$$\boldsymbol{\tau}_{P(kin)} = \mathbf{J}_C^{-T} \cdot {}^P\mathbf{f}_{P(kin)}\,|_B \tag{25}$$

where

$${}^P\mathbf{f}_{P(kin)}\,|_B = \begin{bmatrix} {}^P\mathbf{F}_{P(kin)}^T\,|_B & {}^P\mathbf{M}_{P(kin)}^T\,|_B \end{bmatrix}^T \tag{26}$$

Vector ${}^P\mathbf{F}_{P(kin)}\,|_B$ represents the force vector acting on the centre of mass of the mobile platform, and ${}^P\mathbf{M}_{P(kin)}\,|_B$ represents the moment vector acting on the mobile platform, expressed in the base frame, {B}.

From equation (24) it can be concluded that two matrices playing the roles of the inertia matrix and the Coriolis and centripetal terms matrix are:

$$\mathbf{I}_P\,|_B \cdot \mathbf{T} \tag{27}$$

$$\frac{d}{dt}\left(\mathbf{I}_P\,|_B \cdot \mathbf{T}\right) \tag{28}$$

It must be emphasized that these matrices do not have the properties of inertia or Coriolis and centripetal terms matrices and therefore should not, strictly, be named as such. Nevertheless, throughout the paper the names "inertia matrix" and "Coriolis and centripetal terms matrix" may be used if there is no risk of misunderstanding.

3.2 Actuators Modeling

As the manipulator actuators can only move perpendicularly to the base plane, their angular velocity relative to frame {B} is always zero. So, each actuator can be modeled as a point mass located at its centre of mass.

The linear momentum of each actuator along direction $\mathbf{z}_B$, is obtainable from:

$$q_{A_i} = m_A \cdot \dot{l}_i \tag{29}$$

where m_A is the mass and $\dot{l}_i$ the velocity of actuator i.

Simultaneously considering the six actuators results into:

$$\mathbf{q}_A = \begin{bmatrix} q_{A_1} \\ q_{A_2} \\ \vdots \\ q_{A_6} \end{bmatrix} = m_A \begin{bmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \vdots \\ \dot{l}_6 \end{bmatrix} = m_A \cdot \dot{\mathbf{l}} \tag{30}$$

The use of velocity kinematics and matrix transformation **T** in equation (30) leads to:

$$\mathbf{q}_A = m_A \cdot \mathbf{J}_C \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P\,|_{B|E} \tag{31}$$

The kinetic component of the actuating forces, $\boldsymbol{\tau}_{A(kin)}$, due to actuators translation may be obtained from the time derivative of equation (31):

$$\boldsymbol{\tau}_{A(kin)} = \dot{\mathbf{q}}_A = m_A \cdot \left(\dot{\mathbf{J}}_E \cdot {}^B\dot{\mathbf{x}}_P\,|_{B|E} + \mathbf{J}_E \cdot {}^B\ddot{\mathbf{x}}_P\,|_{B|E}\right) \tag{32}$$

Multiplying equation (32) by $\mathbf{J}_C^T$, the inertial component of the generalized force acting on {P} due to actuators translation, expressed in frame {B}, is obtained as:

$$
\begin{aligned}
{}^{P}\mathbf{f}_{A(kin)}\,|_{B} = {} & m_A \cdot \mathbf{J}_C^T \cdot \dot{\mathbf{J}}_E \cdot {}^{B}\dot{\mathbf{x}}_P\,|_{B|E} + \\
& m_A \cdot \mathbf{J}_C^T \cdot \mathbf{J}_E \cdot {}^{B}\ddot{\mathbf{x}}_P\,|_{B|E}
\end{aligned} \tag{33}
$$

The inertia matrix and the Coriolis and centripetal terms matrix will be:

$$m_A \cdot \mathbf{J}_C^T \cdot \mathbf{J}_E \tag{34}$$

$$m_A \cdot \mathbf{J}_C^T \cdot \dot{\mathbf{J}}_E \tag{35}$$

These matrices represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual mobile platform that is equivalent to the six actuators.

