FORMALIZATION OF SOLUTION FOR NONLINEAR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH IMPULSE CONTROL

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Abstract

The article is devoted to formalization of concept of the solution for the differential equations of neutral type with the generalized effect in the right part. The concept of the solution is formalized by closure of the set of smooth solutions in the space of functions of the bounded variation. Sufficient conditions providing existence so the formalized solution are received. The integral equation describing so the formalized solution is received. Illustrating examples are resulted.

Key words

Differential equations of neutral type, systems with time delay, impulse control, functions of bounded variation.

1 Introduction

The numerous dynamic processes based on transfer of mass, energy, information (for example, hereditary) and etc., are accompanied by the presence of delay. This delay can be caused by the most various reasons - limitation of speed of spreading of interaction (for example, an electric signal), presence inertance of some elements (for example, inductance in electric circuits). The equations with a deflecting argument describe many processes with aftereffect. Such equations appear, for example, when in a considered problem of physical or technical character the force acting on a mass point, depends on speed and position of this point not only at present, but also at some preceding moment. It is possible to give many examples of the dynamic systems with a deflecting argument which meet in such sciences, as biology, medicine, economic statistics, mechanics, etc. There is also a plenty of applications in which the lagging argument is included not only into a variable, but also in its derivative. These are so-called difference-differential equations of neutral type. For example, for a georadar the oscillatory processes are described by linear periodic difference-differential equation of neutral type. In the problems of control as control effect the effects, reducing to the spasmodic change of characteristics of dynamic process, can be used. The questions of formalization of concept of the solution for such systems, which dynamics is described by the ordinary differential equations, were considered in [Zavalishchin, S.T., Sesekin, A.N.,1991; Zavalishchin, S.T., Sesekin, A.N..,1997; Bressan, A., Rampazzo, F.,1991; Sesekin, A.N. 2000; Miller, B.M., Rubinovich, E.Y.,2002]. The singularity of such systems is that in the right part of the differential equations there can be an incorrect operation of multiplication of discontinuous function on distributions [Schwartz, L.,1950-1951]. One of approaches to the overcoming noted incorrectness consists that it is offered to take as the solution pointwise limit of smooth solutions, generated by smooth approximations of the generalized effects entering into the right part of the differential equation. Such approach, according to [Krasovskii, N.N., 1968], is natural from the point of view of the theory of optimal control. In papers [Fetisova, Y.V., Sesekin, A.N. 2005; Fetisova, J.V., Sesekin, A.N. 2009; Sesekin, A.N., Fetisova, J.V. 2010] this approach has been spreaded to the differential equations with the constant and distributed delay. For one class of the differential equations of neutral type this approach to formalization of the discontinuous decision has been considered in [Fetisova, J.V., Sesekin, A.N. 2009]. In the given paper we will lead development of the approach connected with closure of set of smooth decisions in space of functions of the bounded variation on more general class of the differential equations of neutral type.

2 The first variant of differential equation of neutral type

Consider the following Cauchy problem

$$\dot{x}(t) = f(t, x(t), x(t-\tau)) + Q(t, x(t))\dot{x}(t-\tau)$$

$$+B(t, x(t))\ \dot{v}(t), \qquad (1)$$

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0].$$

Here $t \in [t_0, \vartheta]$, $x(t)$ and $v(t)$ are respectively n- and m-vector functions of time, $f(t, x, y)$ is an n-vector function, and $B(t, x)$, $Q(t, x)$ are an $n \times m$, $n \times n$ and $n \times n$ matrix function, $v(\cdot) \in BV_m[t_0, \vartheta]$, where $BV_m[t_0, \vartheta]$ denotes Banach space of m-vector functions of bounded variation, $\tau > 0$ is a constant delay, $\varphi(t)$ is an initial $n-$vector function of bounded variation.

Assume that $f(\cdot, \cdot, \cdot)$ is measurable in t, continuous in rest variables and Lipschitz in x, $B(\cdot, \cdot)$, $Q(\cdot, \cdot)$ are continuous and Lipschitz in the second variable on the set $\{t \in [t_0, \vartheta],\ \|x\| < \infty\}$ where $\|x\| = \left(\sum\limits_{i=1}^{n} |x_i|^2\right)^{1/2}$, and satisfy to the following standard conditions in the same set:

$$\|f(t, x, y, v)\| \le \kappa(1+\|x\|),\ \ \|Q(t, x)\| \le \kappa(1+\|x\|),$$

$$\|B(t, x)\| \le \kappa(1 + \|x\|),$$

where κ is some positive constant.

