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THE LEIER ELEVATOR FOR AN ASTEROID

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Abstract
We consider a space station flight about an asteroid. We claim that if the asteroid rotation about its mass center is a regular precession then the space station can be placed on the cable with ends fixed on the asteroid poles. Thus an original space elevator for such asteroid can be realized. We divide the station equilibria on the cable into two types and analyze one of these types, varying the considered system parameters. We formulate some facts of stability for found equilibria.

Key words
space tether system, space elevator, asteroid, leier, stability

1 Introduction

We consider a space station motion on the cable with ends placed in an asteroid surface. Such cable is called 'the leier constraint' [Rodnikov 2006a; Rodnikov 2006b; Rodnikov 2008a; Rodnikov 2008b; Rodnikov 2009a; Rodnikov 2009b]. In Dutch and in Russian the sailing term 'leier' means the rope with fixed ends. (Various aspects of a particle motion along space tethers have been studied also in [Burov 2003; Buchin, Burov and Troger],etc.). We claim that if the asteroid is dynamically symmetric and if the ends of the cable are fixed in the asteroid poles then there exists a set of the space station equilibria on the cable. (Here poles are the points in which the large principal axis of the asteroid crosses the asteroid surface.) So an original space elevator for the asteroid can be realized.

If the asteroid gravitational potential depends only of the distances up to the asteroid mass center and up to the axis of dynamical symmetry, in particular, if assumptions of the Generalized Restricted Circular Problem of Three Bodies (GRCP3B) [Beletsky, 2007] are fulfilled, then all mentioned equilibria belong to two rotating planes containing the asteroid mass center. Equilibria belonging to the plane composed by the asteroid angular momentum and the asteroid axis of dynamical symmetry is called z-equilibria (ZE). Coplanar Libration Points (CLP) of GRCP3B [Beletsky and Rodnikov 2008b] are examples of ZE. Equilibria belonging to the plane being perpendicular to the asteroid angular momentum is called x-equilibria (XE). Triangular Libration Points (TLP) of GRCP3B [Beletsky and Rodnikov 2008a] are examples of XE. In this paper we study XE differing from TLP. Varying parameters of the asteroid we find the sets of XE for possible values of the cable length. We claim that any XE is stable if the station motion along the cable is forbidden. Moreover, we study existence and stability of ZE and XE if gravitation is infinitesimal.

2 Preliminary notes, designations, parameters

Consider an asteroid with the mass center C and axis of dynamical symmetry Cz. Let $Cx_1y_1z_1$ be a frame of König's axes, i.e. axes moving translationally. Evidently, even in the asteroid vicinity the Sun gravitation is big in comparison with gravitation of the asteroid. But taking into account the force of moving space, one can show, that total influence of the Sun in the asteroid vicinity is very small in comparison with the asteroid gravitation. Thus, we can assume the asteroid motion is a regular precession about Cz_1 with the angular velocity ω and with nutation angle ϑ. Without loss of generality, $0 \le \vartheta \le \pi/2$. Let $Cxyz$ be the coordinate system rotating about Cz_1 with the angular velocity ω. Assume Cx does not leave Cx_1y_1, so Cz_1 belongs to Czy. Denote by F_1 and F_2 the asteroid poles (see fig. 1). Suppose the station S is tethered to the asteroid by the cable F_1SF_2 called the leier . Let O be F_1F_2 midpoint, $F_1S + F_2S = 2a$, $F_1F_2 = 2c$. Evidently, the station does not leave an ellipsoid with foci F_1 and F_2 and with big semi-axis a. Denote by x, y, z coordinates of S in $Cxyz$. Let $x = a\rho\cos\varphi$, $y = a\rho\sin\varphi$, $z = a\zeta$.

Clearly, the station motion along the leier and with leier is completely determined by the asteroid gravitational potential Π, by dimensionless parameters ϑ, $d = OC/c$, $e = c/a$ and by dimensionless variables

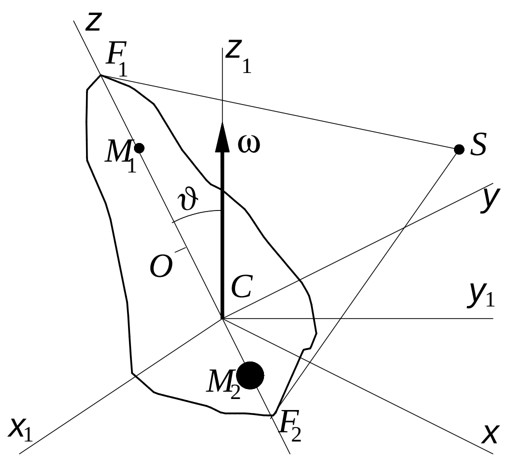

Figure 1.

