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SHUNT IMPEDANCE CONCEPT IN RF CAVITY-MOVING CHARGE ELECTRODYNAMICS

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Abstract
Cavity eigen functions for vector potential make it possible to determine completely the differential equations for amplitudes of the fields induced by charged particle traversing the cavity. For many accelerator based applications such detailed description of cavity excitation problem is not necessary. It is quite sufficient to have expressions or equations for cavity voltage induced by the charge and its phase as well, maximum particle energy gain after RF cavity passage expressed in voltage unites being assumed as voltage amplitude. Usually, equivalent electrical circuit is used instead of RF cavity to make appropriate calculation, stability analysis etc., resonance frequency, quality factor and shunt impedance of the circuit being equal to appropriate parameters of the RF cavity. In this paper, the attempt is undertaken to associate cavity excitation equations with beam charge and its velocity and cavity shunt impedance as well, the deduction being made on the basis of electrodynamics equations. It is shown that complex shunt impedance concept is quite natural generalization of usual shunt impedance for electrical charge-accelerating RF cavity interaction problem. Both shunt impedance module and its phase can be calculated or measured experimentally.

Key words
Cavity, beam, charge, bunch, impedance, interaction.

1 Introduction

As a rule, intensities of modern RF accelerators are sufficiently high in order to take into account a reverse impact of accelerated beam on an accelerating element. One says that beam-RF cavity interaction takes place when electromagnetic fields induced by the beam in the cavity are taken into account while calculating beam dynamics. To analyze the processes resulting from this interaction two approaches are used mainly. The first one is based on Maxwell equations solving. Cavity eigen functions for vector potential are found that together with differential equations for fields amplitudes form the basis for following analysis. As a rule the expressions or equations for fields amplitudes are of interest. These have to have an analytical representation for right hand parts. Since these in turn are functionals of cavity eigen functions that analytical solutions are available for limited number of cavities forms this place the practical limit of field based approach. In other approach mentioned the RF cavity is replaced with the electrical circuit containing active resistance, capacitance and inductance, their values are chosen in such a way to have the resonance frequency, quality factor and shunt impedance the same for the RF cavity and for the circuit. In this approach one has an analytical representation so necessary for analysis but the questions concerning approach justification and some uncertainness arise. In this paper, we use strict approach based on Maxwell equations to derive the expressions for fields amplitudes induced by the charged bunch traversing rf cavity. Transformations have been made to express the formula for the voltage induced by the bunch over cavity external parameters. Thus the equivalence of both approaches has been strictly proved for some cases at list. Similar approach based on Maxwell equations is not known to the author. Complex shunt impedance concept have been introduced and this appeared be fruitful for beam-cavity interaction processes description in RF accelerator based applications problems.

2 Electrodynamics of RF Cavity-Beam Interaction

To find out the fields that induce moving charge in a RF cavity, we will use the method that had been developed in [Lopukchin, 1953]. Vortex electrical $\vec{E}(\vec{r},t)$ and magnetic $\vec{H}(\vec{r},t)$ fields are represented as derivatives of vector potential $\vec{A}(\vec{r},t)$ on time and space coordinates:

$$\left.\begin{array}{l}\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} \\ \vec{H}(\vec{r},t) = \frac{1}{\mu_0} rot \vec{A}(\vec{r},t)\end{array}\right\} \qquad (1)$$

where μ_0 is magnetic permeability of free space. Here ant later SI units are used. Vector potential satisfies the wave equation

$$\Delta\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0 \vec{j}(\vec{r},t) \qquad (2)$$

$\vec{j}(\vec{r},t)$ and c being current density and the light velocity, respectively.

To find out the expressions for vector potential we will use the most direct way. Namely, we represent vector potential as an expansion on the infinite sum of RF cavity eigen functions $\vec{A}(\vec{r},t)$ with time dependent coefficients $g_\lambda(t)$:

$$\vec{A}(\vec{r},t) = \sum_{\lambda=1}^{\infty} g_\lambda(t)\vec{A}_\lambda(\vec{r}) \qquad (3)$$

with the boundary conditions $(\vec{A},\vec{n})$ on cavity surface, where $\vec{n}$ is normal to cavity surface.