3.3 Fixed-length Links Modeling

If the centre of mass of each fixed-length link, cm_L, is located at a constant distance b_{cm} from the fixed-length link to mobile platform connecting point (Figure 2), then its position relative to frame {B} is:

$${}^{B}\mathbf{p}_{L_i}\,|_{B} = {}^{B}\mathbf{x}_{P(pos)}\,|_{B} + {}^{P}\mathbf{p}_i\,|_{B} - \frac{b_{cm}}{L} \cdot \mathbf{a}_i \tag{36}$$

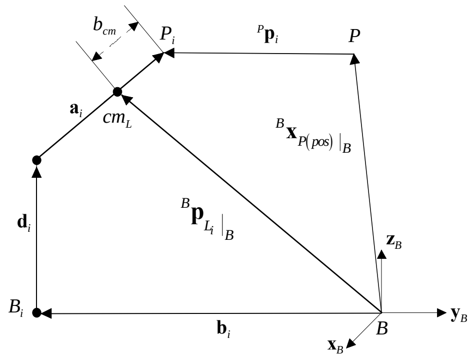


Figure 2. Position of the centre of mass of a fixed-length link *i*.

Equation (36) may be rewritten as:

$$
\begin{aligned}
{}^{B}\mathbf{p}_{L_i}\,|_{B} = {} & \left(1 - \frac{b_{cm}}{L}\right) \cdot {}^{B}\mathbf{x}_{P(pos)}\,|_{B} + \left(1 - \frac{b_{cm}}{L}\right) \cdot {}^{P}\mathbf{p}_i\,|_{B} + \\
& \frac{b_{cm}}{L} \cdot \mathbf{b}_i + \frac{b_{cm}}{L} \cdot \mathbf{d}_i
\end{aligned} \tag{37}
$$

${}^{B}\mathbf{p}_{L_i}\,|_{B}$ being a vector expressed in frame {B}.

The linear velocity of the fixed-length link centre of mass, ${}^{B}\dot{\mathbf{p}}_{L_i}\,|_{B}$, relative to {B} and expressed in the same frame, may be computed from the time derivative of equation (37):

$$
\begin{aligned}
{}^{B}\dot{\mathbf{p}}_{L_i}\,|_{B} = {} & \left(1 - \frac{b_{cm}}{L}\right) \cdot \left({}^{B}\dot{\mathbf{x}}_{P(pos)}\,|_{B} + {}^{B}\boldsymbol{\omega}_P\,|_{B} \times {}^{P}\mathbf{p}_i\,|_{B}\right) + \\
& \frac{b_{cm}}{L} \cdot \dot{l}_i \cdot \mathbf{z}_B
\end{aligned} \tag{38}
$$

Equation (38) can be rewritten as:

$${}^{B}\dot{\mathbf{p}}_{L_i}\,|_{B} = \mathbf{J}_{B_i} \cdot \begin{bmatrix} {}^{B}\mathbf{v}_P\,|_{B} \\ {}^{B}\boldsymbol{\omega}_P\,|_{B} \end{bmatrix} \tag{39}$$

where the jacobian $\mathbf{J}_{B_i}$ is given by:

$$
\mathbf{J}_{B_i} = \left(1 - \frac{b_{cm}}{L}\right) \cdot \begin{bmatrix}
1 & 0 & 0 & 0 & {}^{P}p_i\,|_{Bz} & -{}^{P}p_i\,|_{By} \\
0 & 1 & 0 & -{}^{P}p_i\,|_{Bz} & 0 & {}^{P}p_i\,|_{Bx} \\
\frac{b_{cm}}{L-b_{cm}} J_{Ci1} & \frac{b_{cm}}{L-b_{cm}} J_{Ci2} & \frac{b_{cm}}{L-b_{cm}} J_{Ci3} + 1 & {}^{P}p_i\,|_{By} + \frac{b_{cm}}{L-b_{cm}} J_{Ci4} & -{}^{P}p_i\,|_{Bx} + \frac{b_{cm}}{L-b_{cm}} J_{Ci5} & \frac{b_{cm}}{L-b_{cm}} J_{Ci6}
\end{bmatrix} \tag{40}
$$

being J_{Cij} the elements of line *i* column *j* of matrix $\mathbf{J}_C$.

