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GENERALIZED CONTROLLABILITY SUBSPACES FOR TIME-INVARIANT SINGULAR LINEAR SYSTEMS

M. I. García-Planas
Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya,
Spain
maria.isabel.garcia@upc.edu

Abstract
Controllability subspaces play an important role in geometric control theory for proper linear systems (A, B). In this paper we attempt to extend this concept to singular systems constructing generalized invariant subspaces of controllability for triples of matrices (E, A, B) representing singular systems.

Key words
Singular systems, feedback proportional and derivative, invariant subspaces.

1 Introduction

Let us consider a finite-dimensional singular linear time-invariant system $E\dot{x} = Ax + Bu$, where $E, A \in M_n(C)$, $B \in M_{n\times m}(C)$. For simplicity, we denote the systems as a triples of matrices (E, A, B) and we denote by

$$\mathcal{M} = \{(E, A, B) \mid E, A \in M_n(C), B \in M_{n\times m}(C)\}$$

the set of singular systems. the set of this kind of systems. In the case where $E = I_n$ the system is standard and we denote merely, as a pair (A, B).

For simplicity but without loss of generality, we consider that matrix B has column full rank: $0 < \operatorname{rank} B = m \leq n$.

Invariant subspaces for transformations from $\mathbb{C}^{m+n}$ into $\mathbb{C}^n$ was introduced by Gohberg, Lancaster, Rodman [I. Gohberg, P. Lancaster, L. Rodman, (1986)], as a generalization of similarity called block-similarity. Our objective is to develop a generalization of the concept of invariant subspace for triples of matrices as generalized linear maps defined modulo a subspace.

Remember that a subspace $G \subset \mathbb{C}^n$, is invariant under (A, B) as a map from $\mathbb{C}^{n+m}$ into $\mathbb{C}^n$ if and only if there exists a subspace $\bar{G}$ of $\mathbb{C}^{n+m}$ where the canonical projection of $\bar{G}$ over $\mathbb{C}^n$ is G, $(\pi_{|\mathbb{C}^n}\bar{G} = G)$, and $\begin{pmatrix} A & B \end{pmatrix} \bar{G} \subset \pi_{|\mathbb{C}^n}\bar{G} = G$.

Equivalently (see [I. Gohberg, P. Lancaster, L. Rodman, (1986)] for a proof), a subspace $G \subset \mathbb{C}^n$ is invariant under (A, B) if and only if

$$AG \subset G + \operatorname{Im} B \qquad (1)$$

In this paper, we consider triples of matrices (E, A, B), that we can see as a pair of maps (E, B), (A, B) defined modulo a subspace (see [Mª I. García-Planas, (2006)]), classified under the following equivalence relation: two triples (E, A, B), (E', A', B') are equivalent if and only if the following equality holds:

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & & \\ & P & \\ F_E & F_A & R \end{pmatrix}, \qquad (2)$$

where $Q, P \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $F_E, F_A \in M_{m\times n}(\mathbb{C})$.

Analyzing the definition of invariant subspace for standard systems, we extend this concept to singular systems. The main objective of the paper is to characterize invariant subspaces for singular linear systems, and in particular to study some special invariant subspaces as are the controllability subspaces.

2 Preliminaries

Some basics facts about group action, Grassmannian manifold and controllability character are as follows.

2.1 Equivalence relation as a Lie group action

The equivalence relation defined in (2), can be see as an action over $\mathcal{M}$ under a Lie group action.

Let us consider the following Lie group $\mathcal{G} = Gl(n; \mathbb{C}) \times Gl(n; \mathbb{C}) \times Gl(m; \mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$. The product $\star$ in $\mathcal{G}$ is given by

$$\begin{array}{l} (Q_1, P_1, R_1, F_{E_1}, F_{A_1}) \star (Q_2, P_2, R_2, F_{E_2}, F_{A_2}) = \\ (Q_2Q_1, P_1P_2, R_1R_2, F_{A_1}P_2 + R_1F_{E_2}, F_{E_1}P_2 + R_1F_{A_2}) \end{array} \qquad (3)$$

being $e = (I_n, I_n, I_m, 0, 0)$ its unit element.