ρ, ζ, φ. It can be assumed that Π doesn't depend on φ. Trivially, if the station is on the ellipsoid surface then

$$(\zeta - ed)^2 + \rho^2/(1-e^2) = 1 \tag{1}$$

Note also that $0 < e < 1$.

3 Integrable cases and equilibria types

Using Lagrange's method, we deduce equations of the station motion on the ellipsoid surface. Note that there are two trivial cases when these equations are integrable. If $\vartheta = 0$ then φ is a cyclic variable. If $\vartheta = \pi/2$ then we can integrate equation for the station motions in Cx_1y_1 separately.

In the general case one of conditions for the station equilibria in $Cxyz$ on the ellipsoid surface for any Π has a form

$$\cos\varphi \cdot (\rho \sin\varphi \sin\vartheta + \zeta \cos\vartheta) = 0 \tag{2}$$

It follows from (2) that there exist only two types of the station equilibria. If $\zeta = -\rho \sin\varphi \tan\vartheta$ then the station rotates about Cz_1 in Cx_1y_1 with angular velocity ω. These equilibria are called XE. If $\varphi = \pm\pi/2$ then the station doesn't leave Cyz containing Cz_1. These equilibria are called ZE.

4 X-equilibria in two-particles gravitational field

Now let $\varphi \neq \pm\pi/2$. In this case, conditions for XEs can be written in a form

$$\begin{cases} \frac{\partial\Pi}{\partial\zeta} + 2\lambda d - (1+2\lambda)\,\zeta = 0 \\ \frac{\partial\Pi}{\partial\rho} - \left(1 + \frac{2\lambda}{1-e^2}\right)\rho = 0 \end{cases} \tag{3}$$

Here $\lambda < 0$ for the tense cable, $\lambda = 0$ for the non-tense cable. $\lambda > 0$ is impossible. Suppose the asteroid gravitation field is close to gravitational field of two particles M_1 and M_2 (fig. 1) with masses m_1 and m_2. It follows that

$$\Pi = -\alpha k^3 e^3 \left(\frac{\mu}{r_1} + \frac{1-\mu}{r_2}\right) \tag{4}$$

where $r_1 = SM_1/a$, $r_2 = SM_2/a$, $\alpha = G\,(m_1+m_2)\,/\omega^2 l^3$, G is the gravitational constant, $l = M_1M_2$, $\mu = m_1/\,(m_1+m_2)$, $k = l/c$. Evidently, $\alpha > 0$. Without loss of generality, $m_1 \leq m_2 \iff 0 < \mu \leq 1/2$. Moreover, assume $CM_1 = (1-\mu)l$, $CM_2 = \mu l$. (Factually, here we replace the asteroid with two homogeneous spheres forming a dumbbell. Thus, all assumptions of GRCP3B [Beletsky, 2007] are fulfilled) Substituting (4) into (3) after simplifications one can present inequality $\lambda \leq 0$ in a form

$$\frac{2\zeta - ke\,(1-2\mu)}{e\zeta + d\,(1-e^2)} \leq 0 \tag{5}$$

Eliminating λ from (3)and taking into account (1,4), we obtain equation that doesn't depend on ρ, φ, ϑ. Using this equation for coordinate ζ calculation, we can find sets of XEs for fixed values of μ, d, k, α, ϑ and all admissible values of e. Thus we can define x-equilibria set of the station on the leier for any concrete asteroid but for all admissible cable lengths). There are five types of such sets.

$d = 0$, $\mu = 1/2$ for the first type that is called 'full-symmetric'. In this case from (5) it follows that only $\zeta = 0$ is possible. If $\alpha < 1/8$ then any point of Cx is XE. If $\alpha > 1/8$ then XE set consist of two rays $|x| \geq ke\sqrt{\alpha^{2/3} - 1/4}$ beginning in TLPs [Beletsky, 2007; Beletsky and Rodnikov 2008a]. Analyzing Jacobi's integral, one can show, that each equilibrium belonging to Cx is unstable. Nevertheless, each XE becomes stable if to forbid the station motion along the cable.

