Optimizing Control of Nonlinear Stochastic Systems with Delay: Application for Flying Vehicle

R.A. Aukasov*, N.E. Rodnishev*, N.S. Chernov*

** Tupolev Kazan State Technical University (KSTU-KAI)*
55 Bolshaya Krasnaya Str. Kazan 420015 Russia
(e-mail: Rustam.Aukasov@tatar.ru Nrodnishev@ynd.ru
nikos500@rambler.ru)

Abstract: Necessary optimum control conditions for nonlinear systems to be described by stochastic differential-difference equations with equality-type limitations being determinative for control are explored. An example of an optimum control search of the system being described by stochastic equations is analyzed.

Keywords: control, delay, random disturbance, optimization explosion.

1. INTRODUCTION

Necessary optimum control conditions of nonlinear stochastic systems with delay are analyzed. Using phase space extension, the initial process, which is described by a system of stochastic differential equations with delay, is reduced to a diffusion markovian process. To research the necessary optimum conditions, proof patterns, described in Rodnishev (2001 a,c) are used.

2. THE PROBLEM STATEMENT

It is required to define the optimum control u, which gives minimum to a terminal functional

$$I_0\left(\right) = \int\limits_{\Omega} \Phi_0(x,a) p(t_k, x) dx \qquad (1)$$

characterizing the effectiveness of a controlled system. The behavior of the controlled system over time-interval $[t_0, t_k]$ is described by nonlinear stochastic differential equations with delay

$$dX_i = \varphi_i\left(t, X\left(t\right), X\left(t-\tau\right), u, a\right) dt + \sum_{j=1}^{n} \sigma_{ij}\left(t, X\left(t\right)\right) d\eta_j\left(t\right), \qquad (2)$$

$$X\left(t\right) = \phi\left(t\right), \quad t \in \left[t_0 - \tau, t_0\right].$$

Here t is a time; t_0, t_k – initial and final points of the time-interval being considered $[t_0, t_k]$; τ is a constant delay; $X(t)$ is n-dimensional vector function of phase coordinates state being defined over time-segment $[t_0 - \tau, t_0]$ by a function $\phi\left(t\right)$; $d\eta_j(t)$ are stochastic Stratonovich differentials of uncorrelated Wiener processes $\eta_j(t)$, with intensities G_j^{η}; $u(t)$ is a deterministic $r-$dimensional section-continuous vector control function; a is a deterministic $l-$dimensional vector of controlling parameters, which defines constructive as well as energy parameters of the system; $p(t_k, x)$ – a distribution density of the state vector components of the system at a finite time-point t_k; x is a realization of the state vector; $\Phi_0(x, a)$ is a given function, and

$$\int\limits_{\Omega} = \underbrace{\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty}}_{n}$$

is n - fold multiple integral.

As it is known, the process described by (2), in the general case, is not marcovian, and Kolmogorov–Fokker–Plank (KFP) techniques cannot be applicable to it. So, to bring the process (2) to the markovian one, let us expand the phase space by excluding the delay from the system (2). To do this, let us cover the time-interval $[t_0, t_k]$ with a lattice having increments τ and nodes $t_q = t_0 + q\tau, \quad q = 1, \ldots, N$. Here q is number of an interval $[t_{q-1}, t_q]$ with length $t_q - t_{q-1} = \tau$, N – the number of intervals, $t_k = t_0 + N\tau$. Let us assign to s the current time over interval $[t_{q-1}, t_q] = \tau$ and introduce a state vector of the system

$$X^q(s) = (X_1(t_{q-1}+s), X_2(t_{q-1}+s), \ldots, X_n(t_{q-1}+s))$$

over interval $[t_{q-1}, t_q]$, where $s \in [0, \tau]$; the upper index designates an interval number, the lower index – a number of a component of the states vector. Let us similarly designate a control over interval $[t_{q-1}, t_q]$ as a vector function

$$u^q(s) = \left(u_1(t_{q-1}+s), u_2(t_{q-1}+s), \ldots, u_r(t_{q-1}+s)\right),$$

and additive disturbances

$$\eta^q(s) = \left(\eta_1(t_{q-1}+s), \eta_2(t_{q-1}+s), \ldots, \eta_n(t_{q-1}+s)\right).$$