The linear momentum of each fixed-length link, $\mathbf{Q}_{L_i}\,|_{B}$, can be represented in frame {B} as:

$$\mathbf{Q}_{L_i}\,|_{B} = m_L \cdot {}^{B}\dot{\mathbf{p}}_{L_i}\,|_{B} \tag{41}$$

where m_L is the fixed-length link mass.

Introducing jacobian $\mathbf{J}_{B_i}$ and matrix transformation $\mathbf{T}$ in the previous equation results into:

$$\mathbf{Q}_{L_i}\,|_{B} = m_L \cdot \mathbf{J}_{B_i} \cdot \mathbf{T} \cdot {}^{B}\dot{\mathbf{x}}_P\,|_{B|E} \tag{42}$$

The kinetic component of the force applied to the fixed-length link due to its translation and expressed in {B} can be obtained from the time derivative of equation (42):

$$
\begin{aligned}
{}^{L_i}\mathbf{f}_{L_i(kin)(tra)}\,|_{B} = \dot{\mathbf{Q}}_{L_i}\,|_{B} = {} & m_L \cdot \frac{d}{dt}\left(\mathbf{J}_{B_i} \cdot \mathbf{T}\right) \cdot {}^{B}\dot{\mathbf{x}}_P\,|_{B|E} + \\
& m_L \cdot \mathbf{J}_{B_i} \cdot \mathbf{T} \cdot {}^{B}\ddot{\mathbf{x}}_P\,|_{B|E}
\end{aligned} \tag{43}
$$

When equation (43) is multiplied by $\mathbf{J}_{B_i}^T$, the kinetic component of the force applied to {P} due to each fixed-length link translation is obtained in frame {B}:

$$
\begin{aligned}
{}^{P}\mathbf{f}_{L_i(kin)(tra)}\,|_{B} & = \mathbf{J}_{B_i}^T \cdot {}^{L_i}\mathbf{f}_{L_i(kin)(tra)}\,|_{B} \\
& = m_L \cdot \mathbf{J}_{B_i}^T \cdot \frac{d}{dt}\left(\mathbf{J}_{B_i} \cdot \mathbf{T}\right) \cdot {}^{B}\dot{\mathbf{x}}_P\,|_{B|E} + \\
& \quad m_L \cdot \mathbf{J}_{B_i}^T \cdot \mathbf{J}_{B_i} \cdot \mathbf{T} \cdot {}^{B}\ddot{\mathbf{x}}_P\,|_{B|E}
\end{aligned} \tag{44}
$$

The inertia matrix and the Coriolis and centripetal terms matrix of the translating fixed-length link being:

$$m_L \cdot \mathbf{J}_{B_i}^T \cdot \mathbf{J}_{B_i} \cdot \mathbf{T} \tag{45}$$

$$m_L \cdot \mathbf{J}_{B_i}^T \cdot \frac{d}{dt}\left(\mathbf{J}_{B_i} \cdot \mathbf{T}\right) \tag{46}$$

These matrices represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual mobile platform that is equivalent to each translating fixed-length link.