$\mu = 1/2$, $d \neq 0$ for the second type that is called 'dumbbell-symmetric'. (Here without loss of generality $d > 0$). Possible XEs sets are depicted in fig. 2. In this fig. XEs set for $0 < \vartheta < \pi/2$ is represented by the curve β with ends in TLPs L_1 and L_2. If $\vartheta = 0$ then XEs set is a circle α consisting only of TLPs. If $\vartheta \to \pi/2$ then β goes to the curve ω. Note that if $\mu = 1/2$ then TLPs exist only if $\alpha \geq 1/8$ [Beletsky, 2007]. Therefore, if $\alpha \leq 1/8$ then the circle α vanishes and β becomes an endless curve.

$\mu < 1/2$, $d = 0$ for the third type that is called 'elevator-symmetric'. Only in this case XE sets are presented by infinite curves depicted in fig. 3.

$\mu < 1/2$, $d < 0$ for the fourth type. In this case XE set exists only if $\vartheta \geq \vartheta_{\min}$, where $\vartheta_{\min}$ depends on α, d, μ, k. (see fig. 4.) In this figure S_m presents the unique XE, existing for $\vartheta = \vartheta_{min}$)

$\mu < 1/2$, $d > 0$ for the fifth type. In this case curves consisting of XEs envelop C (see fig. 5). In this fig. the

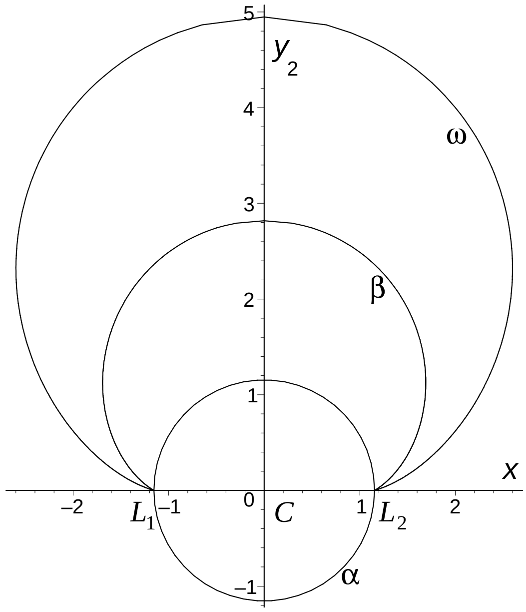

Figure 2.

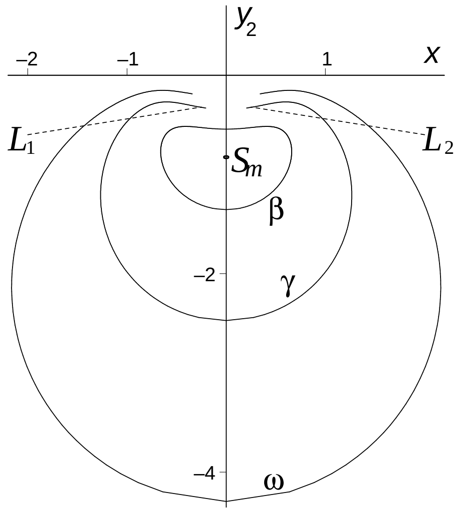

Figure 4.

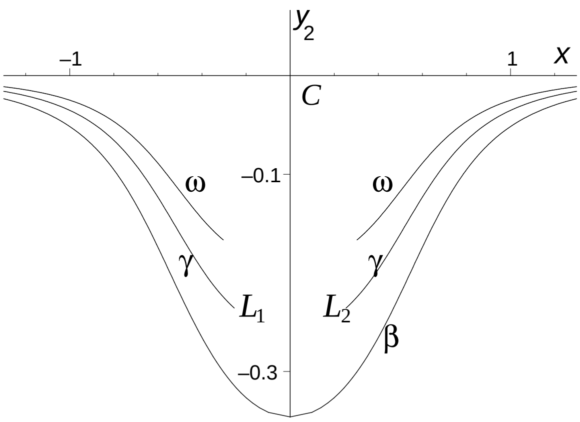

Figure 3.