Let us introduce an extended state vector with phase state components of the system $X_{1,2,\ldots,q}(s) = (X^1(s), X^2(s), \ldots, X^q(s))$ over consecutively adjoining intervals $[t_{q-1}, t_q]$, $q = 1, \ldots, N$. Then, in accordance with Bellman's optimality principle, with control components of the system over consecutively adjoining intervals $[t_{q-1}, t_q], \quad q = 1, \ldots, N$, the initial problem (1)–(2) is reduced to the definition of control $u_{1,2,\ldots,q}(s) = \left(u^1(s), u^2(s), \ldots, u^q(s)\right)$ which gives minimum to the functional

$$I_0\,(u) = \int\limits_{\Omega_N} \Phi_0(x^q, a) p(\tau, x_{1,2,\ldots,N}) dx_{1,2,\ldots,N} \to \min, \quad (3)$$

characterizing the control efficiency of the system, its behavior being described by the stochastic differential equations over the time intervals $[t_0, t_k]$ by consecutively adjoining intervals $[t_{q-1}, t_q]$, $q = 1, \ldots, N$

$$dX_i^m = \varphi_i(t_{m-1} + s, X^m, X^{m-1}, u^m, a)\, ds + \sum_{j=1}^{n} \sigma_{ij}\,(t_{m-1} + s, X^m)\; d\eta^m, \quad (4)$$

$$s \in [0, \tau]\,, \quad X_i^1(t_0) = x_{i0}(s) = \phi_i\,(t_0 - \tau + s)\,,$$
$$X_i^m(t_{m-1}) = X_i^{m-1}(t_{m-2} + \tau),$$
$$(i = 1, \ldots, n), \quad (m = 1, \ldots, q), \quad (q = 1, \ldots, N).$$

Here $\varphi_i(t_{m-1} + s, x^m, x^{m-1}, u^m)$, $\sigma_{ij}^{(}t_{m-1} + s, x^m)$ are given non-random, non-anticipative functions. Right-hand members of (4) uniformly satisfy known requirements (Gikhman and Skorokhod, 1977) about the existence of (4) over the control u^m. The control $u^q(s)$, defined over interval $[t_{q-1}, t_q]$, in accordance with Bellman's optimality principle, does not worsen the optimal control over preceding intervals. Therefore, expanding the state vector of the system over consequently adjoining intervals $[t_{q-1}, t_q]$, the equations (4) consider the control

$$u_{1,2,\ldots,q}(s) = (u^{*1}(s), u^{*2}(s), \ldots, u^{*m}(s), \ldots, u^{*q-1}(s), u^q(s)),$$

where the asterisk indicates the optimum controls defined over preceding intervals. The equations (4) describe a diffusion markovian process over the time interval $[t_0, t_k]$ in a consecutive manner over adjoining segments$[t_{q-1}, t_q]$. The probability density $p(s, x_{1,2,\ldots,q})$ of the process states $X_{1,2,\ldots,q}(s) = \left(X^1(s), X^2(s), \ldots, X^q(s)\right)$ over consecutively adjoining segments $[t_{q-1}, t_q]$ satisfies the KFP- equation. (6), at nodes t_q – conjugation conditions (7).

Thus, expanding the state vector of the system, the stochastic problem (3), (4) is reduced to the equivalent deterministic problem with distributed parameters (5) - (7) relative to the probability density $p(s, x_{1,2,\ldots,q})$ of the state vector of the system:

$$I_0(u) = \int\limits_{\Omega_N} \Phi_0(x^q, a) p(\tau, x_{1,2,\ldots,N}) dx_{1,2,\ldots,N} \to \min, \quad (5)$$

$$\frac{\partial p(s, x_{1,2,\ldots,q})}{\partial s} = L(s, x_{1,2,\ldots,q}, u^q, a) p(s, x_{1,2,\ldots,q}), \quad (6)$$

$$p(s, x_{1,2,\ldots,q}) = \; = p(s, x^1) p(s, x^2|\tau, x^1) p(s, x^3|\tau, x^2) \cdots p(s, x^q|\tau, x^{q-1})$$
$$p(0, x^1) = \delta(x_1 - x_0), p(0, x^q) = p(\tau, x^{q-1}), \quad (7)$$
$$(q = 1, \ldots, N), \quad \in [0, \tau].$$

