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QUANTUM CONTROL FOR BOSE-EINSTEIN CONDENSATES WITH DEMONSTRATION

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Abstract
In this paper, Bose-Einstein-condensates (BEC) is regarded as the target control system, which described by nonlinear Gross-Pitaevskii equation with electromagnetic field. Standing on the viewpoint of mathematics and physics, a complete synthesis for controlling of particles in BEC status will be considered using fundamental analysis based upon variational framework in Hilbert space theoretically, although it's not quite clear to execute with present quantum optical equipments, such as laser cooling, optical lattices. Particularly, the numerical experimental results are simulated to provide the interpretation of controlling process by properly physical parameter selecting.

Key words
Quantum control, Bose Einstein Condensates, Optimal control theory.

1 Physical Background

Beginning with 1990's of last century, physicsits can make the bosonic atoms reach Bose- Einstein condensates in lab experiments for various particles. Sooner-after, extremely rapid growth of quantum physics field supply a great potential in developing of quantum controlling Bose-Einstein-condensates (BEC) by considering significant contributions in the physical and chemical researches area. For instance, optimal control theory apply to molecule formations in a BEC see [Sklarz, 2002]. Genetic-learning algorithm to atomic BEC is published in [Pötting, Cramer and Meystre, 2001]. The remote physical controlling using coefficient concerned with soliton is given in [Radha, Kumar and Porsezian, 2008]. Coherent control see [Holthaus, 2001] in a double well for steering the selftrapping N particle at zero temperature, for single particle refer [Caputo, Kraenkel and Molomed, 2003] in high dimensions. Theoretical study for BEC also be found in [Choi and Bigelow, 2005]. Overall physical investigation for BEC reported in [Morsch and Oberthaler, 2006], and so forth. Actually, there are amount of papers contributed to this area, it is convenient to express concerning ones as cites of this work. In fact, at the viewpoint of physics area, if an ultracold vapor of bosonic atoms are trapped in magnetic well, pure condensates will be created as they are cooled to a temperature below the BEC threshold. After that creation, these BEC are located into a optical lattice potential which can be realized experimentally by a far-detuned, retrore ected laser beam. This phenomenon of macroscopic quantum system consisting of ultra-cold atoms in unique in precision and flexibility for experimental control and manipulation. In order to boost the field of controlling quantum system, very interesting, Bose-Einstein-condensates as a quantum system, it could be considered as the control objective.

Our question arising here is what would be happen if external forcing acting at the particles in BEC? Did optical technology will supply the achievement of the controlling goal? Can laser pulse with high intensity drive the BEC to change their states and transfer energy during this control process. For this purpose, what kinds of ultra-fast (femisecond/attosecond) laser or atom laser pulse would meet our satisfaction? Firstly of all, let us to describe the problem in details. Mathematically, the BEC is usually modeled by the celebrated Gross-Pitaevskii equation, a cubically nonlinear Schrödinger equation (NLS), see [Pitaevskii and Stringari, 2003],

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi\boldsymbol{x}\boldsymbol{x} + v_1(\boldsymbol{x})\psi + v_2(\boldsymbol{x})\psi + N\alpha|\psi|^2\psi, \quad (1)$$

where ψ denote the condensate wave function (i.e. probability amplitudes) of one particle in BEC, m denote the atomic mass, $\hbar$ is the Planck constant, N is the number of atoms in the condensate, and $\alpha =$

$4\pi\hbar^2 a/m$, with $a \in \mathbf{R}$ denoting the characteristic scattering length of the particles. The external potential $v_1(\boldsymbol{x})$ is confining in order to describe the electromagnetic trap needed for the experimental realization of a BEC. Typically it is assumed to be of harmonic form

$$v_1(\boldsymbol{x}) = m\omega_0^2 \frac{|x|^2}{2}, \quad \omega_0 \in \mathbf{R}. \tag{2}$$

In (1), a particular example for the periodic potentials used in physical experiments is then given by [Deconinck, Frigyik and Kutz, 2002; Pitaevskii and Stringari, 2003]

$$v_2(\boldsymbol{x}) = s \sum_{i=1}^{3} \frac{\hbar^2 \boldsymbol{x}^2}{m} \sin^2(x_i x_i), \quad x_i \in \mathbf{R}, \tag{3}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ denotes the wave vector of the applied laser field and $s > 0$ is a dimensionless parameter describing the depth of the optical lattice (expressed in terms of the recoil energy).

Remark 1. *Chemically, the BEC experiment should be available for ^{7}Li, ^{85}Rb and ^{133}Cs, etc. Especially, the well known ^{87}Rb condensates will be considered in our simulation Section 4.*

For the BEC quantum system (1), our goal is to control the target system for minimizing given constrained criteria function with external forces both at magnetic field v_1 and electric field v_2. In details, theoretical control is analyzed for both fields, but the numerical demonstration will be executed in one field case for simplifying.

This article will be organized in following contents. After above introduction of physical model in Section 1, we will propose the BEC quantum state system with the mathematical setting in Hilbert space as theoretic preparation. Section 3 is to state the control theory for BEC quantum dynamics. Section 4 will show the experimental results by numerical simulation for exampled BEC system with selected physical parameters. Section 5 summarize conclusions and drive some discussions.

