# Realization theory methods for the construction of positively-invariant sets

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*Abstract*— **Realization theory for linear input-output operators and frequency-domain methods for the solvability of Riccati operator equations are used for the stability and instability investigation of a class of nonlinear Volterra integral equations in some Hilbert space. The key idea is to consider, similar to the Volterra equation, a time-invariant control system generated by an abstract ODE in some weighted Sobolev space, which has the same stability properties as the Volterra equation.**

## I. Introduction

The first step in the derivation of equations describing the dynamic behavior of observations is very often a Volterra integral equation ([1], [11], [17]) which represents causal or input-output properties of such observations or time-series. Stability, oscillating behavior, and other qualitative properties from a Volterra integral equation can be observed directly by frequency-domain methods developed in [6], [13], [16], [18]. However, for other types of dynamic behavior such as instability and dichotomy it is useful to consider together with the given Volterra integral equation an associated realization as evolution equation in some function spaces. Realization theory in Hilbert and Fréchet spaces developed for linear input-output operators, can be found in [7], [8], [12], [15], [19]. First results where linear realization theory was used for the stability investigation of finite-dimensional nonlinear Volterra equations can be found in the papers [2], [3], [4].

In the present paper we continue these investigations for infinite-dimensional Volterra equations.

## II. Realization of infinite-dimensional Volterra equations as time invariant control systems

Suppose that $Y$ and $U$ are Hilbert spaces and introduce the Fréchet spaces $L^2_{\rm loc}(\mathbb{R};Y)$ and $L^2_{\rm loc}(\mathbb{R};U)$. Assume that

$$\phi : L^2_{\rm loc}(\mathbb{R}_+;Y)\times\mathbb{R}_+\times L^2_{\rm loc}(\mathbb{R}_+;Y)\to L^2_{\rm loc}(\mathbb{R}_+;Y) \quad (1)$$

is a nonlinear operator generating the *Volterra functional equation*

$$y=\phi\,(y,t,h)\;. \quad (2)$$

Assume also that there are a continuous linear operator

$$\mathcal{T} : L^2_{\rm loc}(\mathbb{R};U)\to L^2_{\rm loc}(\mathbb{R};Y) \quad (3)$$

and a nonlinear operator

$$\varphi : L^2_{\rm loc}(\mathbb{R}_+;Y)\times\mathbb{R}_+\to L^2_{\rm loc}(\mathbb{R}_+;U) \quad (4)$$

such that the operator (1) can be written as

$$\phi(y,t,h)=\mathcal{T}\varphi(y,t)+h(t)\,, \quad (5)$$

where $h\in L^2_{\rm loc}(\mathbb{R}_+;Y)$ is considered as *perturbation* or *forcing function.* Thus the Volterra functional equation has the form

$$y=\mathcal{T}u+h\,, \quad (6a)$$

$$u=\varphi(y,t)\,. \quad (6b)$$

Let us discuss for this the realizability of the operator equation (6a), (6b) as abstract differential equation. Important information comes from the linear operator (3) which we call *input-output operator* of the linear part of (6a). For any interval $\mathcal{J}\subset\mathbb{R}$, a Hilbert space $Z$ and any $s\in\mathbb{R}$ denote by $\tau^s$ the *shift operator* acting on functions $f:\mathcal{J}\to Z$ by

$$\tau^s f(t):=\begin{cases} f(t+s) & \text{if} \quad t+s\in\mathcal{J}\,,\\ 0 & \text{if} \quad t+s\notin\mathcal{J}\,.\end{cases}$$

The input-output operator (3) is called *time invariant* if $\tau^t\mathcal{T}=\mathcal{T}\tau^t$ for every $t\in\mathbb{R}$ and is called *causal* if for all $t\geq 0$

$$u(t)=0\,,\ \forall\, t\leq T \quad\Rightarrow\quad \mathcal{T}u(t)=0\,,\ \forall\, t\leq T\,.$$