On the other hand, the angular momentum of each fixed-length link can be represented in frame {B} as:

$$\mathbf{H}_{L_i}\,|_{B} = \mathbf{I}_{L_i(rot)}\,|_{B} \cdot {}^{B}\boldsymbol{\omega}_{L_i}\,|_{B} \tag{47}$$

It is convenient to express the inertia matrix of the rotating fixed-length link in a frame fixed to the fixed-length link itself, $\{L_i\}\equiv\{\mathbf{x}_{L_i}, \mathbf{y}_{L_i}, \mathbf{z}_{L_i}\}$. So,

$$\mathbf{I}_{L_i(rot)}\big|_B = {}^B\mathbf{R}_{L_i} \cdot \mathbf{I}_{L_i(rot)}\big|_{L_i} \cdot {}^B\mathbf{R}_{L_i}^T \tag{48}$$

where ${}^B\mathbf{R}_{L_i}$ is the orientation matrix of each fixed-length link frame, $\{L_i\}$, relative to the base frame, {B}.

Fixed-length links frames were chosen in the following way: axis $\mathbf{x}_{L_i}$ coincides with the fixed-length link axis and points towards the fixed-length link to mobile platform connecting point, meaning that it is coincident with vector $\mathbf{a}_i$; axis $\mathbf{y}_{L_i}$ is perpendicular to $\mathbf{x}_{L_i}$ and always parallel to the base plane, this condition being possible given the existence of a universal joint in the fixed-length link to actuator connecting point that negates any rotation along its own axis; axis $\mathbf{z}_{L_i}$ completes the referential following the right hand rule, and its projection along axis $\mathbf{z}_B$ is always positive. Thus, matrix ${}^B\mathbf{R}_{L_i}$ becomes:

$$ {}^B\mathbf{R}_{L_i} = \begin{bmatrix} \mathbf{x}_{L_i} & \mathbf{y}_{L_i} & \mathbf{z}_{L_i} \end{bmatrix} \tag{49}$$

where

$$\mathbf{x}_{L_i} = \begin{bmatrix} \dfrac{a_{ix}}{L} & \dfrac{a_{iy}}{L} & \dfrac{a_{iz}}{L} \end{bmatrix}^T \tag{50}$$

$$\mathbf{y}_{L_i} = \begin{bmatrix} -\dfrac{a_{iy}}{\sqrt{a_{ix}^2 + a_{iy}^2}} & \dfrac{a_{ix}}{\sqrt{a_{ix}^2 + a_{iy}^2}} & 0 \end{bmatrix}^T \tag{51}$$

$$\mathbf{z}_{L_i} = \mathbf{x}_{L_i} \times \mathbf{y}_{L_i} \tag{52}$$

So, the inertia matrices of the fixed-length links can be written as

$$\mathbf{I}_{L_i(rot)}\big|_{L_i} = \mathrm{diag}([I_{L_{xx}} \quad I_{L_{yy}} \quad I_{L_{zz}}]) \tag{53}$$

where $I_{L_{xx}}$, $I_{L_{yy}}$ and $I_{L_{zz}}$ are the fixed-length link moments of inertia expressed in its own frame.

The angular velocity of each fixed-length link can be obtained from the linear velocities of two points belonging to it. If these two points are taken as the fixed-length link to actuator, and the fixed-length link to mobile platform connecting points, the following expression results:

$$ {}^B\boldsymbol{\omega}_{L_i}\big|_B \times \mathbf{a}_i = {}^B\mathbf{v}_P\big|_B + {}^B\boldsymbol{\omega}_P\big|_B \times {}^P\mathbf{p}_i\big|_B - \dot{l}_i \cdot \mathbf{z}_B \tag{54}$$

As the fixed-length link cannot rotate along its own axis, the angular velocity along $\mathbf{x}_{L_i} = \hat{\mathbf{a}}_i$ is always zero, and vectors $\mathbf{a}_i$ and ${}^B\boldsymbol{\omega}_{L_i}\big|_B$ are always perpendicular.