2 BEC Quantum System

It is very natural to specially consider the optimal control problem for BEC described by Gross-Pitaevskii equation (1), which permit us to convert the problem into mathematical setting in Hilbert space, theoretically. Note that we will present the control theory for BEC in spatial dimension is $\mathbf{R}^3$, and simulate the experimental results in the case of $\mathbf{R}^2$ space.

Let Ω be an open bounded set of $\mathbf{R}^3$ and $Q = (0,T) \times \Omega$ for $T > 0$. Then $(\boldsymbol{x}, t) \in Q$. Regarding $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ are control variables. Introduce two Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with usual norm and inner products (cf. [Lions, 1971], [Wang, 2011b]). Then the embedding in Gelfand triple space $V \hookrightarrow H \hookrightarrow V'$ are continuous, dense and compact. Suppose $\mathcal{V} = L^2(\Omega)$ is the space of laser controls v_1 and v_2. Let $\mathcal{V}_{ad}$ be a closed and convex admissible set of $\mathcal{V}$. Assume initial ground states $\psi(v_1, v_2, 0) = \psi_0$. The objective function associated with (1) is given by

$$J(v_1, v_2) = \frac{\epsilon_1}{2} \int_\Omega \psi_f(v_1, v_2) \psi_{\text{target}}(v_1, v_2) d\boldsymbol{x} + \epsilon_2 \int_{\mathcal{V}} v_1^2 dt + \epsilon_3 \int_{\mathcal{V}} v_2^2 dt. \tag{4}$$

Here $v_1, v_2 \in \mathcal{V}_{ad}$, ψ_{target} is target state, $\psi_f(v_1, v_2)$ is observed final state, respectively. Moreover, $\epsilon_i, i = 1,2,3$ are weighted coefficients for balancing the values of inherent and running costs, respectively.

In generally, our aim is to find quantum optimal control v_1^* or v_2^* in GP system (1). Here v_1^* and v_2^* are called quantum optimal control for system (1) subject to objective function (4), denoted $\boldsymbol{v} = (v_1, v_2), \boldsymbol{v}^* = (v_1^*, v_2^*)$ if necessary. In order to drive the GP equation the optimality system for the OCT fields that allow efficient channeling of the condensate between given initial and desired states. To do this, let us to define two basic concepts, weak solution and solution space, for preparation.

Definition 1. *For the theoretical control study for (1) with objective function (4), referring [Wang, 2006], [Wang and Cao, 2007] to define weak solution's solution space by Hilbert space:*

$$W(0,T;V,V') = \left\{ \psi \;\middle|\; \psi \in L^2(0,T;V), \psi' \in L^2(0,T;V') \right\}.$$

Definition 2. *A function ψ is called weak solution of* (1) *if $\psi \in W(0,T;V,V')$ and satisfy*

$$\int_0^T \int_\Omega i\hbar \psi_t dt d\boldsymbol{x} = -\frac{\hbar^2}{2m} \int_0^T \int_\Omega \Delta \psi_{\boldsymbol{x}\boldsymbol{x}} d\boldsymbol{x} dt + \int_0^T \int_\Omega v_1(\boldsymbol{x}) \psi d\boldsymbol{x} dt + \int_0^T \int_\Omega v_2(\boldsymbol{x}) \psi d\boldsymbol{x} dt + \int_0^T \int_\Omega N\alpha |\psi|^2 \psi d\boldsymbol{x} dt. \tag{5}$$

3 Control Theory for BEC

Above established mathematical setting permit us to study the quantum system (1) in the framework of variational method and quantum mechanics theory. Therefore, using the same manipulation as in [Lions, 1971], [Wang, 2006], [Wang and Cao, 2007] and refer (5), it's easy to obtained the next theorems.

Theorem 1. *For given initial given $\psi_0 \in V$, there exists weak solution $\psi \in W(0,T;V,V')$ for system* (1) *satisfy the weak form (5).*

Theorem 2. *For given* $\psi_0 \in V$*, there exists at least one quantum optimal control pairing* $\boldsymbol{v}^* = (v_1^*, v_2^*)$ *for system* (1) *subject to objective function* (4).

Theorem 3. *For given initial state* $\psi_0 \in V$ *and control problem for system* (1) *associated with (4), then the optimality system is simultaneously chracterized by*

$$\begin{cases} i\hbar\psi_t = -\dfrac{\hbar^2}{2M}\Delta\psi \\ \quad +v_1^*(\boldsymbol{x})\psi + v_2^*(\boldsymbol{x})\psi + N\alpha|\psi|^2\psi \quad \text{in Q}, \\ \psi(v_1^*, v_2^*, 0) = \psi_0 \quad \text{in } \Omega, \end{cases} \tag{6}$$

$$\begin{cases} i\hbar p_t = -\dfrac{\hbar^2}{2M}\Delta p + 2|\psi|\psi p + |\psi|^2 p \quad \text{in Q}, \\ ip_f = \psi_f(v_1^*, v_2^*) - \psi_{\text{target}} \quad \text{in } \Omega, \end{cases} \tag{7}$$