Let us choose two sequences of absolutely continuous functions $v_k(t)$, $\varphi_k(t)$, $k = 1, 2, \ldots$ pointwise converging to $v(t) \in BV_m[t_0, \vartheta]$ and $\varphi(t) \in BV_m[t_0, \vartheta]$ respectively. According to [Kolmanovskii, V., Myshkis, A. 1992] the solution of Cauchy problem (1) exists for every absolutely continuous functions $v_k(t)$ and $\varphi_k(t)$ (the functions $v(t)$ and $v_k(t)$, $k = 1, 2, \ldots$ satisfy the constraint $\underset{[t_0, \vartheta]}{\mathrm{var}}\ v(\cdot) \le a$). Let $x(t) = x_k(t)$ is a solution of Cauchy problem (1) with $v_k(t),\ \varphi_k(t)$.

Definition 1. A vector function of bounded variation $x(t)$ is called the *approximable solution* of Cauchy problem (1), if $x(t)$ is the pointwise limit of the sequence $x_k(t)$, $k = 1, 2, \ldots$ generated by sequences $v_k(t), \varphi_k(t)$ and $x(t)$ does not depend on the choice of $v_k(t)$ and $\varphi_k(t)$.

Theorem 1. *Let all the conditions given above are satisfied. Moreover we assume there exist the partial derivatives $\partial b_{ij}, \partial x_\nu$ of elements of the matrix function $B(\cdot, \cdot)$ and $\partial q_{ij}, \partial x_\nu$ of elements of the matrix function $Q(\cdot, \cdot)$, which satisfy the following equalities*

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}(t, x)}{\partial x_\nu} b_{\nu l}(t, x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t, x)}{\partial x_\nu} b_{\nu j}(t, x),$$

$$\sum_{\nu=1}^{n} \frac{\partial q_{i\mu}(t, x)}{\partial x_\nu} q_{\nu\eta}(t, x) = \sum_{\nu=1}^{n} \frac{\partial q_{i\eta}(t, x)}{\partial x_\nu} q_{\nu\mu}(t, x),$$

$$\sum_{\nu=1}^{n} \frac{\partial q_{i\mu}(t, x)}{\partial x_\nu} b_{\nu l}(t, x) = \sum_{\nu=1}^{n} \frac{\partial b_{il}(t, x)}{\partial x_\nu} q_{\nu\mu}(t, x)$$

(Frobenius condition) $i, \mu, \eta = 1, 2, ..., n;\ j, l = 1, 2, ..., m.$

Then for any vector function of bounded variation $v(t)$ there exists the approximable solution $x(t)$ of (1), which satisfies to the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f(\xi, x(\xi), x(\xi - \tau))\, d\xi$$

$$+ \int_{t_0}^{t} Q(\xi, x(\xi))\, dx^c(\xi) + \int_{t_0}^{t} B(\xi, x(\xi))\, dv^c(\xi)$$

$$+ \sum_{t_i \le t,\ t_i \in \overline{\Omega}_-} \overline{S}(t_i, x(t_i - 0), \Delta x(t_i - 0))$$

$$+ \sum_{t_i < t,\ t_i \in \overline{\Omega}_+} S(t_i, x(t_i), \Delta x(t_i - \tau + 0)),$$

$$+ \sum_{t_i \le t,\ t_i \in \Omega_-} S(t_i, x(t_i - 0), \Delta v(t_i - 0))$$

$$+ \sum_{t_i \le t,\ t_i \in \Omega_+} S(t_i, x(t_i - 0), \Delta v(t_i + 0))$$

Here $v^c(\xi)$ is the continuous part of the function $v(\xi)$ and $x^c(\xi)$ is the continuous part of the function $x(\xi)$,

$$\overline{S}(t, x(t), \Delta x(t - \tau)) = z(1) - x(t),$$

$$\dot z(\xi) = Q(t, z(\xi))\Delta x(t),\ z(0) = x(t),$$

$$S(t, x(t), \Delta v) = z(1) - x(t),$$

$$\dot z(\xi) = B(t, z(\xi))\Delta v(t),\ z(0) = x(t),$$

$\Omega_-(\Omega_+)$ *is the set of points at which* $v(t)$ *is discontinuous from the left (from the right).*$\overline{\Omega}_-(\overline{\Omega}_+)$*is union of points of left (right) discontinuous of initial function* $\varphi(t)$ *and points of* Ω_-, Ω_+ *shifted to the right into a finite number (integer for points of* Ω_-, Ω_+*) of delays* τ*; these points fall in* $[t_0, \vartheta]$.

$$\Delta v(t-0) = v(t) - v(t-0),\ \Delta v(t+0) = v(t+0) - v(t)$$

$$\Delta x(t-0) = x(t) - x(t-0),\ \Delta x(t+0) = x(t+0) - x(t)$$

Proof. Let apply the step method to the equation (1) [Kolmanovskii, V., Myshkis, A. 1992].In this case the function $\dot x(t-\tau)$ is known function at each step. Therefore it is possible apply the corresponding theorem for differential equations with delay and generalized effect in the right part from [Fetisova, Y.V., Sesekin, A.N. 2005] to the equation (1); the justification of Theorem 1 follows from this theorem.