This property enables equation (54) to be rewritten as:

$$ {}^B\boldsymbol{\omega}_{L_i}\big|_B = \frac{1}{L^2} \cdot \left[\mathbf{a}_i \times \left({}^B\mathbf{v}_P\big|_B + {}^B\boldsymbol{\omega}_P\big|_B \times {}^P\mathbf{p}_i\big|_B - \dot{l}_i \cdot \mathbf{z}_B\right)\right] \tag{55}$$

or,

$$ {}^B\boldsymbol{\omega}_{L_i}\big|_B = \mathbf{J}_{D_i} \cdot \begin{bmatrix} {}^B\mathbf{v}_P\big|_B \\ {}^B\boldsymbol{\omega}_P\big|_B \end{bmatrix} \tag{56}$$

where jacobian $\mathbf{J}_{D_i}$ is given by:

$$\mathbf{J}_{D_i} = \frac{1}{L^2} \cdot \begin{bmatrix} -a_{iy}J_{Ci1} & -a_{iy}J_{Ci2} - a_{iz} & a_{iy}(1-J_{Ci3}) & a_{iy}\left({}^Pp_i\big|_{By} - J_{Ci4}\right) + a_{iz}\,{}^Pp_i\big|_{Bz} & -a_{iy}\left({}^Pp_i\big|_{Bx} + J_{Ci5}\right) & -a_{iy}J_{Ci6} - a_{iz}\,{}^Pp_i\big|_{Bx} \\ a_{iz} + a_{ix}J_{Ci1} & a_{ix}J_{Ci2} & -a_{iz}(1-J_{Ci3}) & -a_{ix}\left({}^Pp_i\big|_{By} - J_{Ci4}\right) & a_{iz}\,{}^Pp_i\big|_{Bz} + a_{ix}\left({}^Pp_i\big|_{Bx} + J_{Ci5}\right) & -a_{iz}\,{}^Pp_i\big|_{By} + a_{ix}J_{Ci6} \\ -a_{iy} & a_{ix} & 0 & -a_{ix}\,{}^Pp_i\big|_{Bz} & -a_{iy}\,{}^Pp_i\big|_{Bz} & a_{ix}\,{}^Pp_i\big|_{Bx} + a_{iy}\,{}^Pp_i\big|_{By} \end{bmatrix} \tag{57}$$

Introducing jacobian $\mathbf{J}_{D_i}$ and matrix transformation $\mathbf{T}$ in equation (47) results into:

$$\mathbf{H}_{L_i}\big|_B = \mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P\big|_{B|E} \tag{58}$$

The kinetic component of the generalized force applied to the fixed-length link, due to its rotation and expressed in {B} can be obtained from the time derivative of equation (58):

$$\begin{aligned} {}^{L_i}\mathbf{f}_{L_i(kin)(rot)}\big|_B &= \dot{\mathbf{H}}_{L_i}\big|_B \\ &= \frac{d}{dt}\left(\mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T}\right) {}^B\dot{\mathbf{x}}_P\big|_{B|E} + \\ &\quad \mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T} \cdot {}^B\ddot{\mathbf{x}}_P\big|_{B|E} \end{aligned} \tag{59}$$

When equation (59) is pre-multiplied by $\mathbf{J}_{D_i}^T$ the kinetic component of the generalized force applied to {P} due to each fixed-length link rotation is obtained in frame {B}:

$$\begin{aligned} {}^P\mathbf{f}_{L_i(kin)(rot)}\big|_B &= \mathbf{J}_{D_i}^T \cdot {}^{L_i}\mathbf{f}_{L_i(kin)(rot)}\big|_B \\ &= \mathbf{J}_{D_i}^T \cdot \frac{d}{dt}\left(\mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T}\right) {}^B\dot{\mathbf{x}}_P\big|_{B|E} + \\ &\quad \mathbf{J}_{D_i}^T \cdot \mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T} \cdot {}^B\ddot{\mathbf{x}}_P\big|_{B|E} \end{aligned} \tag{60}$$

The inertia matrix and the Coriolis and centripetal terms matrix of the rotating fixed-length link may be written as:

$$\mathbf{J}_{D_i}^T \cdot \mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T} \tag{61}$$

$$\mathbf{J}_{D_i}^T \cdot \frac{d}{dt}\left(\mathbf{I}_{L_i(rot)}\big|_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T}\right) \tag{62}$$

These matrices represent the inertia matrix and the Coriolis and centripetal terms matrix of a virtual mobile platform that is equivalent to each rotating fixed-length link.