$$\int_Q p(v_1^*)(v_1 - v_1^*)\, d\boldsymbol{x}dt + \int_Q p(v_2^*)(v_2 - v_2^*)\, d\boldsymbol{x}dt + (\boldsymbol{v}^*, \boldsymbol{v} - \boldsymbol{v}^*)_{\mathcal{V}} \geq 0 \tag{8}$$

for all $v_1, v_2 \in \mathcal{V}_{ad}$*. In here,* $p \in W(0, T; V, V')$ *is solution of the adjoint systems (7) corresponding to* ψ *in state systems (6).* p_f *is the terminal state of adjoint state* p *at final time* t_f*. As is well known that the inequality (8) is necessary optimality condition for* $\boldsymbol{v}^* = (v_1^*, v_2^*)$.

The proof of above theoretic results can be found for details in our other papers, omit it in here. It needs notice that by referring [Wang and Cao, 2007] to employ a semi-discrete algorithm (spatial discrete, time continuous) with updated (nonlinear) conjugate gradient method and finite element approach to deal with the numerical simulation, quantum optimal control v_1 and v_2 can be found efficiently. Obviously, the convergence is guaranteed and in the order of $o(h)$ for spatial interval h. For the simplification, neglect the redundant description in context of this paper.

4 Experiments Demonstration

This section is to consider two spatial dimension case experimental demonstration in $\mathbf{R}^2$, set $\Omega = [-10, 10] \times [-10, 10] \in \mathbf{R}^2$. Notice that, for simplicity to analyze the gap of the two term $v_1(t), v_2(t)$ and their physical meanings in (2) and (3), denote external force as one $u(t)$ for being control input. The following Gross- Pitaevskii equation would be considered as experimental example in upcoming simulation demonstration.

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi_{\boldsymbol{xx}} + V(x, y, \lambda)\psi + u(t)\psi + N\alpha|\psi|^2\psi,$$

Here, $V(x, y, \lambda)$ is the external potential coming from electric and magnetic fields. The rest notations are same with Gross-Pitaevskii equation in system (1). It would be convenient to execute numerical experiments in scale of real atomic units. Take time $t_0 = 0.0s$, $T = 0.001s$ and step size $dt = 0.0001s$.The Reduced Planck constant $\hbar = 1.0545715964207855 \times 10^{-34}$. The mass of ^{87}Rb is $m = 1.41923 \times 10^{-25}$ a.u. The coefficient in nonlinear term is $4\pi 2a/m$ and $a = 5.1 \times 10^{-9}$. Take β, η are constants. Let $N_0 = 5$ particles are located at (x_1, y_1),(x_2, y_2),(x_3, y_3),(x_4, y_4),(x_5, y_5) respectively. Their two axes coordinates are given by

$$x_j = d\cos(\frac{2\pi j}{N}),\ y_j = d\sin(\frac{2\pi j}{N}),\ j = 1, 2, 3, , 4, 5.$$

Therefore, one can take the potential function as

$$V(x, y, \lambda) = V_0 \sum_{j=1}^{N} \exp\Big(-\frac{\lambda^2(x - \eta x_j)^2 + j(y - y_j)^2}{2\omega^2}\Big),$$

where $V_0 = -0.6$ and $\omega = 1.0$. Let the iteration number $N = 10$. The appendix function is configured by

$$\Psi(\beta, \eta) = \sum_{j=1}^{N} \big((x - \eta x_j) + i(y - y_j)\big)^2 \times \exp\Big(-\frac{(x - \eta x_j)^2 + i(y - y_j)^2}{8\beta}\Big),$$

Then by using Ψ to construct ground states $\psi_0 = \Psi(0.8, 5)$, and plot in Figures 1-3.

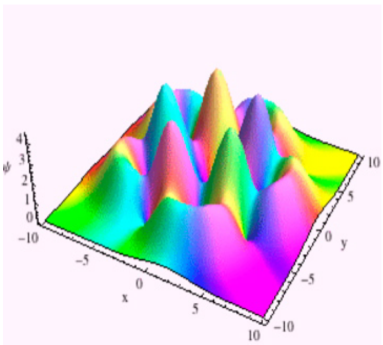


Fig. 1. Plot $\psi_0 = \psi(0)$.

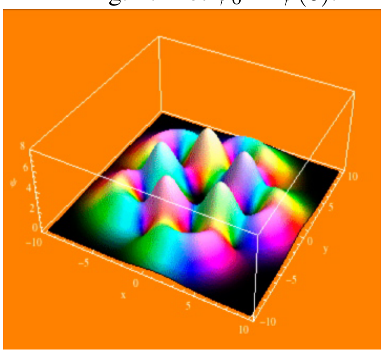


Fig. 2. Boxed plot $\psi_0 = \psi(0)$.

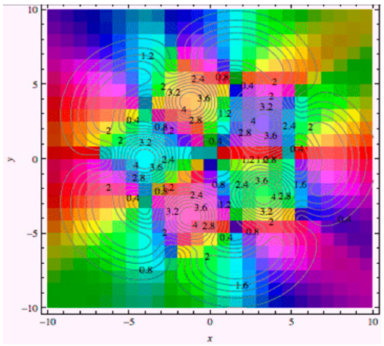


Fig. 3. Contour plot $\psi_0 = \psi(0)$.

