Controllability issues for continuous-spectrum systems and ensemble controllability of Bloch Equations

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Abstract

We study the controllability of the Bloch equation, for an ensemble of non interacting half-spins, in a static magnetic field, with dispersion in the Larmor frequency. This system may be seen as a prototype for infinite dimensional bilinear systems with continuous spectrum, whose controllability is not well understood. We provide several mathematical answers, with discrimination between approximate and exact controllability, and between finite time or infinite time controllability: this system is not exactly controllable in finite time T with bounded controls in $L^2(0,T)$, but it is approximately controllable in L^∞ in finite time with unbounded controls in $L^\infty_{loc}([0,+\infty))$. Moreover, we propose explicit controls realizing the asymptotic exact controllability to a uniform state of spin $+1/2$ or $-1/2$.

Key words. bilinear control systems, Bloch equation, continuous spectrum, controllability of infinite dimensional systems, ensemble controllability, quantum systems.

1 Introduction

1.1 Studied system, bibliography

Most controllability results available for infinite dimensional systems are related to systems with discrete spectra. As far as we know, very few controllability studies consider systems admitting a continuous part in their spectra. In [12] an approximate controllability result is given for a system with mixed discrete/continuous spectrum: the Schrödinger partial differential equation of a quantum particle in an N-dimensional decaying potential is shown to be approximately controllable (in infinite time) to the ground bounded state when the initial state is a linear superposition of bounded states.

In [9, 10, 11] a controllability notion, called ensemble controllability, is introduced and discussed for quantum systems described by a family of ordinary differential equations (Bloch equations) depending continuously on a finite number of scalar parameters and with a finite number of control inputs. Ensemble controllability means that it is possible to find open-loop controls that compensate for the dispersion in these scalar parameters: the goal is to simultaneously steer a continuum of systems between states of interest with the same control input. The articles [9, 10, 11] highlight, for three common dispersions in NMR spectroscopy, the role of Lie algebras and non-commutativity in the design of a compensating control sequence and consequently in the characterization of ensemble controllability.

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Such continuous family of ordinary differential systems sharing the same control inputs can be seen as the prototype of infinite dimensional systems with purely continuous spectra. The goal of this paper is to show that the very interesting controllability analysis of [9, 10, 11] can be completed by functional analysis methods developed for infinite dimensional systems governed by partial differential equations (see, e.g., [7] for samples of these methods).

We focus here on one of the three dispersions cases treated in [9, 10, 11]. We consider an ensemble of non interacting half-spins in a static field $\begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}$ in $\mathbb{R}^3$, subject to a transverse radio frequency field $\begin{pmatrix} v(t) \\ -u(t) \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$ (the control input). The ensemble of half-spins is described by the magnetization vector $M \in \mathbb{R}^3$ depending on time t but also on the Larmor frequency $\omega = -\gamma B_0$ (γ is the gyromagnetic ratio). It obeys to the Bloch equation:

$$\frac{\partial M}{\partial t}(t,\omega) = \begin{pmatrix} 0 & -\omega & v(t) \\ \omega & 0 & -u(t) \\ -v(t) & u(t) & 0 \end{pmatrix} M(t,\omega), \quad (t,\omega) \in [0,+\infty) \times (\omega_*, \omega^*), \tag{1}$$

where $-\infty \leqslant \omega_* < \omega^* \leqslant +\infty$ are given . With the notations

$$\Omega_x := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega_y := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Omega_z := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

the system (1) can be written

$$\frac{\partial M}{\partial t}(t,\omega) = \left(\omega \Omega_z + u(t)\Omega_x + v(t)\Omega_y\right) M(t,\omega), \quad (t,\omega) \in [0,+\infty) \times (\omega_*, \omega^*). \tag{3}$$

It is a bilinear control system in which, at time t,

- the state is $(M(t,\omega))_{\omega \in (\omega_*, \omega^*)}$; for each ω, $M(t,\omega) \in \mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$,
- the two control inputs $u(t)$ and $v(t)$ are real.

Thus, we study the simultaneous controllability of a continuum of ordinary differential equations, with respect to a parameter ω that belongs to an interval (ω_*, ω^*). Notice that, when $v = u = 0$, the spectrum of this system is made by the union of the two segments, $i(\omega_*, \omega^*)$ and $-i(\omega_*, \omega^*)$, belonging to the imaginary axis.