Here in (5), (6)

$$\Omega_N = \bigcup_{m=1}^{N} \Omega_m; \int\limits_{\Omega_m} = \underbrace{\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty}}_{n};$$

$$L(s, x_{1,2,\ldots,q}, u^q, a) p(s, x_{1,2,\ldots,q}) = \sum_{m=1}^{q} \sum_{i=1}^{n} \frac{\partial}{\partial x_i^m} A_i^m(s, x^m, x^{m-1}, u^m) p(s, x_{1,2,\ldots,q}) + \frac{1}{2} \sum_{m=1}^{q} \sum_{i=1}^{n} \frac{\partial^2}{(\partial x_i^m)^2} [B_{ii}^m(s, x^m)\, p(s, x_{1,2,\ldots,q})],$$

where $A_i^m(s, x^m, x^{m-1}, u^m, a)$ are the drift coefficients of the process described in (4),

$$A_i^m(s, x^m, x^{m-1}, u^m, a) = \varphi_i(t_{m-1} + s, x^m, x^{m-1}, u^m, a) + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \sigma_{ij}(t_{m-1} + s, x^m)}{\partial x_i^m} \sigma_{ij}(t_{m-1} + s, x^m)\, G_j^\eta;$$

$B_{ii}^m(s, x^m)$ are coefficients of diffusion:

$$B_{ii}^m(s, x^m) = \sum_{j=1}^{n} (\sigma_{ij}(t_{m-1} + s, x^m))^2 G_j^\eta.$$

3. NECESSARY CONDITIONS OF OPTIMUM PROGRAM CONTROL

The conditions of optimum control of the problem (5)–(7), are set by theorem 1, similarly to Rodnishev (2001a).

Theorem 1. (weak principle of the minimum). Let (p^*, u^{*q}, a^*) is the optimum solution to the problem (5)–(7). Then, there exists an identically nonzero function $\lambda(s, x_{1,2,\ldots,q}) \in C^{1,2}$ that

a) $\lambda(s, x_{1,2,\ldots,q})$ satisfies the solution to the Cauchy problem

$$\frac{\partial \lambda(s, x_{1,2,\ldots,q})}{\partial s} + L^*(s, x_{1,2,\ldots,q}, u^q, a) \lambda(s, x_{1,2,\ldots,q}) = 0, \quad (8)$$
$$s \in [\tau, 0]\,, \quad \lambda\,(\tau, x_{1,2,\ldots,q}) = \Phi_0\,(x^q)\,;$$

b) for almost all $s \in [\tau, 0]$ and all $u^q(s)$

$$M\left(\frac{\partial R}{\partial u^q}\right)(u^q - u^{*q}) \geq 0; \quad (9)$$

c) the parameters a^* satisfy the condition

$$M\left(\frac{\partial_0(x^q, a)}{\partial a}\right) + \int\limits_0^\tau M\left(\frac{\partial R}{\partial a}\right) dt = 0. \quad (10)$$

The optimum control satisfies the expression

$$M\left(\frac{\partial R}{\partial u^q}\right) = 0. \quad (11)$$

In relations (8)–(11)

$$L^*(s, x_{1,2,\ldots,q}, u^q, a) \lambda(s, x_{1,2,\ldots,q}) = \sum_{m=1}^{q} \sum_{i=1}^{n} \frac{\partial \lambda(s, x_{1,2,\ldots,q})}{\partial x_i^m} A_i^m(s, x^m, x^{m-1}, u^m, a)$$

$$+\frac{1}{2}\sum_{m=1}^{q} t\sum_{i=1}^{n}\frac{\partial^2\lambda(s,x_{1,2,\ldots,q})}{(\partial x_i^m)^2}B_{ii}^m(s,x^m),$$
$$R = L^*(s,x_{1,2,\ldots,q},u^q,a)\lambda(s,x_{1,2,\ldots,q}),$$

and $M(\cdot)$ is the expectation operator.