It should be noted that equations (24), (33), (44) and (60) by providing expressions for the kinetic component of the generalized force applied to {P} and expressed in {B}, enable a clear physical meaning to the moments applied to {P}.

3.4 Dynamic Model Gravitational Component

Given a general frame {**x**, **y**, **z**}, with $\mathbf{z} \equiv -\hat{\mathbf{g}}$, the potential energy of a rigid body is given by:

$$P_c = m_c \cdot g \cdot z_c \tag{63}$$

where m_c is the body mass, g is the modulus of the gravitational acceleration and z_c the distance, along **z**, from the frame origin to the body centre of mass.

The gravitational components of the generalized forces acting on {P} can be easily obtained from the potential energy of the different bodies that compose the system:

$$ {}^{P}\mathbf{f}_{P(gra)}\,|_{B|E} = \frac{\partial P_P\left({}^{B}\mathbf{x}_P\,|_{B|E}\right)}{\partial\, {}^{B}\mathbf{x}_P\,|_{B|E}} \tag{64}$$

$$ {}^{P}\mathbf{f}_{A_i(gra)}\,|_{B|E} = \frac{\partial P_{A_i}\left({}^{B}\mathbf{x}_P\,|_{B|E}\right)}{\partial\, {}^{B}\mathbf{x}_P\,|_{B|E}} \tag{65}$$

$$ {}^{P}\mathbf{f}_{L_i(gra)}\,|_{B|E} = \frac{\partial P_{L_i}\left({}^{B}\mathbf{x}_P\,|_{B|E}\right)}{\partial\, {}^{B}\mathbf{x}_P\,|_{B|E}} \tag{66}$$

The three vectors ${}^{P}\mathbf{f}_{P(gra)}\,|_{B|E}$, ${}^{P}\mathbf{f}_{A_i(gra)}\,|_{B|E}$ and ${}^{P}\mathbf{f}_{L_i(gra)}\,|_{B|E}$ represent the gravitational components of the generalized forces acting on {P}, expressed using the Euler angles system, due to, in that order, the mobile platform, each actuator and each fixed-length link. Therefore, to be added to the kinetic force components, these vectors must be transformed to be expressed in frame {B}. This may be done pre-multiplying the gravitational components force vectors by the following matrix:

$$\begin{bmatrix} \Im & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_A^{-T} \end{bmatrix} \tag{67}$$

4 Computational Effort of the Dynamic Model

The computational effort of the dynamic model obtained through the use of the generalized momentum approach is compared with the one resulting from applying the Lagrange method using the Koditschek representation [Lebret et al., 1993; Koditschek, 1984].

As the largest difference between the two methods rests on how the Coriolis and centripetal terms matrices are calculated, the two models are evaluated by the number of arithmetic operations involved in the computation of these matrices. The results were obtained using the symbolic computational software Maple®, and are presented in Table 1.

Table 1. Computational burden of the dynamic model.

	Lagrange			Generalized Momentum		
	Add	Mul	Div	Add	Mul	Div
Mobile platform	310	590	0	94	226	0
Six actuators	3028	4403	30	724	940	18
Translating link	751	1579	6	131	279	4
Rotating link	2180	3711	7	355	664	7
Total operations	20924	36733	108	3734	6824	84

The dynamic model obtained using the generalized momentum approach is computationally much more efficient, and its superiority manifests precisely in the computation of the matrices requiring the largest relative computational effort: the Coriolis and centripetal terms matrices.

The proposed approach was used in the dynamic modeling of a 6-dof parallel manipulator similar to the Stewart platform. Nevertheless, it can be applied to any mechanism.