3 The second variant of differential equation of neutral type

Now consider the following Cauchy problem

$$\dot x(t) = f(t, x(t), x(t-\tau)) + G(t, x(t-\tau))\dot x(t-\tau), \tag{2}$$

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0].$$

Here the vector-function $f(t, x, y)$ satisfies the same hypothesis from previous section.$G(t, x)$ is continuous $n \times n$ matrix of variables t and x and satisfy to restriction

$$\|G(t, x)\| \le \kappa(1 + \|x\|),$$

Moreover we assume there exist the partial derivatives $\partial g_{ij}/\partial x_\nu$ of elements of the matrix function $G(t, x)$, that satisfy to the following conditions:

$$\sum_{\nu=1}^{n} \frac{\partial g_{i\mu}(t, x)}{\partial x_\nu} g_{\nu\eta}(t, x) = \sum_{\nu=1}^{n} \frac{\partial g_{i\eta}(t, x)}{\partial x_\nu} g_{\nu\mu}(t, x) \tag{3}$$

$i, \nu, \mu, \eta = 1, 2, ..., n$. If the condition (3) is realized, then there exists a potential vector function $U(t, x)$ such that $g_{ij}(t, x) = \frac{\partial u_i(t,x)}{\partial x_j}$, where $g_{ij}(t, x)$ are the elements of matrix G, $u_i(t, x)$ are the coordinates of vector $U(t, x)$.

As generalized effect $\dot v(t)$ is missing, now we shall give the following definition of solution of the problem (2).

Definition 2. A vector function of bounded variation $x(t)$ is called the *approximable solution* of Cauchy problem (2), if $x(t)$ is the pointwise limit of the sequence $x_k(t)$, $k = 1, 2, \dots$ generated by a sequence $\varphi_k(t)$; $\varphi(t) \in BV_m[t_0, \vartheta]$ is the pointwise limit of the sequence $\varphi_k(t)$ and $x(t)$ does not depend on the choice of $\varphi_k(t)$.

Theorem 2. *Let all the conditions given above are satisfied. Then for any initial function of bounded variation* $\varphi(t)$ *there exists the approximable solution* $x(t)$ *of* (2), *which satisfies to the integral equation*

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f(\xi, x(\xi), x(\xi - \tau))\, d\xi$$

$$+ U(x(t-\tau) - U(x(t_0 - \tau)) \tag{4}$$

Proof. In the first place we rewrite the differential equation (2) for a sequence of $x_k(t)$ in the integral form

$$x_k(t) = \varphi_k(t_0) + \int_{t_0}^{t} f(\xi, x_k(\xi), x_k(\xi - \tau))\, d\xi$$

$$+ \int_{t_0}^{t} G(\xi, x_k(\xi - \tau))\, dx_k(\xi - \tau), \tag{5}$$

where $x_k(t)$ is an absolutely continuous solution generated by absolutely continuous functions $\varphi_k(t)$. The solution $x_k(t)$ is constructed by the step method in $[t_0, \vartheta]$. Using the condition (3) it follows that

$$\int_{t_0}^{t} G(\xi, x_k(\xi - \tau))\, dx_k(\xi - \tau)$$

can be written as

$$U(x_k(t-\tau)) - U(x_k(t_0 - \tau)).$$

So the equation (5) can be written into the form:

$$x_k(t) = \varphi_k(t_0) + \int_{t_0}^{t} f(\xi, x_k(\xi), x_k(\xi - \tau))\, d\xi$$

$$+U(x_k(t-\tau)) - U(x_k(t_0-\tau)) \qquad (6)$$

The convergent subsequence $x_{k_i}(t)$ can be chosen from the sequence of $x_k(t)$ of bounded variation always. The function $U(t,x)$ is continuous in variables.So passing to the limit as $k_i \to \infty$ in the equation (6),we get the limit of the subsequence $x_{k_i}(t)$ converges to some function $x(t)$ that will be the solution of the equation (4).It is easily proved that the solution of the equation (4) is unique.This fact provides convergence of all sequence $x_k(t)$ to $x(t)$, the proof is completed.