To establish necessary optimum conditions of a strong extremum using the time transformation (Girsanov, 1970) $s\to\mu$

$$s(\mu)=\int_0^\mu w(\mu)d\mu,\mu\in[0,\,1],$$
$$\mu(1)=\tau,\quad w(\mu)\geq 0, \tag{12}$$

let us pass from the problem (5)–(7) to the equivalent problem (13)–(16)

$$I_0(u)=\int_{\Omega_N}\Phi_0(x^q)p(\mu(1),x_{1,2,\ldots,N})dx_{1,2,\ldots,N}\to\min, \tag{13}$$

$$\frac{\partial p(\mu,x_{1,2,\ldots,q})}{\partial s}=w(\mu)L(\mu,x_{1,2,\ldots,q},u^q,a)p(\mu,x_{1,2,\ldots,q}), \tag{14}$$
$$p(\mu,x_{1,2,\ldots,q})=p(\mu,x^1)p(\mu,x^2|\mu(1),x^1)p(\mu,x^3|\mu(1),x^2)\cdot\cdot\cdot\cdot p(\mu,x^q|\mu(1),x^{q-1}),$$
$$p(0,x^1)=\delta(x_1-x_0),p\left(0,x^q\right)=p\left(\mu(1),x^{q-1}\right), \tag{15}$$
$$(\,q=1,\ldots,N),\quad \mu\in[0,1],$$
$$w(\mu)\geq 0. \tag{16}$$

Here

$$u^q(\mu)=\begin{cases}u^q(s(\mu)) & at\ \mu\in R_1,\\ arbitrary & at\ \mu\in R_2.\end{cases}$$

$$R_1=\{\mu\in[0,1]\,:\,w(\mu)>0\},$$
$$R_2=\{\mu\in[0,1]\,:\,w(\mu)=0\}.$$

It is quite clear, that the solution $(p^*,\,u^{*q}a^*,w^*)$ to the problem (13)–(16) is also the solution to a problem different from (13)–(16); the difference is that the control u^{*q} is being fixed and the solution (13)–(16) is being searched over $w(\mu)$. Since the limitation (16) has the appearance of $w(\mu)\in W\subset E_1$ and W is a convex set in E_1 having an internal point (a positive semi-axis) we shall find out that in accordance with (9) for $w^*(\mu)$ the condition

$$M\left(\frac{\partial\overline{R}}{\partial w}\right)(w-w^*)\geq 0\quad , \tag{17}$$

is met when applying the local principal of the minimum (theorem 1) to the problem (13)–(16) at the fixed control u^{*q} relative to control $w(\mu)$ where $\bar{R}=w(\mu)L^*(s,x_{1,2,\ldots,q},u^q,a)\lambda(s,x_{1,2,\ldots,q})$. Taking into account the definition of $\overline{R}$ from (17) we get

$$M[R(\mu,x_{1,2,\ldots,q},u^{*q},a,\lambda)](w-w^0)\geq 0 \tag{18}$$

for almost all $\mu\in[0,1]$ and $w(\mu)\geq 0$. From this, it follows that $M[R(\mu,x_{1,2,\ldots,q},u^{*q},a,\lambda)]=0$ for almost all $\mu\in R_1=\{\mu : w^*(\mu)>0\}$ and $M[R(\mu,x_{1,2,\ldots,q},u^{*q},a,\lambda)]\geq 0$ for almost all $\mu\in R_2=\{\mu : w^*(\mu)=0\}$. Drawing an analogy (Girsanov, 1970), the construction $w^*(\mu)$, $u^{*q}(\mu)$, where $w^*(\mu)$ is given as

$$w^*(\mu)=\begin{cases}\tau-0\,, & \mu\in R_1,\\ 0\quad , & \mu\in R_2=[0,1]\backslash R_1,\end{cases}$$

after the transfer from $\mu\to s:\mu(s)=\inf\{\,\mu : s(\mu)=s\}$, we get:

$$M[R(s,x_{1,2,\ldots,q},u^{*q},a,\lambda)]=0,$$
$$M[R(s,x_{1,2,\ldots,q},u^q,a,\lambda)]\geq 0$$

for almost all $s\in[0,\tau]$.