5 Numerical Simulation

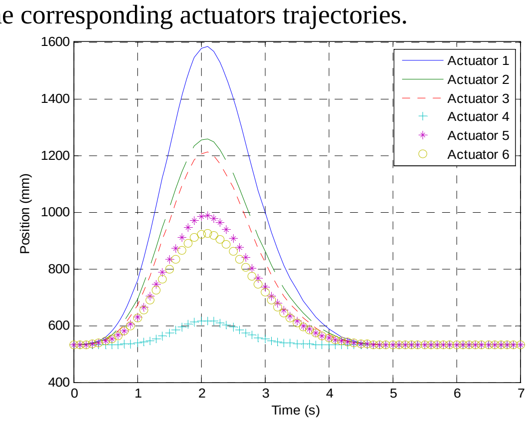
A 6-dof parallel manipulator presenting the kinematic and dynamic parameters shown in Table 2 was considered.

Table 2. Manipulator parameters.

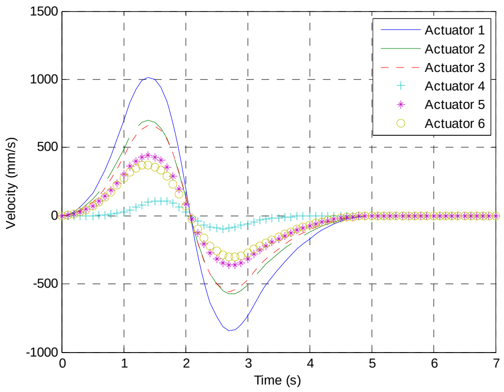
Para.	Value	Para.	Value	Para.	Value
r_B	1.500 m	m_P	1.430 kg	I_{Pzz}	0.4 kg·m^2
r_P	0.750 m	m_A	0.123 kg	I_{Lxx}	0.0 kg·m^2
L	1.837 m	m_L	0.389 kg	I_{Lyy}	0.1 kg·m^2
ϕ_B	15º	I_{Pxx}	0.2 kg·m^2	I_{Lzz}	0.1 kg·m^2
ϕ_P	0º	I_{Pyy}	0.2 kg·m^2	b_{cm}	0.918 m

A trajectory was specified in task space. The moving platform initial position is P_1 = [0, 0, 2000, 0, 0, 0] (mm; deg). The moving platform is then displaced to point P_2 = [-100, -200, 2500, 15, -15, 15] (mm; deg), and finally it returns to point P_1.

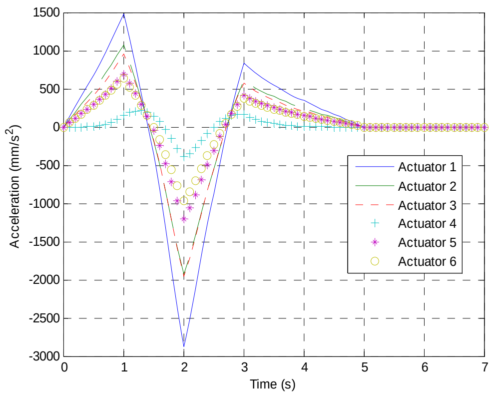
Third order trigonometric splines were interpolated between the specified points, in order to obtain continuous and smooth trajectories. Figure 3 shows the corresponding actuators trajectories.



(a)



(b)



(c)

Figure 3. Actuators trajectories: (*a*) – position; (*b*) – velocity; (*c*) – acceleration.

Figure 4 shows the developed actuators forces, necessary to follow the specified trajectories.

Figures 5 to 7 show the contribution of the mobile platform, the six fixed-length links, and the six actuators to the total developed actuators forces, presented in Figure 4.

It is important to note, the contribution of both the mobile platform and the six fixed-length links are equivalent in magnitude and, therefore, fixed-length links should not be neglected as they are in several related works presented in the literature.