The action of the Lie group $\mathcal{G}$ on $\mathcal{M}$ can be defined as

$$\begin{array}{c} \alpha : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M} \\ ((P, Q, R, F_E, F_A), (E, A, B)) \longrightarrow (E_1, A_1, B_1) \end{array} \quad (4)$$

with

$$\begin{array}{l} E_1 = QEP + QBF_E, \\ A_1 = QAP + QBF_A, \\ B_1 = QBR, \end{array}$$

give rise to the equivalence relation in $\mathcal{M}$ which will be called feedback-equivalence.

From now on, we will make use of the following notation: g=$(P, Q, R, F_E, F_E) \in \mathcal{G}$, and $x = (E, A, B) \in \mathcal{M}$.

Given a triple $x_0 = (E_0, A_0, B_0) \in \mathcal{M}$ we define the maps

$$\alpha_{x_0}(g) = \alpha(g, x_0). \quad (5)$$

The equivalence class of the triple x_0 with respect to the $\mathcal{G}$-action, called the $\mathcal{G}$-orbit of x_0, is the range of the function α_{x_0} and is denoted by

$$\mathcal{O}(x_0) = \operatorname{Im} \alpha_{x_0} = \{\alpha_{x_0}(g) \mid g \in \mathcal{G}\}. \quad (6)$$

Remark 2.1. *The maps α_{x_0} are clearly differentiable and $\mathcal{O}(x_0)$, is a smooth submanifold of $\mathcal{M}$.*

2.2 The Grassmannian manifold

The Grassmannian $G_V(k, n)$ is a space which parameterizes all linear subspaces of a n-dimensional vector space V of a given dimension k. We will denote by $G_{\mathbb{C}}(k, n)$ the Grassmann manifold formed by all k-subspaces in $\mathbb{C}^n$. The Grassmann manifolds are important in the study of the geometry and the topology, especially in the theory of fibre bundles (see [S. Helgason, (1978), S. Kobayashi, K. Nomizu, (1969)]).

Every k-dimensional subspace L is given by a linear transformation of $n \times (n-k)$ matrix G, more precisely. let $M^*_{n\times k}$ be the open subset of $M_{n\times k}(\mathbb{C})$ formed by the matrices G such that $\operatorname{rank} G = k,\ k \leq n$, $Gl(k; \mathbb{C})$ the linear group of invertible matrices in $M_k(\mathbb{C})$.

Let $G = (a^i_j)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}}$ be a matrix in $M_{n\times k}(\mathbb{C})$, denote by a_j the column vector $(a^1_j, \dots, a^n_j)^t$ of $\mathbb{C}^n$, $1 \leq j \leq k$, and $[a_1, \dots, a_k]$ the subspace generates by these vectors $a_1, \dots, a_k$.

Consider now the map $\pi : M^*_{n\times k} \longrightarrow G_{\mathbb{C}}(k, n)$ defined by

$$\pi(G) = [a_1, \dots, a_k]$$

Let L a subspace such that $\pi(G) = L$, in this case we say that G is a matrix representation of the subspace L.

The usual topology in $G_{\mathbb{C}}(k, n)$ is the final topology with respect π. Thus $G_{\mathbb{C}}(k, n)$ is a compact space.

In fact, $\pi(G) = [a_1, \dots, a_k]$ is a differentiable $Gl(k; \mathbb{C})$-principal fiber bundle.

In order to define local sections of π we introduce the following notations.

Let G be a matrix in $M_{n\times k}(\mathbb{C}^n)$ and $I = (i_1, \dots, i_k)$ a family of integers such that $1 \leq i_1 < \dots < i_k \leq n$, we denote by G^I the submatrix of G formed by the following rows $i_1, \dots, i_k$. We denote by I^0 the indices of the remaining rows, and for G^{I^0} the submatrix of G formed by these rows, in the same order.

We define the local sections as follows:

$$V_I = \{G \in M^*_{n\times k} \mid \det\ G^I \neq 0\},$$

$$\mathcal{U}_I = \pi(V_I) \subset G_{\mathbb{C}}(k, n),$$

$$\sigma_I : \mathcal{U}_I \longrightarrow V_I$$

verifying the conditions

$$\pi \circ \sigma_I = Id, \qquad (\sigma_I(L))^I = Id_k,$$

that is, fixing I, $\sigma_I(L)$ is the unique matrix representation G of L such that $G^I = Id_k$

If $G_1 \in V_I$ is a matrix representation of any $L \in \mathcal{U}_I$, then:

$$\sigma_I(L) = \sigma_I(\pi(G_1)) = G_1(G^I_1)^{-1}$$

From the above local section, is easy to find a family of local charts defining the differentiable structure of the grassmannian manifold.