Similarly, the target states $\psi_f = \Psi(0.2, 3)$ is plotted in Figures 4-6.

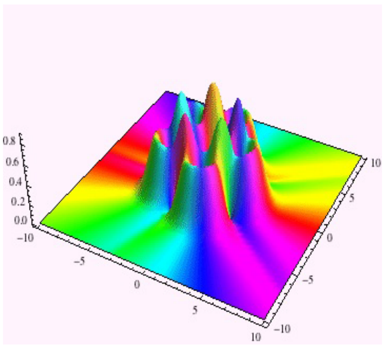


Fig. 4. Plot of ψ_f.

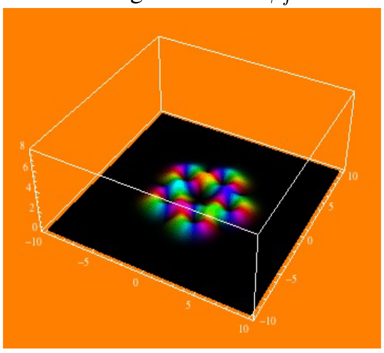


Fig. 5. Boxed plot ψ_f.

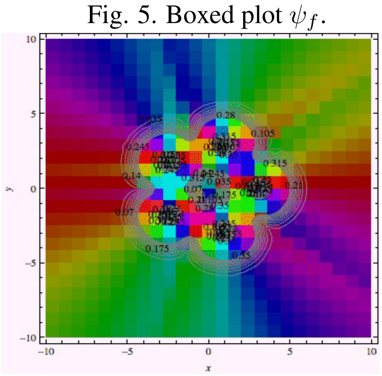


Fig. 6. Contour plot ψ_f.

The initial control is given by $u_0(t) = 1.0 \times 10^{15} \sin(1.5 \times 10^4 t)$ and show in Figure 7.

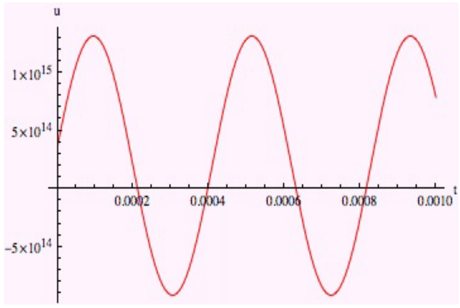
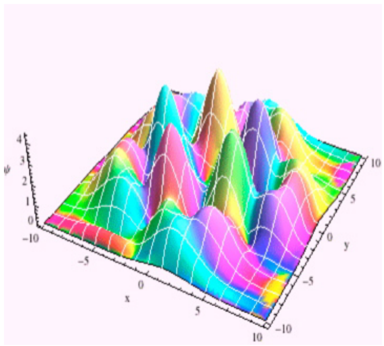
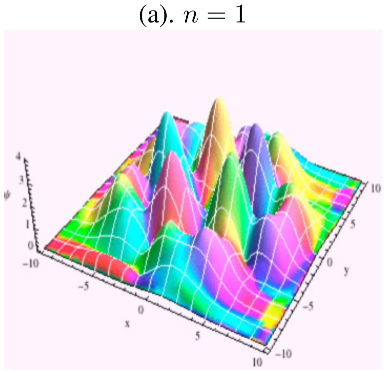


Fig. 7. Plot $u_0(t)$.

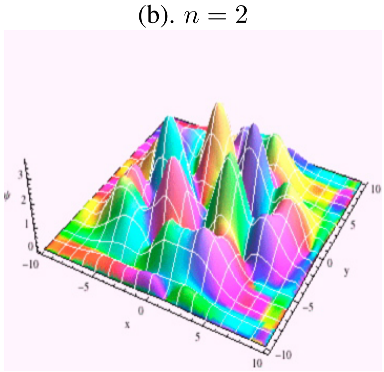
Through iteration times $n = 10$ to show the states change of BEC under external forcing $V(x, y, \lambda)$. One can find BEC states changing in Figure 8 at each iteration step from (a). $n = 1$ to (j). $n = 10$ with meshes.



