DYNAMIC MODES OF POPULATION WITH A SIMPLE AGE STRUCTURE UNDER LIMITATION OF BIRTH RATE

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Abstract

In this paper we research a mathematical model of dynamics for the population number. We considered the population of the two age classes by the beginning of the next season: the younger, one including not reproductive individuals, and the senior class, consisting of the individuals participating in reproduction. The birth parameter is represented the exponential functions of the both age groups numbers. According to this supposition the density-dependent factors restrict the development of population. Analytical and numerical analysis of the model is made. We investigate the dynamic modes of the model. It is shown that density-dependent factors of regulation for the population number can lead to generation of fluctuations and chaotic dynamics behavior of the population.

Key words

Population model, age structure, density-dependent factors, stability, bifurcations, chaos.

1 Introduction

In this paper we consider evolutionary scenarios of the origination of oscillatory and chaotic population dynamics in the species with the simple age structure. We take into account nonlinear interactions, observed in nature, between different age groups in the population, influencing the populations birth rate.

It is really the birth rate of many animals, small mammals in particular, substantially decreases with the population growth. Usually, this is the manifestation of the stress-syndrome leading to the decline of sexual activity and fertility of individuals. Sometimes, even dissolution of already existing embryos is observed. Such observations are most common for the populations subject to large magnitude size oscillations, such as lemmings, field-voles etc.

2 Mathematical model

We consider the population which, by the end of each reproduction season, consists of two age groups: juveniles (immature individuals) and adults (participants of the reproduction process). We assume that the time between two reproduction seasons is enough for the juveniles to become adults [Frisman, Skaletskaya, 1994, Frisman, 1994]. The dynamic equations for our model are as follows

$$\left.\begin{array}{l} x_{n+1} = a \cdot y_n \cdot \exp(-\alpha \cdot x_n - \beta \cdot y_n) \\ y_{n+1} = s \cdot x_n + v \cdot y_n \end{array}\right\} \quad (1)$$

where x is a number of juveniles, y is a number of adults, n is a reproductive season number, s $(0 \leq s < 1)$ and v $(0 \leq v < 1)$ are the survival rates of juveniles and adults respectively, a is the maximum possible birth rate, α and β are the intensities of the birth rate decline because of the growth of juvenile and adult numbers respectively.

3 The model research

Substitutions $\beta x \to x$ and $\beta y \to y$ transform (1) to

$$\left.\begin{array}{l} x_{n+1} = a \cdot y_n \cdot \exp(-\gamma \cdot x_n - y_n) \\ y_{n+1} = s \cdot x_n + v \cdot y_n \end{array}\right\} \quad (2)$$

where $\gamma = \alpha/\beta$. Analysis of this system is simplified by the introduction of the parameters $r = as$ and $b = \gamma/s$. Here r characterizes reproductive potential of the population, and parameter b reflects the difference in the birth rate limitation due to the numbers of juvenilesand adults. The system (2) may have only one non-trivial stationary solution

$$\left.\begin{array}{l}\overline{x} = \frac{1-v}{s(b-vb+1)} \cdot \ln\frac{r}{1-v} \\ \overline{y} = \frac{1}{b-vb+1} \cdot \ln\frac{r}{1-v}\end{array}\right\} \qquad (3)$$

which exists if $r \neq 0$, $r \geq 1-v$, $0 \leq v < 1$, $v < (1+b)/b$. Stability of the solution (3) depends on the eigenvalues determined by the equation $\lambda^2 + p\lambda + q = 0$.

The standard method of finding the stability domain is based on the following theorem: Solutions of the equation $\lambda^2 + p\lambda + q = 0$ belong to the circle $|\lambda| < 1$ if and only if

$$|p| - 1 < q < 1 \qquad (4)$$

[Shapiro, Luppov, 1983]. It is also shown in [Shapiro, Luppov, 1983].] that the inequalities (4) define in the plane (p, q) a triangle of stability. Its boundaries are given by the lines:

1) $q = -1 - p$, on this line one of the eigenvalues λ^* is equal to 1;

2) $q = p - 1$, on this line one of the eigenvalues λ^* is equal to -1;

3) $q = 1$, on this line eigenvalues are complex numbers ($\lambda_1 \cdot \lambda_2 = 1$), and on the segment $(-2 < p < 2)$, limiting the stability triangle, they are also conjugate: $\lambda_1 = \exp(i\varphi)$, $\lambda_2 = \exp(-i\varphi)$.