This implies that $\mathcal{T}$ in (3) is defined by its restriction

$$\mathcal{T} : L^2_{\rm loc}(\mathbb{R}_+;U)\to L^2_{\rm loc}(\mathbb{R}_+;Y)\,. \quad (7)$$

For any interval $\mathcal{J}\subset\mathbb{R}$, a Hilbert space $Z$ and a parameter $\rho\in\mathbb{R}$ we introduce the weighted spaces $L^2_\rho(\mathcal{J};Z)$ and $W^{1,2}_\rho(\mathcal{J};Z)$ by

$$L^2_\rho(\mathcal{J};Z):=\left\{f\in L^2_{\rm loc}(\mathcal{J};Z)\;\middle|\;\int_{\mathcal{J}} e^{-2\rho t}\,|\,f(t)\,|^2_Z\,dt<\infty\right\}$$

$$\text{and}\quad W^{1,2}_\rho(\mathcal{J};Z):=\{f\in L^2_\rho(\mathcal{J};Z)\,|\,\dot f\in L^2_\rho(\mathcal{J};Z)\}\,.$$

($\dot f$ denotes the distribution derivative.) Let us assume that $\mathcal{T}$ from (3) can be considered for some $\rho\in\mathbb{R}$ as bounded linear operator

$$\mathcal{T} : L^2_\rho(\mathbb{R};U)\to L^2_\rho(\mathbb{R};Y)\,. \quad (8)$$

Assume for this that the input-output operator $\mathcal{T}$ from (8) can be represented as convolution operator

$$(\mathcal{T}u)(t):=\int_0^t K(t-s)\,u\,(s)\,ds\,, \quad (9)$$

where $K(\cdot)$ is a certain kernel called ([19]) the *weighting pattern* of $\mathcal{T}$.

Assume that the map $t\in\mathbb{R}_+\mapsto\mathcal{L}(U,Y)$ is twice piecewise-differentiable and satisfies the following condition: There exists a $\rho_0>0$ and a constant $\gamma>0$ such that

$$\|\,K(t)\,\|_{\mathcal{L}(U,Y)}\leq\gamma e^{-\rho_0 t}\,,\quad \forall\, t>0\,, \quad (10)$$

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and

$$\int_0^\infty \left[\, \| \dot K(t) \|^2_{\mathcal{L}(U,Y)} + \| \ddot K(t) \|^2_{\mathcal{L}(U,Y)} \right] e^{2\rho_0 t} dt < \infty\,. \quad (11)$$

Under these conditions we can choose a state space realization of (9) which was used in [4] for the special case $U = Y = \mathbb{R}^n$ :

$$Z_0 := W^{1,2}_{-\rho}(0,\infty;Y), \quad \text{where} \quad 0 < \rho < \rho_0 \quad \text{is arbitrary}\,, \quad (12)$$

$$D(A) := \left\{ \xi(s) \in W^{1,2}_{-\rho}(0,\infty;Y) \mid \int_0^\infty e^{2\rho s} \| \ddot\xi(s) \|^2_Y \, ds < \infty \right\}, \quad (13)$$

$$(A\,\xi)(s) := \dot\xi(s)\,, \quad \forall\, \xi \in D(A)\,, \quad (14)$$

$$(B\eta)(s) := K(s)\eta\,, \quad \forall \eta \in U\,, \quad (15)$$

$$(Cz)(s) := z(0)\,, \quad \forall\, z \in Z_0\,. \quad (16)$$

Thus we have defined a time-invariant control system

$$\dot z = Az + Bu\,, \quad (17a)$$

$$y = Cz\,, \quad (17b)$$

where $A$ from (14) is a closed linear operator that acts in $Z_0$ given in (12) and which has the dense domain of definition $D(A)$ from (13). It is clear that $A$ is the generator of some $C_0$ semigroup $\{S(t)\}_{t\ge 0}$. The map (15) defines a linear bounded operator $B : U \to Z_0$. If $z_0 \in D(A)$ the generalized solution $z(\cdot, z_0)$ of (17a) starting in $Z_0$ is a continuous function $t \mapsto z(t, z_0) \in Z_0$ which can be represented in integral form as