Thus, using the reduction of the problem (5)–(7) in the form (13)–(16) and applying the theorem 1 to it, we get necessary optimum conditions of the strong extremum, which is formulated as the principle of minimum by

Theorem 2 (the strong local minimum). Let $(p^*,\,u^{*q},a^*)$ be the optimum solution to the problem (5)–(7). Then there exists an identically nonzero function $\lambda(s,x_{1,2,\ldots,q})\in C^{1,2}$ that

a) $\lambda(s,x_{1,2,\ldots,q})$ satisfies the solution to the boundary problem (8),

b) for almost all $s\in[0,\tau]$, the minimum $M[R(s,x_{1,2,\ldots,q},u^q,a,\lambda)]$ with respect to the variable u^q corresponds to the optimum control u^{*q}.

4. NECESSARY CONDITIONS OF OPTIMUM CONTROL WITH FEEDBACK

From theorem 2, at fixing the realizations of control vector $X^q(s)$, the conditions of optimum control with feedback are ensued as limiting (Rodnishev, 1991a). Optimum control $u^{*q}=u^{*q}(s,x^q)$ is defined as a local control; this local control is connected with the program control $u^{*q}(t)=u^{*q}(s,x^q,t)$, $t\in[s,\tau]$ and its corresponding state $x^q=X^q(s)$ relative to the fixed initial point (s,x^q) with the expression

$$u^{*q}(t)=u^{*q}(s,x^q,t)||_{t=s}=u^{*q}(s,x^q)$$

over each time point $s\in[0,\tau]$. The solution to the equation (6) is defined by the probability density of the transition $p(s,x^q,\tau,X^q(\tau))$ relative to the point (s,x^q). As the efficiency assessment of the control, the following criteria is considered:

$$I_0^q(s,x^q)=\min M_{s,x^q}\left(\Phi_0\left[X^q(\tau),a\right]\right). \tag{19}$$

This criteria represents the function of a point $x^q(s)=X^q(s)$ of the phase space of the system at the time point s, which characterizes the effectiveness of control $u^q(t)$ over the time segment $[s,\tau]$ under the condition that the representation point in the phase space was in the state $X^q(s)=x^q$ at the time point s. The functional (18), relatively to the point (s,x^q) and the probability density of the transition $p(s,x_q,t,y_q)$, is considered at it as a conditional mathematical expectation at the time point τ under the condition that the system was in the state $X^q(s)=x^q$ at the time point s.

The theorem 3 establishes the necessary optimum control conditions $u^q=u^q(s,x^q)$.

Theorem 3. Let $u^{*q} = u^{*q}(s, x^q)$ be optimum control, which delivers minimum to the criteria (18) at each (s, x_q). Then there exists such a function $\lambda(s, x_{1,2,...,q}) \in C^{1,2}$ that

a) $\lambda(s, x_{1,2,...,q})$ satisfies the Bellman's equation

$$\frac{\partial \lambda(s, x_{1,2,...,q})}{\partial s}+ \min_{u^q} L^*(s, x_{1,2,...,q}, u^q, a)\lambda(s, x_{1,2,...,q}) = 0, \quad s \in [\tau, 0], \quad \lambda(\tau, x_{1,2,...,q}) = \Phi_0(x^q); \tag{20}$$

b) the optimal control $u^q = u^q(s, x^q)$ satisfies the condition

$$L^*(u^{*q}(s, x^q), \cdot)\lambda(s, x_{1,2,...,q}) = = \min_{u^{(q)}} L^*(u^q(s, x^q), \cdot)\lambda(s, x_{1,2,...,q}), \ ) \text{ at all} s \in [0, \tau]\,; \tag{21}$$

c) the parameters a^* satisfy the condition

$$\frac{\partial \Phi_0(x^q, a)}{\partial a} + \int_0^{\tau} \frac{\partial R}{\partial a} dt = 0. \tag{22}$$

