Controllability issues for continuous-spectrum systems and ensemble controllability of Bloch Equations

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Abstract
We study the controllability of the Bloch equation, for an ensemble of non-interacting half-spins, in a static magnetic field, with dispersion in the Larmor frequency. This system may be seen as a prototype for infinite dimensional bilinear systems with continuous spectrum, whose controllability is not well understood. We provide several mathematical answers, with discrimination between approximate and exact controllability, and between finite time or infinite time controllability: this system is not exactly controllable in finite time with bounded controls in $L^2(0,T)$, but it is approximately controllable in $L^\infty$ in finite time with unbounded controls in $L^\infty_{loc}([0,+\infty))$. Moreover, we propose explicit controls realizing the asymptotic exact controllability to a uniform state of spin $+1/2$ or $-1/2$.

Key words. bilinear control systems, Bloch equation, continuous spectrum, controllability of infinite dimensional systems, ensemble controllability, quantum systems.

1 Introduction

1.1 Studied system, bibliography
Most controllability results available for infinite dimensional systems are related to systems with discrete spectra. As far as we know, very few controllability studies consider systems admitting a continuous part in their spectra. In [12] an approximate controllability result is given for a system with mixed discrete/continuous spectrum: the Schrödinger partial differential equation of a quantum particle in an N-dimensional decaying potential is shown to be approximately controllable (in infinite time) to the ground bounded state when the initial state is a linear superposition of bounded states.

In [9, 10, 11] a controllability notion, called ensemble controllability, is introduced and discussed for quantum systems described by a family of ordinary differential equations (Bloch equations) depending continuously on a finite number of scalar parameters and with a finite number of control inputs. Ensemble controllability means that it is possible to find open-loop controls that compensate for the dispersion in these scalar parameters: the goal is to simultaneously steer a continuum of systems between states of interest with the same control input. The articles [9, 10, 11] highlight, for three common dispersions in NMR spectroscopy, the role of Lie algebras and non-commutativity in the design of a compensating control sequence and consequently in the characterization of ensemble controllability.

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The two control inputs can be seen as the prototype of infinite dimensional systems with purely continuous spectra. The goal of this paper is to show that the very interesting controllability analysis of [9, 10, 11] can be completed by functional analysis methods developed for infinite dimensional systems governed by partial differential equations (see, e.g., [7] for samples of these methods).

We focus here on one of the three dispersions cases treated in [9, 10, 11]. We consider an ensemble of non interacting half-spins in a static field \( \begin{pmatrix} 0 & 0 \\ 0 & B_0 \end{pmatrix} \) in \( \mathbb{R}^3 \), subject to a transverse radio frequency field \( \begin{pmatrix} v(t) \\ -u(t) \end{pmatrix} \) in \( \mathbb{R}^3 \) (the control input). The ensemble of half-spins is described by the magnetization vector \( M \in \mathbb{R}^3 \) depending on time \( t \) but also on the Larmor frequency \( \omega = -\gamma B_0 \) (\( \gamma \) is the gyromagnetic ratio). It obeys to the Bloch equation:

\[
\frac{\partial M}{\partial t}(t,\omega) = \begin{pmatrix} 0 & -\omega & v(t) \\ \omega & 0 & -u(t) \\ -v(t) & u(t) & 0 \end{pmatrix} M(t,\omega), \quad (t,\omega) \in [0, +\infty) \times (\omega_*,\omega^*),
\]

(1)

where \(-\infty \leq \omega_* < \omega^* \leq +\infty\) are given. With the notations

\[
\Omega_x := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega_y := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Omega_z := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(2)

the system (1) can be written

\[
\frac{\partial M}{\partial t}(t,\omega) = (\omega \Omega_x + u(t)\Omega_x + v(t)\Omega_y) M(t,\omega), \quad (t,\omega) \in [0, +\infty) \times (\omega_*,\omega^*).
\]

(3)

It is a bilinear control system in which, at time \( t \),

- the state is \( (M(t,\omega))_{\omega \in (\omega_*,\omega^*)} \); for each \( \omega \), \( M(t,\omega) \in S^2 \), the unit sphere of \( \mathbb{R}^3 \),

- the two control inputs \( u(t) \) and \( v(t) \) are real.

Thus, we study the simultaneous controllability of a continuum of ordinary differential equations, with respect to a parameter \( \omega \) that belongs to an interval \( (\omega_*,\omega^*) \). Notice that, when \( v = u = 0 \), the spectrum of this system is made by the union of the two segments, \( i(\omega_*,\omega^*) \) and \(-i(\omega_*,\omega^*) \), belonging to the imaginary axis.

The pioneer articles [9, 10, 11] provide convincing arguments indicating why the system (3) is ensemble controllable (i.e. approximately controllable in \( L^2((\omega_*,\omega^*),S^2) \)) with unbounded and also bounded controls, when \( \omega_* \) and \( \omega^* \) are finite. Here, we provide several mathematical results that complete these ensemble controllability results with discriminations between approximate or exact controllability and between finite or infinite time (asymptotically) controllability.