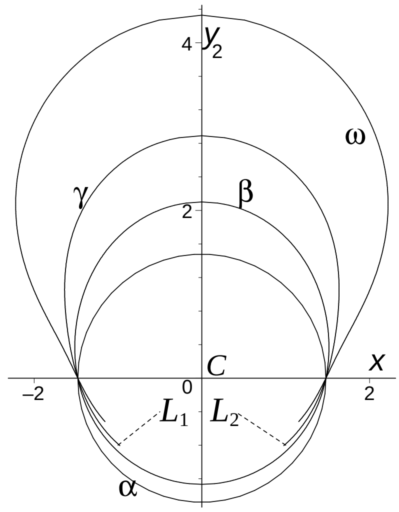

Figure 5.

circle ω is XEs set for $\vartheta = \pi/2$. Here each set of XE is like stationary orbit more than in other cases.

Figures 3,4,5 are depicted for $\alpha > 1/8$. In this case for any α, d, μ, k there exists ϑ_* such that if $\vartheta > \vartheta_*$ then sets of XEs are represented by curves γ with ends in TLPs L_1 and $L2$. γ goes to ω if $\vartheta \to \pi/2$. If $\vartheta \leq \vartheta_*$ then sets of XEs are represented by endless curves β. In the opposite case $\alpha \leq 1/8$ there are only endless curves β. In all figures the axis Cy_2 is projection of Cy to Cx_1y_1. Note also that in first, second and third type for fixed value of e there exist not more than 2 XEs . In fourth and fifth type for fixed value of e there exist not more than 4 XEs.

Using A.P.Ivanov's theorem [Ivanov 1984; Ivanov 1997]and analysing Jacobi's integral of the studied problem, we get the following criterion for stability of XEs that are not ZEs: x-equilibrium is stable if the station motion along the cable is forbidden, if the cable is tense and if $0 < \vartheta < \pi/2$.

5 Equilibria for zero gravitation

Now assume $\Pi = 0$. This situation takes place if, for example, some load coast along on the leier tethered to an extended space station that factually is an artificial asteroid. It can be checked that if gravitational is infinitesimal then for fixed e there exist 4 ZEs and 2 or 0 XEs. For coordinates of two first ZEs we have $\rho/\zeta = \pm\tan\vartheta$. These equilibria are the common points of the axis of precession and the ellipsoid (1). Evidently, the leier is not tense in these points. It can be proved that equilibria in the Cz_1 are unstable. For the third ZE we have $\rho/(\zeta - ed) = \left(1 - e^2\right)\cot\vartheta$. This equilibrium is stable for any d, e and $\vartheta \neq 0$. The fourth ZE is $\rho/(\zeta - ed) = -\left(1 - e^2\right)\cot\vartheta$. This ZE is stable only if

$$d < \frac{e \sin \vartheta}{\sqrt{1 - e^2 \sin^2 \vartheta}} \qquad (6)$$

Note that the tangential plane to (1) is parallel to Cz_1 in the last two ZEs. In our case XEs exist only if (6) is fulfilled. These equilibria are always unstable. Their coordinates are

$$\begin{array}{l} x = \pm a \frac{\sqrt{1-e^2}}{e \sin \vartheta} \sqrt{e^2 \sin^2 \vartheta - d^2 \left(1 - e^2 \cos^2 \vartheta\right)}, \\ y = \frac{ad\left(1-e^2\right)}{e} \cot \vartheta, \qquad z = -\frac{ad\left(1-e^2\right)}{e} \end{array} \qquad (7)$$

Note also that the cable is tense for the last 4 equilibria. (In this section we assume without loss of generality $d \geq 0$).

6 Conclusion

In this paper a space station equilibria on a cable with ends placed in poles of a dynamically symmetric asteroid are studied. These equilibria are divided into two classes. One of these classes is studied in the assumption, that the asteroid gravitational potential can be replaced with potential of two particles. This class is shared into five types of equilibria sets depending on values of the system parameters. It is proved that the studied equilibria are stable if the station 'is pasted' to the cable. Criteria of existence and stability for equilibria of both classes are deduced in the assumption that the asteroid gravitation is infinitesimal.

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