The pioneer articles [9, 10, 11] provide convincing arguments indicating why the system (3) is ensemble controllable (i.e. approximately controllable in $L^2((\omega_*, \omega^*), \mathbb{S}^2)$) with unbounded and also bounded controls, when ω_* and ω^* are finite. Here, we provide several mathematical results that complete these ensemble controllability results with discriminations between approximate or exact controllability and between finite or infinite time (asymptotically) controllability.

1.2 Controllability issues

Let us recall a famous non controllability result for infinite dimensional bilinear systems due to Ball, Marsden and Slemrod [1]. This result concerns general systems of the form

$$\frac{dw}{dt} = \mathcal{A}w + p(t)\mathcal{B}w \tag{4}$$

where the state is w and the control is $p : [0,T] \to \mathbb{R}$.

Theorem 1 *Let X be a Banach space with $dim(X) = +\infty$, $\mathcal{A}$ generate a C^0-semigroup of bounded operators on X and $\mathcal{B} : X \to X$ be a bounded operator. For $w_0 \in X$, $w(t; p, w_0)$*

denotes the unique solution of (4) with $p \in L^1_{loc}([0,+\infty))$ and $w(0) = w_0$. The reachable set from w_0

$$\mathcal{R}(w_0) := \{w(t;p,w_0); t \geqslant 0, p \in L^r_{loc}([0,+\infty)), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, it has an empty interior in X. Thus (4) is not controllable in X with controls in $\cup_{r>1} L^r_{loc}([0,+\infty))$.

We cannot apply directly here this result since the spaces $X = L^2((\omega_*,\omega^*),\mathbb{S}^2)$ or $C^0([\omega_*,\omega^*],\mathbb{S}^2)$ where the Cauchy problem is well-defined are not vector spaces. In order to get an interesting result for the Bloch equation, one needs extensions of the above result to Banach manifolds. (This has been done in [14] when the manifold is the unit sphere of a Hilbert space.) For (2), the situation is similar to the one described in Theorem 1. In Theorem **??**, we show that for any analytic initial condition $M_0(\omega)$, the reachable set in finite time $T > 0$ from M_0 with controls in $L^2(0,T)$ only contains analytic functions of ω. Thus, the reachable set (from an analytic initial dara) has an empty interior in $L^2((\omega_*,\omega^*),\mathbb{S}^2)$, which is a natural space for the Cauchy problem.

However, for (2), the obstruction to exact controllability given by Theorem **??** has much stronger consequence than the obstruction described by Theorem 1 which is, in fact, a rather weak non controllability result. Indeed, it does not prevent the reachable set from being dense in X (approximate controllability in X). For example, this is the case for the 1D beam equation

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)u_{xx} = 0, & x \in (0,1),\ t \in (0,+\infty), \\ u = u_x = 0 \text{ at } x = 0,1, \end{cases}$$

in which the state is (u,u_t) and the control is p. Theorem 1 ensures that this system is not exactly controllable in $H^2_0 \times L^2(0,1)$ with controls in $L^r_{loc}([0,+\infty))$, $r > 1$. However, it is proved in [3] that this system is exactly controllable in $H^{5+} \times H^{3+}(0,1)$ with controls in $H^1_0(0,T)$, at least locally around a stationnary trajectory. Similarly, Turinici's generalization [14] of Theorem 1 applies to 1D Schrödinger equations of the form

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - u(t)\mu(x)\psi, & x \in (0,1),\ t \in (0,+\infty), \\ \psi(t,0) = \psi(t,1) = 0 \end{cases}$$

where the state is ψ, the control is u and $\mu \in C^\infty([0,1])$. It proves that this system is not exactly controllable in $H^2((0,1),\mathbb{C})$ with controls in $L^2_{loc}([0,+\infty))$. However it is proved in [2, 4] that this system, with $\mu(x) = (x - 1/2)$ is exactly controllable in $H^7((0,1),\mathbb{C})$ with controls in $H^1_0(0,T)$, locally around the eigenstates, for T large enough.