5. EXAMPLE

As an example, let us consider the problem of optimum control synthesis, which is formulated in the following way. It is required to define the control $u = u(t, x)$ under the condition $|\,u\,| \leq 1$, which gives minimum to the functional:

$$M((x(3))^2) \to \min, \tag{23}$$

which characterizes the state deviation of the system $X(t)$, relative to the mathematical expectation $m_x(t) = 0$ at $t = 0$. The functioning of the controlled system with the constant delay $\tau = 1$ at $t \in [\,0; 3\,]$ is described by the equation:

$$\dot{x}(t) = -3.2x(t) + 3.2x(t-1) + 3.2u(t) + \xi(t). \tag{24}$$

To exclude the delay, let us introduce notations:

$$\begin{aligned}
&s \in [0; 1],\\
&x^{(1)}(s) = x(s);\ x^{(2)}(s) = x(1+s);\ x^{(3)}(s) = x(2+s);\\
&u^{(1)}(s) = u(s);\ u^{(2)}(s) = u(1+s);\ u^{(3)}(s) = u(2+s);\\
&\xi^{(1)}(s) = \xi(s);\ \xi^{(2)}(s) = \xi(1+s);\ \xi^{(3)}(s) = \xi(2+s).
\end{aligned}$$

Then, the problem (22)–(24) is reduced to the following sequence of problems:

$$\begin{gathered}
M((x^{(1)}(1))^2) \to \min,\\
\dot{x}^{(1)}(s) = -3.2x^{(1)}(s) + 3.2u^{(1)}(s) + \xi^{(1)}(s),\\
x^{(1)}(0) = 0,\ t \in [0; 1].
\end{gathered} \tag{25}$$

$$\begin{gathered}
M((x^{(3)})(1)^2) \to \min,\\
\dot{x}^{(1)}(s) = -3.2x^{(1)}(s) + 3.2u^{*(1)}(s) + \xi^{(1)}(s),\\
\dot{x}^{(2)}(s) = -3.2x^{(2)}(s) + 3.2x^{(1)}(s) + 3.2u^{(2)}(s) + \xi^{(2)}(s),\\
x^{(1)}(0) = 0,\ x^{(2)}(0) = x^{(1)}(1),\ t \in [1; 2].
\end{gathered} \tag{26}$$

$$\begin{gathered}
M((x^{(3)}(1))^2) \to \min\\
\dot{x}^{(1)}(s) = -3.2x^{(1)}(s) + 3.2u^{*(1)}(s) + \xi^{(1)}(s)\\
\dot{x}^{(2)}(s) = -3.2x^{(2)}(s) + 3.2x^{(1)}(s) + 3.2u^{*(2)}(s) + \xi^{(2)}(s),\\
\dot{x}^{(3)}(s) = -3.2x^{(3)}(s) + 3.2x^{(2)}(s) + 3.2u^{(3)}(s) + \xi^{(3)}(s),\\
x^{(1)}(0) = 0,\ x^{(2)}(0) = x^{(1)}(1),\ x^{(3)}(0) = x^{(1)}(2),\ t \in [2; 3].
\end{gathered} \tag{27}$$

where $u^{*(1)}(s)$ and $u^{*(2)}(s)$ are known functions of the variable s being defined by the solutions (25) and (26).