Starting from the equation (2) and taking into account (3) one can easily obtain the equations for cavity vector eigen functions

$$\Delta\vec{A}_\lambda(\vec{r}) + k_\lambda^2 \vec{A}_\lambda(\vec{r}) = 0 \qquad (4)$$

and appropriate time dependent coefficients (fields amplitudes)

$$\frac{d^2 g_\lambda(t)}{dt^2} + \omega_\lambda^2 g_\lambda(t) = \int\limits_V \vec{j}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV \qquad (5)$$

Here $k_\lambda = \omega_\lambda/c$ are eigen values of boundary problems (4), the specific solutions for RF cavities are called cavity modes, ω_λ being the eigen angular frequencies of appropriate modes. Integration in formula (5) is assumed to be performed over cavity volume. Last equation can be generalized up to the next one

$$\frac{d^2 g_\lambda(t)}{dt^2} + \frac{\omega_\lambda}{Q_\lambda}\frac{dg_\lambda}{dt} + \omega_\lambda^2 g_\lambda(t) = \int\limits_V \vec{j}(\vec{r},t)\vec{A}_\lambda(\vec{r})dV \qquad (6)$$

if losses in cavity walls as well as electromagnetic power flow over apertures in cavity surface are taking into account. Here O_λ stands for cavity quality factor for λ mode:

$$Q_\lambda = \frac{\omega_\lambda W_\lambda}{P_\lambda} \qquad (7)$$

where W_λ is the electromagnetic energy in the mode λ, stored in cavity volume and P_λ represents the total RF power losses that besides ohm losses in cavity walls includes the external losses due to cavity coupling with external circuits. It I supposed that eigen functions are normalized by the condition

$$\int\limits_V A^2 dV = \mu_0 c^2 = \frac{1}{\epsilon_0} \qquad (8)$$

Here ϵ_0 is electric permeability.

For the following analysis we will use the cavity excitation in the form with small RF losses, and this has no any influence on generality of results to be obtained. Then, all calculations will be made for a single charged particle with charge value q of zero dimensions in all directions entering cavity at the moment $t = 0$. In such a case the total current density

$$\vec{j}(\vec{r},t) = q\vec{v}(\vec{r},t)\delta(x,y,vt) \qquad (9)$$

where $\vec{v}(\vec{r},t)$ stands for particle velocity being assumed constant within the cavity, and $\delta()$ is Dirac delta function, $\delta(0) = \infty$ and $\delta(x) = 0$ for all other x :

$$\int\limits_{-\infty}^{\infty} \delta(x)dx = 1 \qquad (10)$$

We suppose also the case that is the most interesting for accelerator based applications– the particle moves along cavity axis where

$$x = 0, y = 0.z = vt \qquad (11)$$

In such assumptions:

$$\frac{d^2 g(t)}{dt^2} + \omega^2 g(t) = \int\limits_0^L \delta(z - vt)qvA(z)dz \qquad (12)$$

From here and to the paper end we omit mode indexes that does lad to ambiguity. It follows from last relation that

$$\begin{array}{l}\frac{d^2 g(t)}{dt^2} + \omega^2 g(t) = J(t) \\ J(t) = qvA(vt)\eta(t)\eta(L - vt)\end{array} \qquad (13)$$

were $A(z) = A_z(0,0,z)$ and $\eta()$ is Heaviside step function, $\eta(x) = 1$ for all $x \geq 0$ and $\eta(x) = 0$ for $x < 0$ The solution of the equation (13) that satisfies initial conditions $g(0) = \dot{g}(0) = 0$ (corresponding equal to zero electric and magnetic components of induced field) can be represented in the form [Stepanov, 2006]:

$$g(t) = \frac{1}{\omega}\int_0^{L/v} J(\tau)\sin(\omega\tau)d\tau = \frac{\sin(\omega t)}{\omega}J_1 - \frac{\cos(\omega t)}{\omega}J_2 \tag{14}$$

Here

$$J_1 = \int_0^{L/v} J(t)\cos(\omega\tau)d\tau, J_2 = \int_0^{L/v} J(t)\sin(\omega\tau)d\tau \tag{15}$$

Note that solution for field amplitude in the form (14) is valid for time interval $t > L/v$.

3 Induced voltage over cavity external parameters

In accelerator based applications, such for example as self consistent beam dynamics [Kurakin V. and Kurakin P., 2012] or stability analysis [Kurakin, 2010], voltages amplitudes and its phases on so called accelerating gap are of value instead of details fields description. Now, we proceed to this for the problem under discussion. First, let us find out probe particle with charge e energy gain $\mathcal{E}$ after passage of the cavity assuming field amplitude being $g(t) = a\sin(\omega t + \phi)$, where a is constant. One can derive easily:

$$\begin{aligned}\mathcal{E}(\phi) &= e\int_o^L E(z,t)dz = -e\int_o^L \dot{g}(t)A(z)dz = \\ &-\tfrac{ea\omega}{q}\cos\phi J_1 + \tfrac{ea\omega}{q}\sin\phi J_1\end{aligned} \tag{16}$$

Representing rf cavity in the form equivalent thin gap of zero length (accelerating gap) with applied rf voltage one can conclude that appropriate voltage amplitude U_m is equal to

$$U_m = \frac{\mathcal{E}_{max}}{q} = \frac{a\omega}{q}(J_1^2 + J_2^2)^{1/2} \tag{17}$$