For a fixed subspace $L_0 \in G_{\mathbb{C}}(k, n)$ we can consider the following map

$$\begin{array}{l} \pi' : Gl(n, \mathbb{C}) \longrightarrow G_{\mathbb{C}}(k, n) \\ \pi'(P) = PL_0 \end{array}$$

that is to say PL_0 is the subspace $\pi(PG)$ where G is any matrix representation of L_0.

This map defines a fiber bundle over $Gr_{\mathbb{C}}(k, n)$, whose local sections are defined as

$$\sigma'_I : \mathcal{U}_I \longrightarrow Gl(n, \mathbb{C}), \qquad \sigma'_I = s_I \circ \sigma_I$$

where $s_I : V_I \longrightarrow {\pi'}^{-1}(\mathcal{U}_I)$ is defined by

$$s_I(G) = (G \mid G_1), \quad G^I_1 = 0,\ G^{I^0}_1 = Id_{n-k}$$

denoting by $\mathcal{H}$ the fiber of L_0, that is, the subset of $Gl(n;\mathbb{C})$ formed by matrices that leave L_0 fixed:

$$\mathcal{H}{=}\pi'^{-1}(L_0) = \{P \in Gl(n;\mathbb{C}) \mid PL_0 = L_0\}$$

and if we consider a basis $\{v_1,\dots,v_n\}$ for $\mathbb{C}^n$ such that $\{v_1,\dots,v_k\}$ is a basis for L_0 then

$$\mathcal{H} = \left\{ \begin{pmatrix} P_1 & C \\ 0 & P_2 \end{pmatrix} \mid P_1 \in Gl(k;\mathbb{C}), P_2 \in Gl(n-k;\mathbb{C}) \right\}.$$

The fiber bundle $\pi' : Gl(n;\mathbb{C}) \longrightarrow G_{\mathbb{C}}(k,n)$ is in fact a $\mathcal{H}$-principal bundle with respect the actions on $GL(n;\mathbb{C})$ and $G_{\mathbb{C}}(k,n)\times\mathcal{H}$ respectively

$$\begin{aligned} &S\cdot P = SP,\\ &(L,P)P' = (L,(P')^{-1}P). \end{aligned}$$

2.3 Controllability

An important structural property for singular systems is the concept of controllability.

Definition 2.1. *The system (E,A,B) is called controllable if, for any $t_1 > 0$, $x(0) \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$, there exists a control input $u(t) \in C_p^{h-1}$ such that $x(t_1) = w$.*

This definition is a natural generalization of controllability definition for standard systems.

Proposition 2.1. *A system (E,A,B) is controllable if and only if*

$$\left.\begin{array}{l} \text{rank}\begin{pmatrix} E & B \end{pmatrix} = n, \\ \text{rank}\begin{pmatrix} sE-A & B \end{pmatrix} = n \;\forall s\in\mathbb{C} \end{array}\right\}.$$

First condition for controllability ensures the following corollary.

Corollary 2.1. *Let (E,A,B) be a controllable systems. Then (E,A,B) is standardizable.*

Remember that a system is called standardizable if it can be reduced to an standard one, that is to say, if and only if there existsF_E such that $E+BF_E$ is invertible.

For any triple of matrices $(E,A,B)\in\mathcal{M}$ we can associate the following matrices

$$C_r = \begin{pmatrix} E & & & & B & & & \\ A & E & & & & B & & \\ & \ddots & \ddots & & & & \ddots & \\ & & & E & & & & B \\ & & & A & & & & & B \end{pmatrix} \in M_{nr\times(n(r-1)+mr)}(\mathbb{C})$$

We have the following proposition.

Proposition 2.2 ([M$^{\underline{a}}$ I. García-Planas, (2009)]). *A triple (E,A,B) is controllable if and only if the matrix C_{n-1} has full rank.*

3 Invariant (E,A,B)-subspaces

In this section we try to generalize definition of invariant subspace under (A,B)-map, to the case of triples of matrices.

Let (E,A,B) be a standardizable triple in $\mathcal{M}$, so there exists feedback F_E such that $E+BF_E$ is invertible, and it permit us to obtain the following standard system $((E+BF_E)^{-1}A,(E+BFE)^{-1}B)$.