$$z(t, z_0) = S(t)z_0 + \int_0^t S(t-\tau)Bu(\tau)\, d\tau \quad (18)$$

$$\text{with} \qquad \| S(t) \| \le \alpha\, e^{-\varkappa t}\,, \qquad \forall\, t > 0 \quad (19)$$

where $\alpha$ and $\varkappa$ are positive numbers, and which satisfies for any $u \in L^2(\mathbb{R}_+;U)$ the output relation

$$C \int_0^t S(t-\tau)Bu(\tau)\, d\tau = \int_0^t K(t-\tau)u(\tau)\, d\tau\,. \quad (20)$$

Note that if $z(t, z_0) \in D(A)$ for $t > 0$ then $z(\cdot, z_0)$ is an ordinary strong solution of (17a).

## III. Solvability of the Riccati operator equation for the realizations of a class of Volterra equations

Let us assume that $F_1 = F_1^* \in \mathcal{L}(Y,Y), F_2 \in \mathcal{L}(U,Y)$ and $F_3 = F_3^* \in \mathcal{L}(U,U)$ are bounded linear operators and introduce the bilinear form

$$j(x, y\,;\, u, v) := (F_1 x, y)_Y + (F_2 u, x)_Y + (F_2 v, y)_Y + (F_3 u, u)_U\,, \qquad \forall\, x, y \in Y\,,\ \forall\, u, v \in U\,. \quad (21)$$

A direct calculation shows that

$$j(x, y\,;\, u, v) = j(y, x\,;\, v, u),\ \forall\, x, y \in Y, \forall\, u, v \in U\,. \quad (22)$$

Introduce the linear operator

$$\mathcal{K}(u, h)\,(t) := (\mathcal{T} u)\,(t) + h(t)\,, \quad u \in L^2(\mathbb{R}_+;U)\,, \quad h \in W^{1,2}_{-\rho}(\mathbb{R}_+;Y)\,,$$

and consider for $T > 0$ and a parameter $\nu, |\nu| \le \nu_0$, the bilinear functional

$$J^T_\nu(u, h) := \int_0^T \Big[ j(\mathcal{K}(u,h), \mathcal{K}(u,h); u, u) - \nu \| z(t,h,u) \|^2_{W^{1,2}_{-\rho}(\mathbb{R}_+;Y)} \Big]\, dt\,, \quad (23)$$

which is for $\rho \in (0, \rho_0)$ a continuous map $L^2(0,T;U) \times W^{1,2}_{-\rho}(0,+\infty;Y) \to \mathbb{R}$.

The next theorem contains a version of the operator Riccati equation. For the case $U = Y = \mathbb{R}^n$ this theorem was proved in [4].

*Theorem 1:* Let $\chi(p)$ be the Laplace transform of the absolutely continuous function $K$. Suppose that $F_1 = F_1^* \ge 0,\ F_3 = F_3^* > 0,\ F_3^{-1}$ exists,

$$\chi(p) \in \mathcal{L}(U,Y), \forall\, p \in \mathbb{C}\,,$$

and

$$\Pi(i\omega) := \chi^*(i\omega) F_1 \chi(i\omega) + 2\,\mathrm{Re}(F_2^* \chi(i\omega)) + F_3 > 0, \quad \forall\, \omega \in \mathbb{R}\,. \quad (24)$$

Then there exists a sufficiently small $\nu_0 > 0$ such that for any $\nu \in [0, \nu_0]$ we have (the index $\nu$ is omitted):

1) For any $h \in W^{1,2}_{-\rho}(0,+\infty;Y)$ there exists a $\tilde u(h) \in L^2(0,+\infty;U)$ such that

$$J^T_0(\tilde u(h), h) < J^T_0(u, h)\,, \quad \forall\, u \in L^2(0,T;U)\,,\ \| u - \tilde u(h) \|_{L^2(0,T;U)} > 0 \,.$$