The conclusion of [2, 3, 4] is that, sometimes, the negative result of Theorem 1 is only due to a bad choice of functional spaces that do not allow the controllability; but positive controllability results may be expected in different functional spaces. Therefore, one may still hope to prove the exact controllability of the Bloch equation in some well chosen functional spaces. We will see in this article that it is not the case: the Bloch equation is not exactly controllable in a much stronger sense than the one of Theorem 1.

Indeed, we will prove that, when $(\omega_*,\omega^*) = (-\infty,+\infty)$, the reachable set (in finite time and with small controls) from $M_0 \equiv e_3$ is a **submanifold** of some functional space, that does not coincide with one of its tangent spaces. When the domain (ω_*,ω^*) is a bounded interval of $\mathbb{R}$, we will see that there exist **analytic** targets, arbitrarily close to e_3 that cannot be reached exactly from e_3 with bounded controls in $L^2(0,T)$. Thus, the non controllability of (3) is not related to a regularity problem and this equation corresponds to a very different situation from [2, 3, 4].

1.3 Outline and open problems

In a first section, we study the linearized system of (3) around the steady-state $(M \equiv e_3, (u,v) \equiv 0)$ with $-\infty < \omega_* < \omega^* < +\infty$. This system is shown to be approximately controllable in $C^0([\omega_*,\omega^*],\mathbb{R}^3)$, in any finite time T, with unbounded controls $(u,v) \in C_c^\infty((0,T),\mathbb{R}^2)$. But it is not exactly controllable neither in finite time nor in infinite time. Moreover, for any reachable target, there exists only one control which steers the control system to the target.

In a second section, we study the exact controllability of the nonlinear system (3), locally around $M \equiv e_3$, in finite time. First, we prove that the simultaneous exact controllability with respect to ω in the whole space $\mathbb{R}$ (i.e. $\omega_* = -\infty$, $\omega^* = +\infty$) does not hold with bounded controls. Indeed, for every time $T > 0$, the reachable set from $M_0 \equiv e_3$ with bounded controls in $L^2(0,T)$ is a strict submanifold (of some functional space) that does not coincide with one of its tangent space. Then, with an analyticity argument, we deduce that the simultaneous exact controllability with respect to ω in a bounded interval (ω_*,ω^*), $-\infty < \omega_* < \omega^* < +\infty$, does not hold neither.

The exact controllability of (3) being impossible with bounded controls, then, we investigate the exact controllability of (3) with unbounded controls.

In a third section, completing the arguments of [9, 10, 11], we prove the ensemble controllability of (3): any measurable initial condition $M_0 : (\omega_*,\omega^*) \to \mathbb{S}^2$ can be steered approximately in $L^2(\omega_*,\omega^*)$ to e_3. This approximate controllability indeed holds for stronger norms, for instance $\|.\|_{L^\infty}$ and $\|.\|_{H^s}$, $\forall s \in (0,1)$. The controls used to realize this motion are sequences of pulses presented in [9] (but one may also use controls in $L^\infty_{loc}([0,+\infty))$) and the proof relies on non-commutativity and functional analysis as in [13]

In a fourth section, we propose other explicit unbounded controls realizing the asymptotic local (exact) controllability to e_3, simultaneously with respect to ω in a bounded interval. Here, the proof relies on Fourier analysis.

Let us emphasize that the behavior of the nonlinear system around e_3 is very different from the one of the linearized system around e_3. Indeed,

- first, the linearized system is not asymptotically zero controllable whereas the nonlinear system is asymptotically locally controllable to e_3,
- then, as seen in the first section, for the linearized system and for any reachable target, only a single control works, whereas for the nonlinear system and for any initial condition, many controls allow to reach exactly e_3 (in infinite time).

Thus, the nonlinearity allows to recover controllability.

Finally, let us mention some open problems.

In the second section, we proved the non exact controllability to e_3 with bounded controls, in finite time, because the reachable set is a submanifold. The equation of this submanifold and the validity of the same negative result in infinite time (i.e. the non asymptotic exact controllability to e_3 with bounded controls) are open problems.

In the last section, we prove the exact controllability to e_3 with unbounded controls, in infinite time. The validity of the same result in finite time is also open.

The complete article is available at :

http://www.cmla.ens-cachan.fr/fileadmin/Membres/beauchard/Bloch10.pdf

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