4 Examples

Let's give the following Cauchy problem

$$\left.\begin{array}{r}\dot{x}(t) = y(t) + x(t)\,\dot{x}(t-1),\\ \dot{y}(t) = y(t)\delta(t),\end{array}\right\}. \qquad (7)$$

the initial function is given by

$$\varphi(t) = \begin{pmatrix}\varphi_x(t)\\ \varphi_y(t)\end{pmatrix}, \qquad (8)$$

where

$$\varphi_x(t) = \begin{cases}0, & t=-1;\\ 1, & t\in(-1,0].\end{cases}, \varphi_y(t) = 1,\ t\in[-1,0]. \qquad (9)$$

For the Cauchy problem (7) Frobenius condition from the Theorem 1 is satisfied. The solution of Cauchy problem (7) on the interval $[0,1]$ can be written as

$$x(t) = e^{\chi(t)} + e\,t\,\chi(t),\ \ y(t) = e^{\chi(t)},$$

where

$$\chi(t) = \begin{cases}0, & t\le 0,\\ 1, & t>0.\end{cases}$$

Now we consider the following Cauchy problem:

$$\left.\begin{array}{l}\dot{x}(t) = y(t) + y(t)\,\dot{x}(t-1),\\ \dot{y}(t) = x(t)\delta(t),\end{array}\right\} \qquad (10)$$

the initial function is given by (8), (9). The Frobenius condition is broken for the Cauchy problem (10). Not difficult to show that the various approximations of functions $\varphi_x(t)$ and $\delta(t)$ will lead to the limit of a sequence of $x_k(t)$ will depend on the method of approximation of $\varphi_x(t)$ and $\delta(t)$.
Now give the example illustratingthe Theorem 2.
Now we consider the following Cauchy problem:

$$\dot{x}(t) = x(t-1)\dot{x}(t-1)$$

The initial function defined on $[-1,0]$ is

$$\varphi(t) = \chi(t+\frac{1}{2}),$$

The potential function $U(t,y)$ has the form:

$$U(t,y) = \frac{1}{2}y^2.$$

We can write the solution on $[-1,0]$ in the following form:

$$x(t) = \chi(t+\frac{1}{2}) + \frac{1}{2}\chi(t-\frac{1}{2})$$

5 Conclusion

Formalization of concept of the solution for one class of the nonlinear differential equations of the neutral type having great value for applied problems of impulse optimal control is lead. Sufficient conditions of existence of the solution are received and the integral equation to which satisfies so the formalized solution is established.

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References

Zavalishchin, S.T., Sesekin, A.N. (1991). *Impulse Processes: Models and Applications*. Moscow: Nauka, (In Russian.).

Zavalishchin, S.T., Sesekin, A.N. (1997). *Dynamic Impulse Systems: Theory and Applications*. Kluwer Academic Publishers, Dordrecht.

Bressan, A., Rampazzo, F. (1991). Impulsive Control Systems with Commutative Vector Fields. *JOTA*, vol. 71, no. 1, pp. 67–83.

Sesekin A.N. (2000) Dynamic Systems with Nonlinear Impulse Structure. *Proceedings of the Steklov Institute of Mathematics.* M.: MAIK, "Nauka/Interperiodika", pp. 159-173.

Miller, B.M., Rubinovich, E.Y. (2002). *Impulsive Control in Continuous and Discrete Continuous Systems*. Kluwer Academic / Plenum Publishers.

Schwartz, L. (1950–1951). *Theory des Distributions*. Paris: Hermann.

Krasovskii N.N. (1968) *Theory of Motion of Control. Linear Systems.* Nauka, Moscow.

Fetisova Y.V., Sesekin A.N. (2005) Discontinuous solutions of differential equations with time delay. *WSEAS transactions on systems.* Vol. 4, no. 5. pp. 487–492.

Fetisova Y.V., Sesekin A.N. (2009) Differential equations with time delay in a space of functions of bounded variation. *Functional Differential Equations.* V. 16, no 2. pp. 273-278.

Sesekin A.N., Fetisova J.V. (2010) Functional Differential Equations in Space of Functions of Bouded Variation. *Proceeding of the Steklov Instiute of Mathematics.* pp. 258–265.

Kolmanovskii, V., Myshkis,A. (1992) *Applied Theory of Functinal Differential Equations*. Kluwer Academic Publishers, Dordrecht.