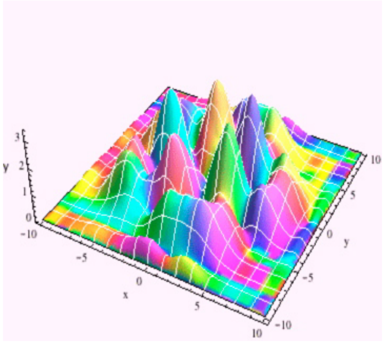
(a). $n = 1$



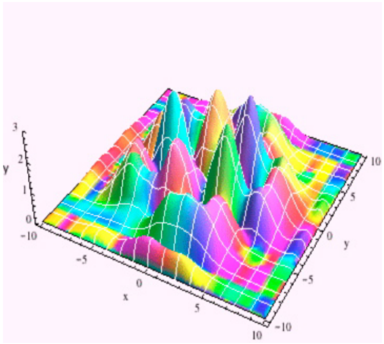
(b). $n = 2$



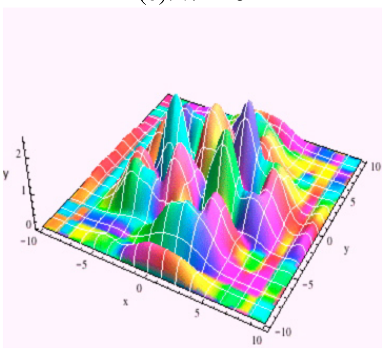
(c). $n = 3$



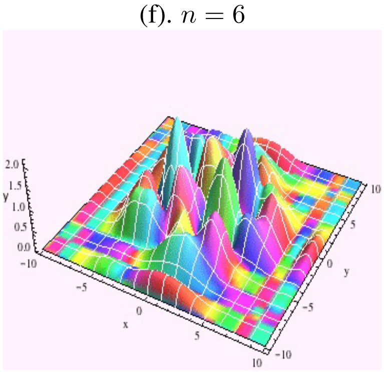
(d). $n = 4$



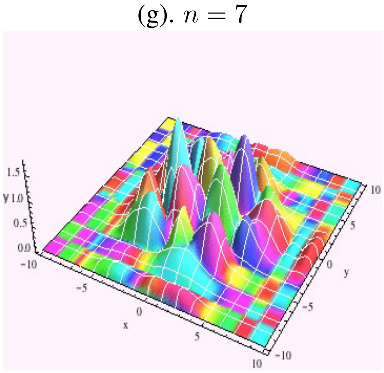
(e). $n = 5$



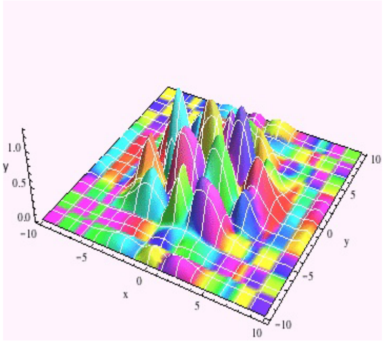
(f). $n = 6$



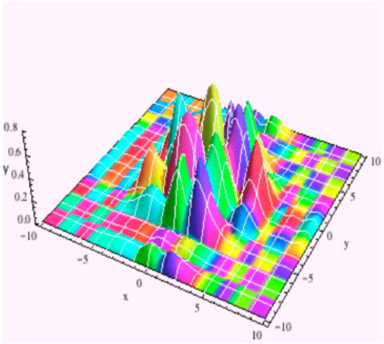
(g). $n = 7$



(h). $n = 8$



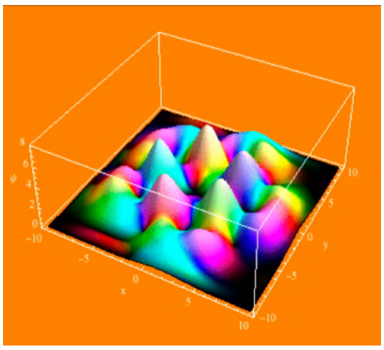
(i). $n = 9$



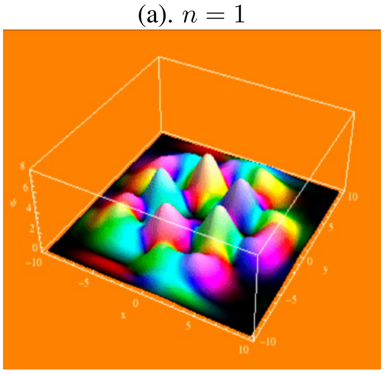
(j). $n = 10$

Fig. 8. Plot $\psi(t)$ for $t \in [0, 0.001], dt = 0.0001$.

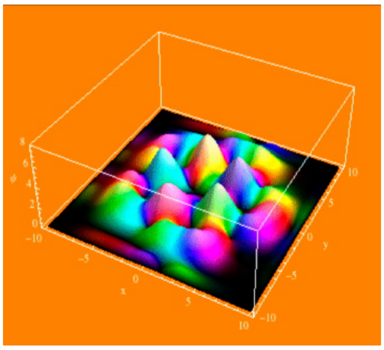
Their interesting box plots are shown in graphics (a)–(j) of Figure 9 for each step $n = 1, 2, ..., 10$.