This can be expressed in terms of cavity shunt impedance R and cavity quality factor Q_0:

$$R = \frac{U_m^2}{P_0^2}, \qquad Q_0 = \frac{\omega W}{P_0} \tag{18}$$

where P_0 stands for cavity walls power losses and W is electromagnetic energy stored in the cavity volume.

$$W = \frac{\epsilon_0}{2}\int_V E_m^2 dV = \frac{a^2\omega^2\epsilon_0}{2} \tag{19}$$

Taking into account normalization condition one arrives finally at relations

$$W = \frac{a^2\omega^2}{2}, \qquad J_1^2 + J_2^2 = \frac{R}{Q_0}\frac{\omega q^2}{2} \tag{20}$$

In beam-cavity interaction problem, electromagnetic field in the cavity is the superposition of three components. These is the electromagnetic field that external RF generator excites in the cavity, and this generator feeds cavity with rf power for charged bunches acceleration. The second one is the field induced in the cavity by the charge traversing it. At last, there is electromagnetic field induced in the cavity by all for-running charges. One may consider that all these fields act independently provided the charge does change its velocity while traversing the cavity, and this is assumed. The subject of our interest is the third source of RF fields. Charge self action is noticeable for large charge value, while one has to take into account RF generator fields for the case when the generator is not isolated from the power flow from the cavity, and this is not assumed. In this paper, we concentrate our attention on the fields, induced by preceding bunches and acting on the currently traversing cavity bunch.

With these remarks, let us calculate energy loss for the particle traversing cavity filled with the field induced by previous charge, both radiating charge and probe particle being spaced by time interval equal to period of rf oscillations.

$$\begin{aligned}\mathcal{E}_{lost} &= -ev\int_o^{L/v} \dot{q}(t)A(vt)dt = \\ &-ev\int_0^{L/v}\int_0^{L/v} J(\tau)\tau\cos(\omega t - \omega\tau)A(vt)d\tau dt \\ &= -\tfrac{e}{q}(J_1^2 + J_2^2)\end{aligned} \tag{21}$$

Together with last expression this gives

$$\mathcal{E}_{lost} = -\frac{eq\omega}{2}\frac{R}{Q_o} \tag{22}$$

In terms of thin gap this means that bunch with charge q induces rf voltage of amplitude U_m

$$U_m = \frac{q\omega}{2}\frac{R}{Q_0} \tag{23}$$

and rf phase π. Furthermore, taking into account field damping we arrive finally at the expression for rf field, induced by charged bunch on equivalent thin gap

$$U = -\frac{q\omega}{2}\frac{R}{Q_0}\exp(-\frac{\omega t}{2Q})\cos(\omega t) \tag{24}$$

Very often, current value I averaged over RF period is used instead of charge value

$$U = -\pi I \frac{R}{Q_0} \exp(-\frac{\omega t}{2Q}) \cos(\omega t) \qquad (25)$$

Two last formulae show clearly that both induced voltage amplitude as well as induced RF voltage phase can be expressed over cavity external parameters, namely cavity shunt impedance , cavity quality factor Q_0 and coupling coefficient $\beta = Q/Q_0 - 1$ In other words, in electro dynamical approach detailed cavity description is not required for the calculation the voltage induced by charged bunch.

Of course, one might derive two last formula using electrical circuit approach but quit natural question arises if similar approach reflects reality in beam-cavity interaction process. The calculations have been made can be considered as the equivalence of both approaches in this particular case of beam interaction with any cavity acceleration mode.

In stored energy accelerator [Kurakin V. and Kurakin P., 2012] the energy spread that arises from beam loading effect might be compensated by additional cavity installed on beam path. This cavity operates at the frequency shifted by the value

$$\frac{\Delta\omega}{\omega} = \frac{U_{lost}}{2\pi U_m (N-1)} \qquad (26)$$

where U_{lost}, U_m and N are the energy (expressed in Volt units) that charge induces on equivalent accelerating gap, compensating cavity voltage amplitude and number of bunches in bunches train respectively. It had been shown by solving Maxwell equations that bunch energy lost in accelerating cavity might be expressed in terms of external parameters by expression similar to formulae (23) for specific type of the cavity namely pill box (cylindrical resonator) cavity. Here we can manifest that the key conclusion of paper [Kurakin V. and Kurakin P., 2012] is justified for any type of accelerating cavity.

For many accelerator based applications expression (25) that links induced voltage with bunch charge value and cavity external parameters is quit sufficient. It allows to anybody to calculate induced voltage for any charged bunch sequence as well as for any charge distribution within any bunch. Appropriate expression is simply an appropriate sum as well as appropriate integrals within bunch length that can be written for any particular problem. Formula (25) is the key expression for any particular application.