2) There exists a bounded self-adjoint operator

$$M_T = M_T^* : W^{1,2}_{-\rho}(0,+\infty;Y) \to W^{1,2}_{-\rho}(0,+\infty;Y)$$
$$(M_T h, h)_{W^{1,2}_{-\rho}(0,+\infty;Y)} = J^T_0(\tilde u(h), h), \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y)\,.$$

3) In the case $T = \infty$ the operator $M := M_\infty$ satisfies the following Riccati operator equation

$$S(h,g) := (Ah, Mg)_{W^{1,2}_{-\rho}(0,+\infty;Y)} + (Mh, Ag)_{W^{1,2}_{-\rho}(0,+\infty;Y)} - (L^* h, L^* g)_U + (F_1 Ch, Cg)_Y - \nu (h,g)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,, \quad \forall\, h, g \in D(A)\,, \quad (25)$$

where

$$N := \sqrt{F_3},$$
$$L := (MB + C^* F_2) N^{-1} \in \mathcal{L}(U, W^{1,2}_{-\rho}(0,+\infty;Y))$$

and $(L^* h, v)_U = (h, Lv)_{W^{1,2}_{-\rho}(0,+\infty;Y)}\,, \quad \forall\, h \in W^{1,2}_{-\rho}(0,+\infty;Y), \forall\, v \in U.$

## IV. Stability and instability of infinite-dimensional Volterra equations by their state-space realizations

Consider the Volterra integral equation

$$y(t) = h(t) + \int_0^t K(t-\tau)\, \varphi\,(y(\tau), \tau)\, d\tau\,, \quad (26)$$

where $K(t) \in \mathcal{L}(U,Y)(U,Y$ Hilbert spaces ) is twice piecewise-differentiable satisfies (10) and (11), and has therefore a state-space realization (12) - (17b). Suppose that

$$\varphi : Y \times \mathbb{R}_+ \to U \tag{27}$$

is a continuous function.

Instead of one fixed nonlinearity $\varphi$ we consider a family $\mathcal{N}$ of continuous maps (27), such that for any $\varphi \in \mathcal{N}$ and any $h \in D(A)$ with $D(A)$ from (13) the nonlinear integral equation (26) has a unique solution $y(\cdot,h,\varphi)$ and this solution is continuous. Suppose also that there are linear bounded operators $G_1 = G_1^* \in \mathcal{L}(Y,Y), G_1 \leq 0, G_2 \in \mathcal{L}(U,Y)$ and $G_3 = G_3^* \in \mathcal{L}(U,U), G_3 < 0, G_3^{-1}$ exists, such that for any $\varphi \in \mathcal{N}$ we have

$$(G_1 y, y)_Y + 2\,(G_2\varphi(y,t), y)_Y + (G_3\varphi(y,t), \varphi(y,t))_U \geq 0, \quad \forall\, t \geq 0,\ \forall\, y \in Y\,. \tag{28}$$

Now we consider together with the state-space equation (17a), (17b) and the nonlinearity $\varphi \in \mathcal{N}$ the nonlinear evolution system

$$\dot{z} = Az + B\varphi(y,t),\ y = Cz\,. \tag{29}$$

In order to describe the absolute stability or instability behaviour of (26) with the help of the state-space realization (29) we need an additional assumption on the class $\mathcal{N}$. Let us assume that there is a linear bounded operator $R : Y \to U$ such that the "nonlinearity" $\varphi = Ry$ belongs to $\mathcal{N}$.