### 1.2 Controllability issues

Let us recall a famous non controllability result for infinite dimensional bilinear systems due to Ball, Marsden and Slemrod [1]. This result concerns general systems of the form

\[
\frac{dw}{dt} = Aw + p(t)Bw
\]

(4)

where the state is \( w \) and the control is \( p : [0, T] \to \mathbb{R} \).

**Theorem 1** Let \( X \) be a Banach space with \( \text{dim}(X) = +\infty \), \( A \) generate a \( C^0 \)-semigroup of bounded operators on \( X \) and \( B : X \to X \) be a bounded operator. For \( w_0 \in X, \ w(t;p,w_0) \)
which is, in fact, a rather
applies to 1D Schrödinger equations of the form
is only
ensures that this system is not
situation from $[0, T]$ is not related to a regularity problem and this equation corresponds to a very different
Indeed, we will prove that, when $(\omega, \omega^*) = (-\infty, +\infty)$, the reachable set (in finite time and with small controls) from $M_0 \equiv e_3$ is a submanifold of some functional space, that does not coincide with one of its tangent spaces. When the domain $(\omega, \omega^*)$ is a bounded interval of $\mathbb{R}$, we will see that there exist analytic targets, arbitrarily close to $e_3$ that cannot be reached exactly from $e_3$ with bounded controls in $L^2(0, T)$. Thus, the non controllability of (3) is not related to a regularity problem and this equation corresponds to a very different situation from [2, 3, 4].

1.3 Outline and open problems
In a first section, we study the linearized system of (3) around the steady-state $(M \equiv e_3, (u, v) \equiv 0)$ with $-\infty < \omega < \omega^* < +\infty$. This system is shown to be approximately
controllable in $C^0([\omega_*, \omega^*], \mathbb{R}^3)$, in any finite time $T$, with unbounded controls $(u, v) \in C^\infty([0, T), \mathbb{R}^2)$. But it is not exactly controllable neither in finite time nor in infinite time. Moreover, for any reachable target, there exists only one control which steers the control system to the target.

In a second section, we study the exact controllability of the nonlinear system (3), locally around $M \equiv e_3$, in finite time. First, we prove that the simultaneous exact controllability with respect to $\omega$ in the whole space $\mathbb{R}$ (i.e. $\omega_* = -\infty$, $\omega^* = +\infty$) does not hold with bounded controls. Indeed, for every time $T > 0$, the reachable set from $M_0 \equiv e_3$ with bounded controls in $L^2(0, T)$ is a strict submanifold (of some functional space) that does not coincide with one of its tangent space. Then, with an analyticity argument, we deduce that the simultaneous exact controllability with respect to $\omega$ in a bounded interval $(\omega_*, \omega^*)$, $-\infty < \omega_* < \omega^* < +\infty$, does not hold neither.

The exact controllability of (3) being impossible with bounded controls, then, we investigate the exact controllability of (3) with unbounded controls.

In a third section, completing the arguments of [9, 10, 11], we prove the ensemble controllability of (3): any measurable initial condition $M_0 : (\omega_*, \omega^*) \to S^2$ can be steered approximately in $L^2(\omega_*, \omega^*)$ to $e_3$. This approximate controllability indeed holds for stronger norms, for instance $\| \cdot \|_{L^\infty}$ and $\| \cdot \|_{H^s}$, $\forall s \in (0, 1)$. The controls used to realize this motion are sequences of pulses presented in [9] (but one may also use controls in $L^\infty_{loc}([0, +\infty))$) and the proof relies on non-commutativity and functional analysis as in [13].

In a fourth section, we propose other explicit unbounded controls realizing the asymptotic local (exact) controllability to $e_3$, simultaneously with respect to $\omega$ in a bounded interval. Here, the proof relies on Fourier analysis.

Let us emphasize that the behavior of the nonlinear system around $e_3$ is very different from the one of the linearized system around $e_3$. Indeed,

- first, the linearized system is not asymptotically zero controllable whereas the nonlinear system is asymptotically locally controllable to $e_3$,

- then, as seen in the first section, for the linearized system and for any reachable target, only a single control works, whereas for the nonlinear system and for any initial condition, many controls allow to reach exactly $e_3$ (in infinite time).

Thus, the nonlinearity allows to recover controllability.

Finally, let us mention some open problems.

In the second section, we proved the non exact controllability to $e_3$ with bounded controls, in finite time, because the reachable set is a submanifold. The equation of this submanifold and the validity of the same negative result in infinite time (i.e. the non asymptotic exact controllability to $e_3$ with bounded controls) are open problems.

In the last section, we prove the exact controllability to $e_3$ with unbounded controls, in infinite time. The validity of the same result in finite time is also open.

The complete article is available at :


References