Applying definition (1) given in the introduction, a subspace $G \subset \mathbb{C}^n$ is invariant under $((E+BF_E)^{-1}A,(E+BF_E)^{-1}B)$ if and only if

$$(E+BF_E)^{-1}AG \subset G + \operatorname{Im}(E+BF_E)^{-1}B$$

and we can deduce the following proposition.

Proposition 3.1. *Let G be a vector subspace of $\mathbb{C}^n$ and (E,A,B) a standardizable triple of matrices. Then the following conditions are equivalent:*

a) For any feedback $F_E \in M_{m\times n}(\mathbb{C})$ standardizing the triple,

$$(E+BF_E)^{-1}AG \subset G + \operatorname{Im}(E+BF_E)^{-1}B$$

b) For any derivative feedback $F_E \in M_{m\times n}(\mathbb{C})$ standardizing the triple,

$$AG \subset (E+BF_E)G + \operatorname{Im}B$$

c) For any proportional feedback $F_A \in M_{m\times n}(\mathbb{C})$, $(A+BF_A)G \subset EG + \operatorname{Im}B$

d) $AG \subset EG + \operatorname{Im}B$.

Proof. $a) \Rightarrow b)$ Let v be a vector in G, then $Av \in AG$. Applying condition a), we have $(E+BF_E)^{-1}Av \in G + \operatorname{Im}(E+BF_E)^{-1}B$, that is to say $(E+BF_E)^{-1}Av = u + (E+BF_E)^{-1}Bw$ and $Av = (E+BF_E)u + Bw \in (E+BF_E)G + \operatorname{Im}B$.

$b) \Rightarrow a)$ Let x be a vector in $(E+BF_E)^{-1}AG$, then there exists $u\in G$ such that $x = (E+BF_E)^{-1}1AGu$, so $(E+BF_E)x = AGu$. Taking into account that $AGu \in (E+BF_E)G + \operatorname{Im}B$, there exists $v\in G$, $w\in\mathbb{C}^m$ such that $AGu = (E+BF_E)v + Bw$ then $(E+BF_E)x = (E+BF_E)v+Bw$ and $x = v+(E+BF_E)^{-1}Bw \in G + Im(E+BF_E)^{-}1B$

$b) \Rightarrow c)$ Let x be a vector in $(A+BF_A)G$, then $x = (A+BF_Au = Au + BF_Au \in AG + \operatorname{Im}B$, by b) $Au = (E+BF_E)v + Bw$. So, $x = Ev + B(w + F_Au + F_Ev) \in EG + \operatorname{Im}B$.

$c) \Rightarrow d)$ Let x be a vector in AG, $x = Au$ so $Au = Au + F_Au - F_Au = (A+BF_A)u$. Taking into account that $(A+BF_A)u \in (A+BF_E)G$, we have $(A+BF_A)u = (E+BF_E)v$. So, $Au = (E+BF_Ev - F_Au = Ev + B(F_Ev - F_Au) \in EG + \operatorname{Im}B$.

$d) \Rightarrow b)$ Let x be a vector in AG, then $x = EGu+Bw$ so $x = EGu + BF_Eu - BF_Eu + Bw = (EG + BF_E)u + B(-F_Eu + w)$.

This proposition permit us to generalize the definition of invariant subspace to the triples of matrices.

Definition 3.1. *A subspace $G \subset \mathbb{C}^n$ is invariant under (E, A, B) if and only if*

$$AG \subset EG + \operatorname{Im} B \tag{7}$$

We observe that, if $E = I_n$, this definition coincides with definition of (A, B)-invariant subspace.

We can construct invariant subspaces in the following manner. Let $H \subset \mathbb{C}^n$ be a subspace, we define

$$G_{k+1} = K \cap \{x \in \mathbb{C}^n \mid Ax \in EG_k + \operatorname{Im} B\},\ G_0 = K,$$

limit of recursion exists and we will denote by $G(H)$. This subspace is the supremal (E, A, B)-invariant subspace contained in H. Taking $H = \mathbb{C}^n$, we will write it as G^*.

Example 3.1. *Let (E, A, B) be a triple with $E = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $H = \{(x, y, z) \mid x = 0\}$,*

Computation of G_1:

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mu \\ \nu \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$[(x, y, 0)] \cap H = [(0, 1, 0)] = G_1$.