In compliance with the theorem 2, we will search for the optimum control $u^{*(1)}(s)$ in problem (25) considering the conditions of the minimum:

$$R(s, x^{(1)}, u^{(1)}, \lambda) = -3.2x^{(1)} \frac{\partial \lambda}{\partial x^{(1)}} +3.2u^{(1)} \frac{\partial \lambda}{\partial x^{(1)}} + \frac{1}{2} \frac{\partial^2 \lambda}{\left(\partial x^{(1)}\right)^2}.$$

It follows from this, that the optimal control is defined by the expression:

$$u^{*(1)}(s) = -\text{sign} \frac{\partial \lambda}{\partial x^{(1)}}$$

where $\lambda = \lambda\left(s, x^{(1)}\right)$ for $s \in [1; 0]$ is defined by the solution to equation (29) in the reverse time:

$$\frac{\partial \lambda}{\partial s} = 3.2x^{(1)} \frac{\partial \lambda}{\partial x^{(1)}} + 3.2 \left|\frac{\partial \lambda}{\partial x^{(1)}}\right| - \frac{1}{2} \frac{\partial^2 \lambda}{\left(\partial x^{(1)}\right)^2}, \quad \lambda\left(1, x^{(1)}\right) = \left(x^{(1)}(1)\right)^2. \tag{28}$$

To define the solution (28) in the straight time, let us introduce a substitute $\tau = 1 - s$; then we get the following:

$$\frac{\partial \lambda}{\partial \tau} = -3.2x^{(1)} \frac{\partial \lambda}{\partial x^{(1)}} - 3.2 \left|\frac{\partial \lambda}{\partial x^{(1)}}\right| + \frac{1}{2} \frac{\partial^2 \lambda}{\left(\partial x^{(1)}\right)^2}, \quad \lambda\left(0, x^{(1)}\right) = \left(x^{(1)}\,(1)\right)^2. \tag{29}$$

We will search for the solution (29) $\lambda = \lambda\left(\tau, x^{(1)}\right)$ as a linear quadratic form with indefinite coefficients

$$\lambda(\tau, x_1) = k_0(\tau) + k_1(\tau)x^{(1)} + k_{11}(\tau)\left(x^{(1)}\right)^2;$$

with initial conditions, which, in the general case, according to (Krasovsky, 1974), are defined by the formula:

$$k_{1,2,..,m}(0) = \frac{1}{(m-1)!} \left(\frac{\partial^m \lambda}{\partial x^{(1)} \partial x^{(2)} ... \partial x^{(m)}}\right)_{x^{(1)} = x^{(2)} = ... = x^{(m)} = 0}. \tag{30}$$

Substituting derivative values $\lambda = \lambda\left(\tau, x^{(1)}\right)$ in equation (29) and equating the coefficients at $x^{(1)}$, we get the system of ordinary differential equations for defining the coefficients $k_0(\tau)$, $k_1(\tau)$, $k_{11}(\tau)$ over the segment [0; 1] for $\frac{\partial \lambda}{\partial x^{(1)}} \geq 0$,

$$\begin{aligned}
\dot{k}_0 &= -3.2k_1 + k_{11},\\
\dot{k}_1 &= -3.2k_1 - 6.4k_{11},\\
\dot{k}_{11} &= -6.4k_{11}
\end{aligned} \tag{31}$$

and for $\frac{\partial \lambda}{\partial x^{(1)}} < 0$

$$\begin{aligned}
\dot{k}_0 &= 3.2k_1 + k_{11},\\
\dot{k}_1 &= -3.2k_1 + 6.4k_{11},\\
\dot{k}_{11} &= -6.4k_{11},
\end{aligned} \tag{32}$$

which are connected with the function derivatives at the origin of coordinates by the initial conditions (30):

$$k_0(0) = k_1(0) = 0, \quad k_{11}(0) = 2.$$

Having solved (30) and (32), we get

$k_1(\tau) = 4e^{-6.4\tau} - 4e^{-3.2\tau}$ if $\frac{\partial\lambda}{\partial x^{(1)}} \geq 0$;

$k_1(\tau) = 4e^{-3.2\tau} - 4e^{-6.4\tau}$ if $\frac{\partial\lambda}{\partial x^{(1)}} < 0$;

$$k_{11}(\tau) = 2e^{-6.4\tau}.$$

Taking into consideration

$$\frac{\partial\lambda}{\partial x^{(1)}} = k_1(\tau) + 2k_{11}(\tau)x^{(1)},$$

and definition $k_1(\tau)$, $k_{11}(\tau)$, we get the optimum control $u^{*(1)} = u^{*(1)}(\tau, x^{(1)})$ of the problem (25):