*Theorem 2:* Suppose that $\chi(\cdot)$ is the Laplace transform of $K$ and the operator function $(I - \chi(p)R)^{-1}$ has poles in the right half-plane and the frequency-domain condition (24) is satisfied with $F_i = -G_i, i = 1,2,3$. Then there exists a bounded linear self-adjoint operator

$$P : W^{1,2}_{-\rho}(0,+\infty;Y) \to W^{1,2}_{-\rho}(0,+\infty;Y) \qquad \text{such that}$$
$$\mathcal{C} := \{h \in W^{1,2}_{-\rho}(0,+\infty;Y) \,|\, (Ph,h)_{W^{1,2}_{-\rho}(0,+\infty;Y)} < 0\}$$

is a quadratic cone $\mathcal{C} \neq \emptyset$ in $W^{1,2}_{-\rho}(0,+\infty;Y)$ with the following properties:

a) There exists a constant $\beta > 0$ such that for any $h \in \mathcal{C}$ and any $\varphi \in \mathcal{N}$

$$\lim_{t\to\infty} e^{-\beta t} \int_0^t \|\, \varphi(y(s,h,\varphi),s)\,\|_U^2\, ds = \infty\,. \tag{30}$$

b) Any solution $y\,(\cdot,h,\varphi)$ of (26) which does not satisfy (30) has the property $\int_0^\infty \|\, \varphi(y(s,h,\varphi),s)\|_U^2\, ds < \infty$ and, consequently,

$$\varphi(y(\cdot,h,\varphi),\cdot) \in L^2(0,\infty;U)$$
$$\text{and} \quad y(\cdot,h,\varphi) \in L^2(0,\infty;Y)\,. \tag{31}$$

## V. PDE's with Boundary Control

Let us investigate the question how to suppress vibrations in a fluid conveying tube. We consider for this a system of equations which is described in [5], [9], [10], [14], [16].

The motion of an incompressible fluid is given for $t > 0$ in the acoustic approximation by

$$\frac{\partial v}{\partial t} = a_1\,\frac{\partial w}{\partial x}\,, \qquad \frac{\partial w}{\partial t} = a_2\,\frac{\partial v}{\partial x}\,,\ x \in (0,1)\,, \tag{32}$$

where $a_1$ and $a_2$ are positive parameters, $v$ denotes the relative velocity of the fluid and $w$ denotes the pressure. The boundary conditions are given for $t > 0$ by

$$w(t,1) = 0, \qquad \Big(\frac{1}{2}w(t,0) - v\,(t,0)\Big) = -u\,(t)\,, \tag{33}$$

where $u(\cdot)$ is a function ("boundary control") which describes the relative displacement of the piston of a servomotor. The equation of the turbine for $t > 0$ is

$$T_a\,\frac{dq}{dt} + q(t) = u(t) + \frac{3}{2}\,w(0,t)\,. \tag{34}$$

Here $q$ denotes the relative angular speed of the turbine, $T_a$ is a positive parameter. The regulator is described by the equation

$$T_r^2\,\frac{d^2\zeta}{dt^2} + T_k\,\frac{d\zeta}{dt} + \delta\zeta + k\varphi(\dot{\zeta}) + q(t) = 0\,, \tag{35}$$

where $\zeta$ represents the displacement of the clutch of the regulator and $T_r, T_k, \delta$ and $k$ are positive parameters. The friction term is given by a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ defined through a parameter $\kappa > 0$ by

$$\varphi(y) = \begin{cases} 1 & \text{if} \quad y \geq \kappa\,, \\ \frac{1}{\kappa}\,y & \text{if} \quad y \in (-\kappa,\kappa)\,, \\ -1 & \text{if} \quad y \leq -\kappa\,, \end{cases}$$

and thus satisfying the property

$$\varphi(y)\,y \geq 0\,, \quad \forall\, y \in \mathbb{R}\,. \tag{36}$$

The equation of the servomotor is

$$T_s\,\frac{du}{dt} = \eta(t)\,, \tag{37}$$

where $T_s$ is a positive parameter and $\eta$ denotes the displacement of the slide value. The last condition is for $t > 0$ and with a positive parameter $\beta$