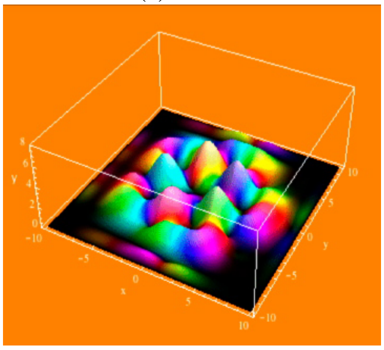
(a). $n = 1$



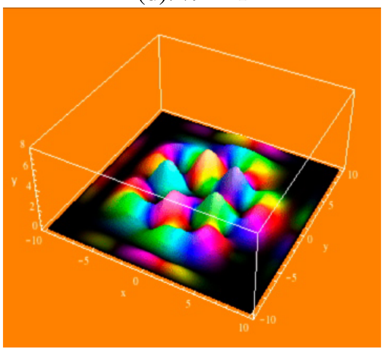
(b). $n = 2$



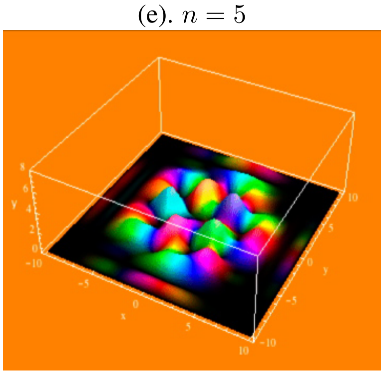
(c). $n = 3$



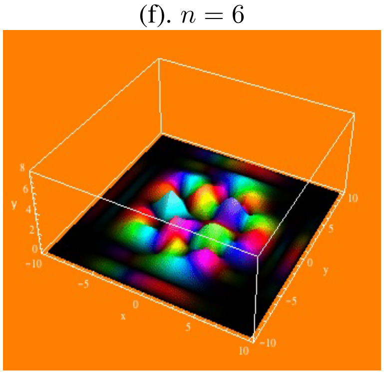
(d). $n = 4$



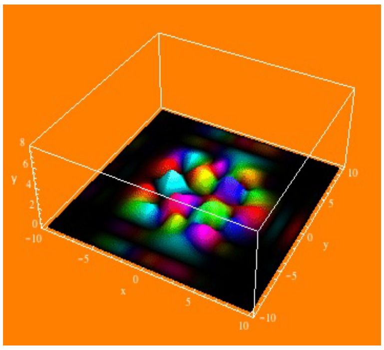
(e). $n = 5$



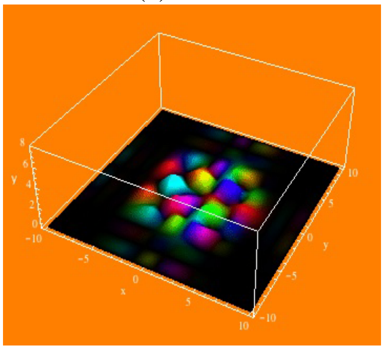
(f). $n = 6$



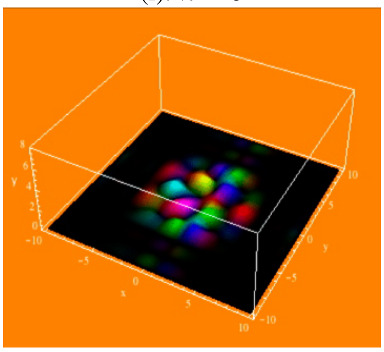
(g). $n = 7$



(h). $n = 8$



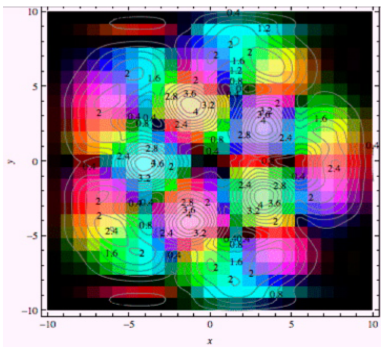
(i). $n = 9$



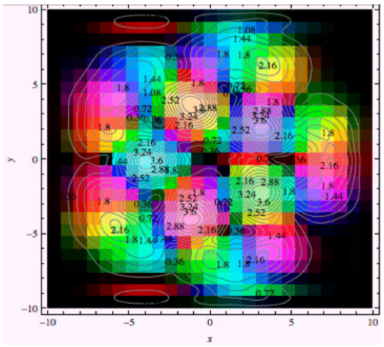
(j). $n = 10$

Fig. 9. Boxed plot $\psi(t)$ for $t \in [0, 0.001], dt = 0.0001$.

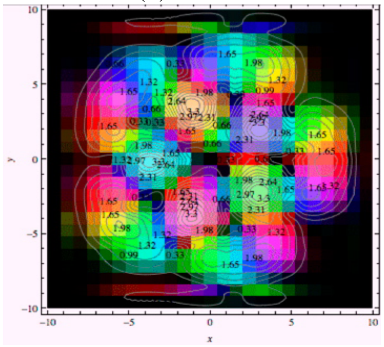
For each iteration from $n = 1$ to $n = 10$, the contour plots with meshes are listed in the graphics (a)-(j) of Figure 10.