In our case

$$p = \frac{b(1-v)}{1+b(1-v)} \cdot \ln\frac{r}{1-v} - v,$$

$$p = \frac{1-v}{1+b(1-v)} \cdot \left(bv - b - 1 - (1+bv) \cdot \ln\frac{r}{1-v}\right).$$

And therefore the boundaries of the stability domain for the equilibrium point (3) are as follows:

$$\lambda^* = 1 : r = 1 - v,$$

$$\lambda^* = -1 : r = (1-v)\exp\frac{2v \cdot (b+B)}{(b-B)(1-v)}, \qquad (5)$$

$$q = 1 : r = (1-v)\exp\frac{(2-v)(b+B)}{B(1-v)}, \qquad (6)$$

where $B = 1 - vb$.

Fig.1 shows the variation of stability domain of the system (2) non-zero equilibrium in coordinates $(v; r)$ for different values of parameters b, α and β.

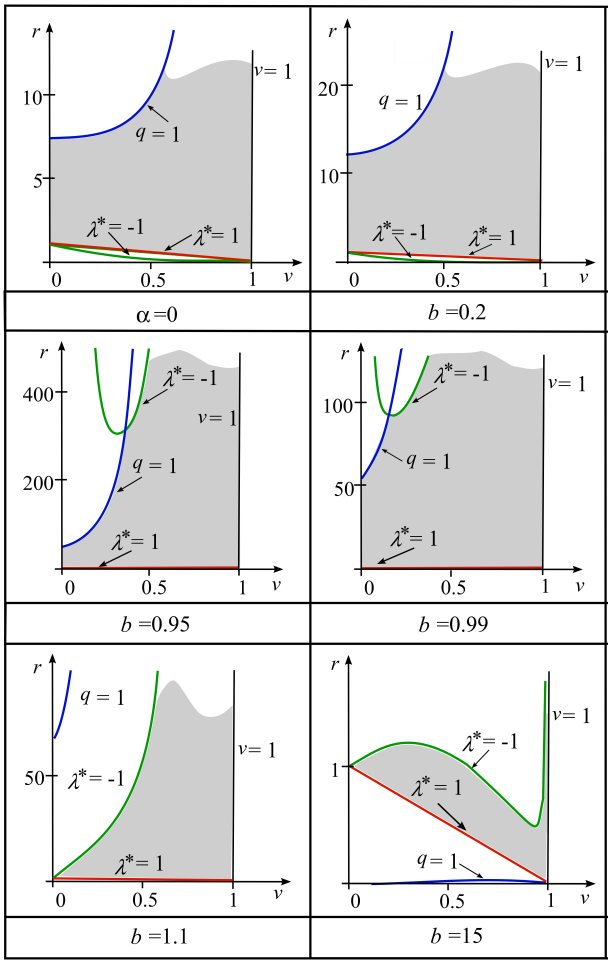

Figure 1. Domains of existence and stability of the non-zero equilibrium of the system (1).

4 Results

Note that the decline of the birth rate with the increase of adult number only ($\alpha = 0$) may lead to the emergence of fairly complex oscillations of the population size. Modest decrease in the birth rate with the increase of juvenile number leads (at $b \leqslant 3/4$) to substantial growth of the parameter domain corresponding to the stable population equilibrium. Qualitatively, the pattern of stability loss does not change in this situation. This loss happens either with the growth of parameter r (the sooner the smaller the parameter r is) or with the decrease of the survival rate (if reproduction potential of the population is high). Loss of stability under this model happens when the eigenvalues are conjugates and $|\lambda|$ transitions through 1. This loss is accompanied by the emergence of limiting invariant curves which disintegrate with further moving of the parameters v and r away from the stability boundary, and then form very complex limiting structures.