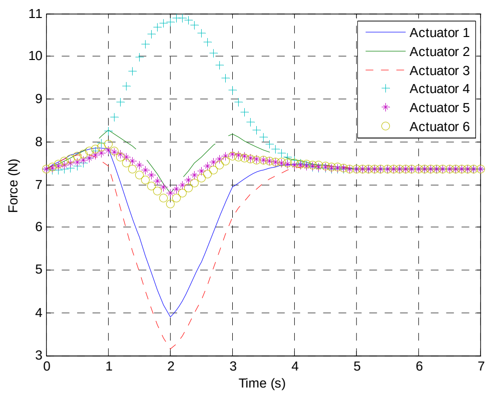


Figure 4. Developed actuators forces.

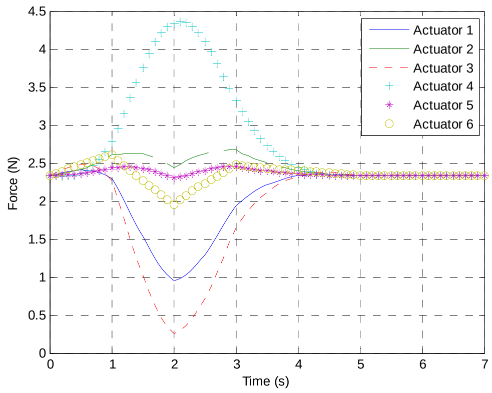


Figure 5. Mobile platform contribution to the developed actuators forces.

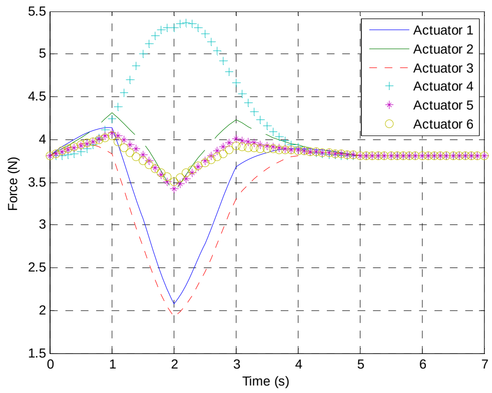


Figure 6. Fixed-length links contribution to the developed actuators forces.

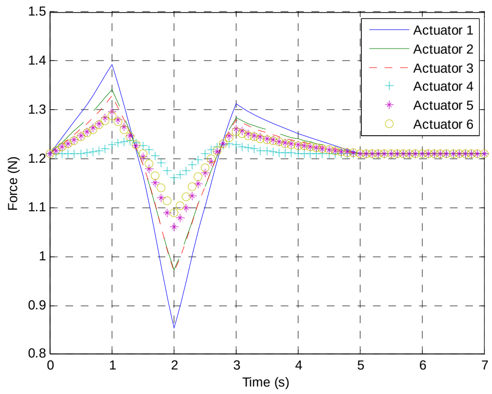


Figure 7. Actuators contribution to the developed actuators forces.

6 Conclusion

Dynamic modeling of parallel manipulators presents an inherent complexity. Despite the intensive study in this topic of robotics, mostly conducted in the last two decades, additional research still has to be done in this area.

In this paper an approach based on the manipulator generalized momentum is explored and applied to the dynamic modeling of parallel manipulators. The generalized momentum is used to compute the kinetic component of the generalized force acting on the mobile platform. Each manipulator rigid body may be considered and analyzed independently. Analytic expressions for the rigid bodies' inertia and Coriolis and centripetal terms matrices are obtained, which can be added, as they are expressed in the same frame. Having these matrices, the kinetic component of the generalized force acting on the mobile platform may be easily computed. This component can be added to the gravitational part of the generalized force, which is obtained through the manipulator potential energy.

The proposed approach is completely general and can be used as a dynamic modeling tool applicable to any mechanism.

The obtained dynamic model was found to be computationally much more efficient than the one resulting from applying the Lagrange method using the Koditschek representation. Its superiority manifesting precisely in the computation of the matrices requiring the largest relative computational effort: the Coriolis and centripetal terms matrices.

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