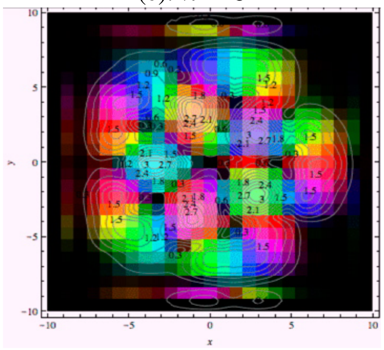
(a). $n = 1$



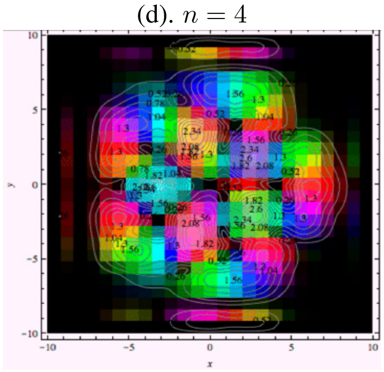
(b). $n = 2$



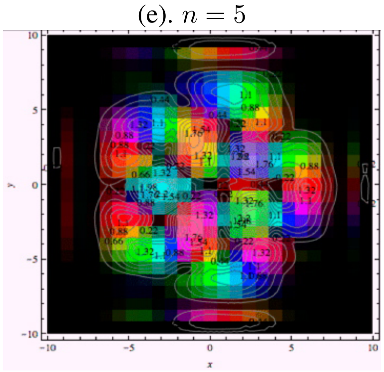
(c). $n = 3$



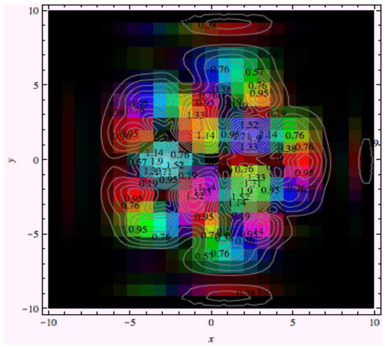
(d). $n = 4$



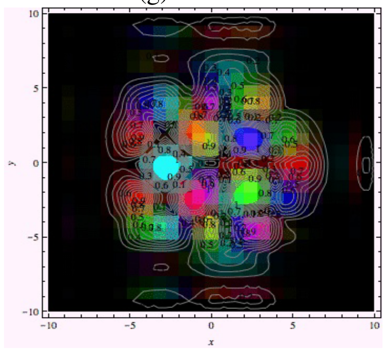
(e). $n = 5$



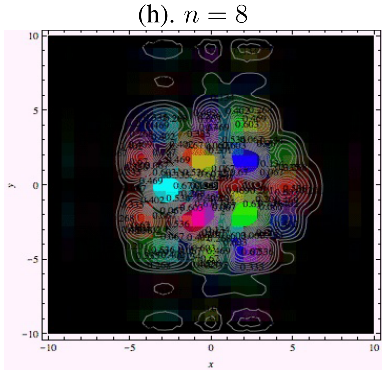
(f). $n = 6$



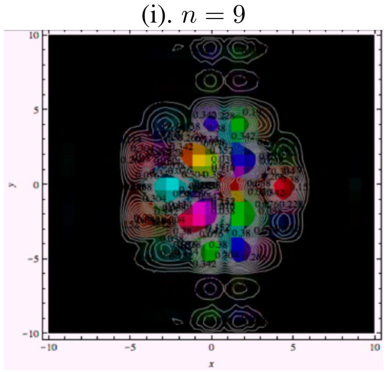
(g). $n = 7$



(h). $n = 8$



(i). $n = 9$



(j). $n = 10$

Fig. 10. Contour plot $\psi(t)$ for $t \in [0, 0.001], dt = 0.0001$.

The control inputs at each step $n = 1$ to $n = 10$ are calculated by nonlinear conjugate gradient method in the list:

$$\begin{aligned}
u_1 &= 1.0 \times 10^{15} \sin(15000t);\\
u_2 &= 3.51545 \times 10^{14} + 1.0 \times 10^{15} \sin(15000t);\\
u_3 &= 9.28201 \times 10^{14} + 1.0 \times 10^{15} \sin(15000t);\\
u_4 &= 2.79538 \times 10^{23} + 1.0 \times 10^{15} \sin(15000t);\\
u_5 &= -8.56925 \times 10^{23} + 1.0 \times 10^{15} \sin(15000t);\\
u_6 &= -2.77424 \times 10^{23} + 1,0 \times 10^{15} \sin(15000t);\\
u_7 &= -8.86984 \times 10^{23} + 1,0 \times 10^{15} \sin(15000t);\\
u_8 &= -3.65048 \times 10^{23} + 1,0 \times 10^{15} \sin(15000t);\\
u_9 &= -3.65048 \times 10^{23} + 1,0 \times 10^{15} \sin(15000t);\\
u_{10} &= -1.67563 \times 10^{23} + 1,0 \times 10^{15} \sin(15000t).
\end{aligned}$$

Finally, it is clearly to find that with external force $u_8 = u_9$, the each state is holding for a while (0.0001s) and reach condensates for 5 particles. Therefore, quantum optimal control force is obtained as

$$u^* = -3.65048 \times 10^{23} + 1.0 \times 10^{15} \sin(15000t).$$

Its graphics of the optimal quantum control function u^* see in Figure 11.

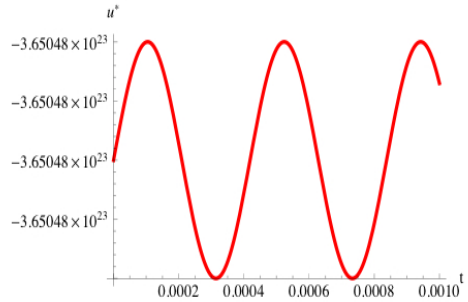


Fig. 11. Plot of u^* (red curve).