Fig.2 presents the bifurcation diagram showing dependence of limiting distribution of number of juveniles (x) on parameter r. This diagram provides visual concept of the mode of population dynamics at $b \leqslant 3/4$. Fig.2 also presents portraits of the limiting trajectories

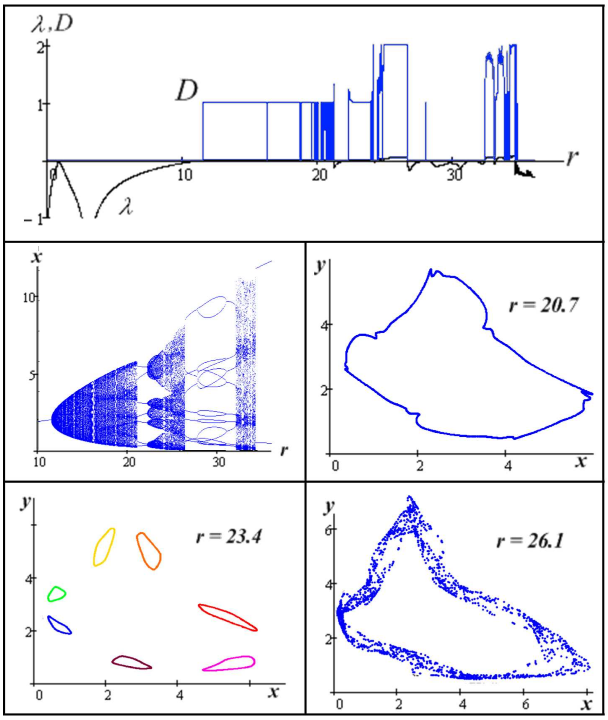

Figure 2. Changes in the attractor dimension (D) and 1st Lyapunov coefficient (λ) with the change of parameter r; bifurcation diagram of the dynamic variable x versus parameter r at $b = 0.2$ and $v = 0.1$; attractors for the system (2) for selected values of parameter r..

of the system (2), i.e. attractors, corresponding to particular values of parameter r. In order to visualize the domains of regular, quasi-periodic, and chaotic dynamics, the graphs of the 1st Lyapunov coefficient (λ) and attractor dimension (D) are also shown .

The bifurcation diagram and the graph of the attractor dimension nicely complement each other. Loss of stability leads to the limit cycle (invariant curve) with dimension 1. Further there are series of transitions of invariant curves, cycles of finite length, and attractors with various dimensions, including an attractor with maximum dimension 2.

Further increase of birth rate restriction by the number of juveniles $(3/4 < b < 1)$ leads to contraction of the stability domain (fig.1). There appears a domain of parameters v and r such that transition into it is accompanied by loss of equilibrium stability with an emergence of 2-cycles and the transition of one of eigenvalues through -1.

At $b \geqslant 1$ the stability domain quickly contracts with the parameter b growth; loss of stability only happens if $\lambda = -1$ and is accompanied by a 2-cycles appearance.

If the birth rate is limited by the juvenile number only $(\beta = 0)$, the number of both age groups oscillates with growing magnitude and 2-year period. The number of the juveniles drops to zero at minimums and exponentially grows in years of maximum. Oscillations of the adults are asynchronous to the oscillations of juveniles; both minimums and maximums are growing exponentially, but at different rates, and the magnitude of the oscillation is growing (fig.3).

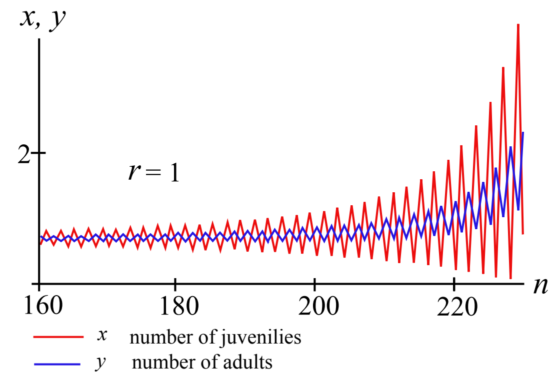

Figure 3. Oscillations of the numbers of juveniles and adults when the number of adults does not limit the population growth.

5 Conclusion

Our analysis shows that birth rates decrease with the number of adults is an efficient mechanism for controlling the population size. Through the growth of the individual reproductive potential it can lead to oscillations of population size with fairly complex temporal structure. If the birth rate also is controlled by the number of juveniles then such a mechanism positively affects stability only if the dependence on the number of juveniles is modest and is weaker than the dependence on the number of adults. If these requirements are met, the stability domain increases substantially.

The regulation of the birth rate by the number of juveniles appears to be inefficient; small increase of the reproductive potential allows the population to start growing exponentially, which leads to formation of new restricting mechanisms.

Acknowledgements

This work is in part supported by the Far Eastern Branch of Russian Academy of Sciences (nos. 09-I-OBN-12); Russian Foundation for the Fundamental Research and (no. 09-04-00146-a).

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