$$\eta(t) - \zeta(t) + \beta u(t) = 0\,. \tag{38}$$

A direct computation shows ([9], [16]) that the transfer function of the linear part of (32) – (38) which connects the (formal) Laplace transforms of $-\varphi$ and $\dot{\zeta}$ is given by

$$\chi(p) = k\,\frac{p\,(T_a p + 1)\,(T_s p + \beta)\,(\sinh p\tau + \alpha\cosh p\tau)}{(T_r^2 p^2 + T_k p + \delta)\,Q(p) + R(p)}\,, \tag{39}$$

where

$$\alpha = 2\sqrt{\frac{a_1}{a_2}}\,, \quad \tau = 1/\sqrt{a_1 a_2}\,, \tag{39 a}$$

$$Q(p) = (T_s p + \beta)\ (T_a p + 1)(\alpha\cosh p\tau + \sinh p\tau)\,,$$
$$R(p) = 2\cosh p\tau - 2\sinh p\tau\,.$$

Note that $\chi(p)$ can be written with some $c > 0$ as

$$\chi(p) = \frac{k}{T_r^2 p + c} + \chi_1(p)\,. \tag{40}$$

where

$$\chi_1(p) = \frac{k\,(c - T_k)\,p\,Q(p) - \delta Q(p) - R(p)}{(T_r^2\,p + c)P(p)}\,. \tag{41}$$

The representation (41) shows that $\chi_1(p)$ is analytic in some halfplane $\{\mathrm{Re}\,p > -\varepsilon\}$ with $\varepsilon > 0$. From this it follows that $\chi_1(p)$ has a Laplace original $K_1(t)$ which is absolute continuous, satisfies the inequalities

$$|\, K_1(t)\,| \leq \mathrm{const}\, e^{-\varepsilon_0 t} \tag{42}$$

with some $\varepsilon_0 > 0$ and such that $K_1$ and $\dot{K}_1$ belong to $L^2(0,\infty;\mathbb{R})$. Since first part of (40) has the Laplace original $\frac{k}{T_r^2}e^{-ct/T_r^2}$ the whole original of $\chi(p)$ can be represented as $K(t) = K_1(t) + \frac{k}{T_r^2}e^{-ct/T_r}$. It is shown in [9], [16] that any solution component $y(t) := \dot{\zeta}(t)$ from (35) can be written as

$$y(t) = h(t) + \int_0^t K(t-\tau)\varphi(y(\tau))\, d\tau\,, \tag{43}$$

where again $K$ is absolute continuous, satisfies an inequality of type (42) and $h, \dot{h}$ belong to $L^2(0,\infty;\mathbb{R})$.

The quadratic constraints (28) can be described in $Y = U = \mathbb{R}$ by the inequality

$$\varphi(y)\,(y - \kappa\,\varphi(y)) \geq 0\,, \quad \forall\, y \in \mathbb{R}\,, \tag{44}$$

i.e., (28) is satisfied with $G_1 = 0, G_2 = \frac{1}{2}$ and
$G_3 = -\kappa < 0\,.$

Using the transfer function (40) and the constraints (44) we can verify the frequency-domain condition (24). A direct computation shows (see [16], [9]) that if

$$T_k(\alpha^2 - 1)\beta^2 \leq \left(\frac{55}{32} + \alpha^2\right)(\beta\, T_a + T_s) \tag{45}$$

is satisfied and then the condition

$$\alpha(T_k\beta^2 - (\beta\, T_a + T_s)) \geq 3\,\tau\beta\,. \tag{46}$$

is necessary and sufficient for the frequency-domain condition (24). The stability and instability domains of the denominator of $\chi(p)$ where investigated in [10] and characterized in the $(T_k, T_r^2)$-plane by domains $\Omega_{\rm st}$ and $\Omega_{\rm unst}$, respectively. It follows now from Theorem 2 that under the conditions (44) - (46) for parameters from $\Omega_{\rm st}$ the solutions of the integral equation (43) have the properties described by Theorem 2.

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