4 RF Cavity Complex Shunt Impedance Concept

As it has been shown, cavity external parameters allow to find out appropriate integral parameters of physical quantity, but these nothings say about real quantities values inside the cavity. In other words, one can consider cavity as a black box while induced voltage in defined above sense as this black box response under external exposure, charged bunch in the case under consideration. It is quite clear from formulae (14). It follows that the phase of oscillations depends on two quantities J_1 and J_2, and these two quite different functionals can not be expressed over one quantity. These parameters might be used for detailed description of beam-cavity interaction and the outlook on relation (14) prompts to represent it in the form

$$\begin{aligned} g(t) &= \tfrac{\sin\omega t}{\omega} J_1 - \tfrac{\cos\omega t}{\omega} J_2 \\ &= \tfrac{D}{\omega}(\sin\omega t \sin\psi - \cos\omega t \cos\psi) \end{aligned} \qquad (27)$$

where

$$D = (J_1^2 + J_2^2)^{\frac{1}{2}}, \sin\psi = \frac{J_1}{D}, \cos\psi = \frac{J_2}{D} \qquad (28)$$

and formula for fields amplitude takes the form:

$$g(t) = -\frac{D}{\omega}\cos(\omega t + \psi) \qquad (29)$$

Thus, the pare of quantities J_1 and J_2 or D and ψ is needed for detailed description of beam-cavity interaction, and this pare as it followed from formulae written above might be considered as the real and imaginary parts or the module and the phase of complex quantity:

$$\hat{D} = Re\hat{D} + iIm\hat{D} = D\exp i\psi \qquad (30)$$

where i is imaginary unit. D is expressed over cavity shunt impedance, and finally expression for field amplitude looks like

$$g(t) = -q\sqrt{\frac{R}{2\omega Q_0}}\cos(\omega t + \psi) \qquad (31)$$

It is often much more convenient to deal with complex quantities keeping in mind that physical sense has its real part. Then, denoting

$$\hat{Z} = R\exp(i2\psi) \qquad (32)$$

we arrive at relation

$$g(t) = -Re\sqrt{\frac{\hat{Z}}{2\omega Q_0}}\exp i\omega t \qquad (33)$$

In these notations, it is quite natural to refer to $\hat{Z}$ as complex shunt impedance. Its module coincides with usual cavity shunt impedance. To establish physical sense its phase let us rewrite expression (16) for energy gain for the probe particle entering a cavity at $t = 0$ using ψ definition

$$\begin{aligned} U(\phi) &= \frac{a\omega}{qv} D(-\cos\phi\sin\psi + \sin\phi\cos\psi) \\ &= \frac{a\omega}{qv} D\sin(\phi - \psi) \end{aligned} \tag{34}$$

It follows from last expression that probe particle has zero energy gain after cavity passage if enters the cavity when field phase in the cavity is equal to ψ.

Complex cavity shunt impedance can be calculated for any particular mode according formulae above or established experimentally. To measure R and ψ the following experiment has to be done. RF cavity installed on probe beam path is fed with power P . Cavity RF phase is adjusted to have the maximum energy gain U_m at its exit. Appropriate combination (18) of values obtained gives cavity shunt impedance module. Adjusting phase shifter to position corresponding to zero energy gain at cavity exit one gets information concerning phase ψ .

It is important to emphasize that the complex shunt impedance concept arises from field approach in the problem of beam-cavity interaction. It has not appropriate analogue at all in electrical circuit approach since is connected with a real process in real RF fields with real field pattern.

5 Conclusion

In electrodynamics of RF cavity-beam interaction problem, the right side expressions for electromagnetic fields amplitudes are the integrals of beam currents and cavity eigen functions over cavity volume. This limits these equations applications for those accelerator based problems where explicit representation of right side terms is needed. On the other hand, for many similar problems integral quantities are of importance instead of detailed ones. For our specific task this is the RF voltage induced by moving charge inside a cavity. The solution for this voltage has been obtained and this one has been expressed in terms of cavity external parameters: cavity shunt impedance and quality factor. In other words, the bridge has been thrown between electro dynamical and electrical circuit approach in beam-cavity interaction problem.

It has been shown, that two external parameters are needed instead single one, called cavity shunt impedance, in order to have explicit expression for induced amplitudes and thus to have explicit expressions for fields distribution inside a cavity. The concept of complex shunt impedance has been introduced to the problem under attention, and solution for field amplitude had been expressed in terms of this cavity parameter. The physical sense both for the module and the phase as well of complex shunt impedance has been clarified. The first one is simply cavity shunt impedance in widely used sense, while the other fixes the phase at which the probe particle, entering cavity, traverses it without additional energy gain. It had been shown, that complex shunt impedance components can be calculated or measured experimentally.

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