$u^{*(1)}(\tau) = 1$, if

$$\frac{\partial\lambda}{\partial x^{(1)}} = 4(e^{-3.2\tau} - e^{-6.4\tau} + x^{(1)}e^{-6.4\tau}) < 0;$$

$$u^{*(1)}(\tau) = -1,$$

if

$$\frac{\partial\lambda}{\partial x^{(1)}} = 4(e^{-6.4\tau} - e^{-3.2\tau} + x^{(1)}e^{-6.4\tau}) \geq 0.$$

Similarly, after the substitution of the optimum control $u^{*(1)}$ in (26), the optimum control $u^{*(2)} = u^{*(2)}(s, x^{(2)})$ of the problem (26) is defined; after the substitution of the optimal control $u^{*(1)}$, $u^{*(3)}$ in (27), the optimum control $u^{*(3)} = u^{*(3)}(s, x^{(3)})$ of the problem (27) is defined.

6. CONCLUSIONS

The formulated principle of the minimum lets us research the problem of optimum control of nonlinear stochastic systems with constant delay on the unified methodological basis. The extension of the phase space of the initial stochastic problem of the optimum control with delay makes it possible to reduce it to the sequence of optimization problems of control over time intervals with the length of delay. This enables to use the theory of diffusion markovian processes in order to search for the optimum control of the stochastic systems with delay, which enables to use the Kolmogorov-Fokker-Plank equation techniques.

It is worth mentioning that the suggested approach to defining the optimum control of nonlinear stochastic systems with delay leads to considerable dimensional increase of each problem; it also requires finding of analytical solutions to Kolmogorov-Fokker-Plank equation and Bellman equation o being conjugated.

As it is known, the explicit solutions to these problems can only be obtained for linear systems (Krasovsky, 1974; Kazakov, 1977) and for few types of nonlinear systems (Chernous'ko and Kolmanovsky, 1978; Kolosov, 1984) of no higher than second order. Therefore, to solve the nonlinear stochastic systems of higher order with delay, it is necessary to use approximate numerical methods of searching for the optimum control (Rodnishev, 2001b) relative to statistics – semi-invariants (coamulants) (Bodner et al., 1987).

REFERENCES

Gihman, I.I. and Skorohod, A.V. (1977) *Controlled stochastic processes.* Naukova dumka. Kiev. In Russian.

Girsanov, I.V. (1970) *Lectures on the mathematical theory of extremum problems.* MSU. Moscow. In Russian.

Bodnerm, V.A., Rodnishev, N.E., and Urikov, E.P. (1987) *Optimization of terminal stochastic systems.* Mashinostroenie. Moscow. In Russian.

Kazakov, I.I. (1977) *Statistical dynamics of systems with variable structure.* Nauka. Moscow. In Russian.

Kolosov, G.A. (1984) *Synthesis of optimal automatics systems at random disturbances indignations.* Nauka. Moscow. In Russian.

Krasovsky, A.A.(1974) *Phase space and the statistical theory of dynamic systems.* Nauka. Moscow. In Russian.

Rodnishev, N.E. (2001a) Optimizing control of nonlinear stochastic systems with limitations. *Automation and Remote Control,* (2), 87 – 101. In Russian.

Rodnishev, N.E. (2001b) Approximate search method of optimal control of non-lineal stochastic systems with limitations. *Automation and Remote Control,* (3), 63-71. In Russian.

Rodnishev, N.E. (2001 c) The necessary conditions of optimum control for abrupt non-lineal stochastic systems with limitations. *Izv. RAN, ser. Theory and Control Systems* , (6), 38-49. In Russian.

Chernous'ko, F.L. and Kolmanovsky, V.B. (1978) *Optimum control at random disturbances.* Nauka. Moscow. In Russian.