The values of cost function (4) at each step $n = 1$ to $n = 10$ are computed as

$$\begin{aligned}
&J_1 = 8.52417 \times 10^{26},\ J_2 = 9.77937 \times 10^{26},\\
&J_3 = 1.77561 \times 10^{27},\ J_4 = 7.81416 \times 10^{43},\\
&J_5 = 7.34322 \times 10^{44},\ J_6 = 7.69643 \times 10^{45},\\
&J_7 = 7.86741 \times 10^{44},\ J_8 = 1.3326 \times 10^{62},\\
&J_9 = 1.3326 \times 10^{62},\ J_{10} = 2.80775 \times 10^{63}.
\end{aligned}$$

At the step $n = 8$ and $n = 9$, the cost reach a constant value $J_8 = J_9$. hence, the optimal cost function value attain

$$J(u^*) = 1.3326 \times 10^{62}$$

in the iteration process.

It is easily to know that the external energy is increasing with the tiny time change one by one step. Obviously, the energy is not conserved in the whole process due to the reason of strong forces coming from electric or magnetic field through optical trapping at low temperature. The yield error values for criteria cost function (4) see below for each iteration step from $n = 1$ to $n = 10$.

$$\begin{aligned}
&eJ_1 = 8.52417 \times 10^{26}, eJ_2 = 1.2552 \times 10^{26},\\
&eJ_3 = 7.97676 \times 10^{26}, eJ_4 = 7.81416 \times 10^{43},\\
&eJ_5 = 6.56178 \times 10^{44}, eJ_6 = 6.96211 \times 10^{45},\\
&eJ_7 = 6.90969 \times 10^{45}, eJ_8 = 1.3326 \times 10^{62},\\
&eJ_9 = 2.67449 \times 10^{63}.
\end{aligned}$$

In above control process, cost functions values $J(u)$ is displayed in Figures 12. Notice the straight line between point 8 and 9.

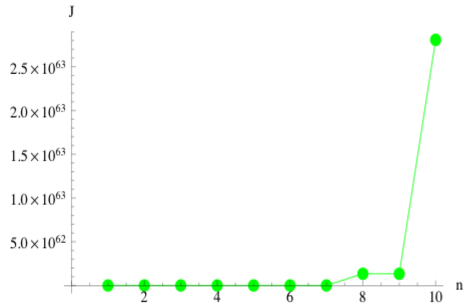


Fig. 12. Plot $J(u)$ (green points)

Furthermore, the error functions values $eJ(u)$ are displayed in Figures 13.

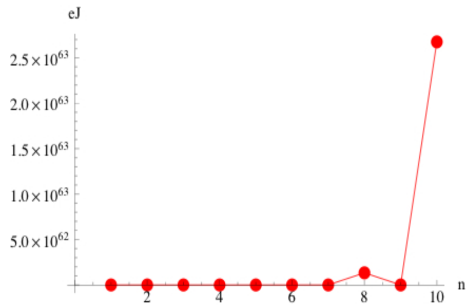


Fig. 13. Plot $eJ(u)$ (red points).

The total occupied CPU maximum memory 380230768 bytes, the used running time 17764.5 second.

Theoretically, the Bose-Einstein condensates could be controlled for per particle within BEC condensation status by tracking (or acting) per particle with external forcing, and change their intensities with the time increase at tiny changing. Experimentally, how to realize the quantum control of particle at BEC would also be a challenge direction for experimental physics to verify the efficiency and feasibility.

5 Conclusions

In summary, controlling for BEC is solved regarding the quantum dynamics to seek the optimal solution. This research exploration extremely acquire the real laboratory evidence for quantum controlling achievement. The attempt progress would become a promising research direction, see [Wang, 2009a], [Wang, 2009b], [Wang, Cao and Luo, 2009], [Wang, 2010], [Wang, 2011a]and [Wang, 2011c].

As to our future works, by observing the literatures of researches on controlling of BEC in physical and chemical fields, see relevant contributed papers [Chacon, Bote and Carretero-Gonzalez, 2008], [Deconinck, Frigyik and Kutz, 2002], [Hohenester, Rekdal, Borzi and Schmiedmayer, 2007], [Parker, Proukakis, Barenghi and Adams, 2004], [Perez-Garcia and Garcia-March, 2007], [Bulatov, Vugmeister and Rabitz, 1999], [Robert, Claussen, Cornish, Donley, Cornell and Wieman,2001], [Rodas, Michinel and Perez-Garcia, 2005], [Stickney, Anderson and Zozulya, 2007], [Trotzky, 2008]. What we interested is controlling the BEC theoretically and computationally. On the other hand, decoherence effects, which also play a role in atom condensate, can be naturally incorporated into OCT calculations. It has been quested in PhysCon 2009 conference. Future perspective will be combining the predictions with real laboratory experiments with toiled advanced optical technologies. Furthermore, the application of controlling Bose-Einstein condensates quantum system for the physical properties, such as superconductivity, would be a fresh direction need effort.

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