Computation of G_2:

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}$$

$[(x, y, 0)] \cap H = [(0, 1, 0)] = G_2 = G_1$. *Then $G = G_1$.*

Obviously $AG \subset EG + \operatorname{Im} B$.

Proposition 3.2. *Let (E, A, B) be a triple of matrices. A subspace $G \subset \mathbb{C}^n$ is invariant under (E, A, B) if and only if is invariant under $(E + BF_E, A + BF_A, B)$ for all feedbacks F_E, $F_A \in M_{m\times n}(\mathbb{C})$.*

Proof. Suppose that $AG \subset EG + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $Ax = Ey + Bw$ so, for any $F_E, F_A \in M_{m\times n}(\mathbb{C})$, we have

$$\begin{aligned} Ax + BF_Ax - BF_Ax &= Ey + BF_Ey - BF_Ey + Bw \\ (A + BF_A)x &= (E + BF_E)y + B(F_Ax - F_Ey + w). \end{aligned}$$

Consequently, for all $x \in G$, $(A + BF_A)G \subset (E + BF_E)G + \operatorname{Im} B$.

Reciprocally, suppose that $(A + BF_A)G \subset (E + BF_E)G + \operatorname{Im} B$, then for all $x \in G$, there exists $y \in G$, $v = Bw \in \operatorname{Im} B$ such that $(A + BF_A)x = (E + BF_E)y + Bw$ so, $Ax = Ey - BF_Ax + BF_Ey + Bw$ and $Ax = Ey + B(-F_Ax + F_Ey + w)$. Then, for all $x \in G$ we have $AG \subset EG + \operatorname{Im} B$.

Proposition 3.3. *Let $(E_1, A_1, B_1), (E_2, A_2, B_2) = (QE_1P + QB_1F_E; QA_1P + QB_1F_A; QB_1R)$ be two equivalent triples. Then $G \subset \mathbb{C}^n$ is an invariant subspace under (E_1, A_1, B_1) if and only if $P^{-1}G$ is invariant under (E_2, A_2, B_2).*

Proof. Suppose that $A_1G \subset E_1G + \operatorname{Im} B$. Then $A_2P^{-1}G = (QA_1P + QB_1F_{A_1})P^{-1}G = Q(A_1G + B_1F_{A_1}P^{-1}G) \subset Q(E_1G + \operatorname{Im} B_1) = Q((Q^{-1}E_2P^{-1} - Q^{-1}B_2R^{-1}F_EP^{-1})G + \operatorname{Im} Q^{-1}B_2R^{-1}) = Q(Q^{-1}(E_2P^{-1} - B_2R^{-1}F_EP^{-1})G + Q^{-1}\operatorname{Im} B_2R^{-1}) = QQ^1((E_2P^{-1} - B_2R^{-1}F_EP^{-1})G + \operatorname{Im}B_2R^{-1}) \subset (E_2 - B_2R^{-1}F_E)P^{-1}G + \operatorname{Im} B_2$ Now, it suffices to apply proposition 2.

So, if G is a (E, A, B)-invariant subspace of dimension k, then each subspace G_1 in $G_{\mathbb{C}}(k, n)$ is an invariant subspace for any triple (E_1, A_1, B_1) equivalent to (E, A, B).

Corollary 3.1. *Let $G \in G_{\mathbb{C}}(k, n)$ be a (E_1, A_1, B_1)-invariant subspace and $(E_2, A_2, B_2) = (QE_1P + QB_1F_E; QA_1P + QB_1F_A; QB_1R)$ an equivalent triple with $P \in \mathcal{H}$ the fiber of G in the fiber bundle $\pi' : Gl(n; \mathbb{C}) \longrightarrow G_{\mathbb{C}}(k, n)$. Then G is (E_2, A_2, B_2)-invariant.*

Example 3.2. *Let (E_1, A_1, B_1) be the triple in the example 3.1 and G the invariant subspace obtained in it. Let $(E_2, A_2, B_2) = (QE_1P + QB_1F_E; QA_1P + QB_1F_A; QB_1R)$ and equivalent triple with $P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $F_E = F_A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 \end{pmatrix}$, It is easy to observe that $P \in \mathcal{H}$.*

Clearly

$$\begin{pmatrix} 3 & 3 & 0 \\ 2 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2\lambda \\ 0 \end{pmatrix}.$$

Consequently, G is a (E_2, A_2, B_2)-invariant subspace.

Let $G \in G_{\mathbb{C}}(k, n)$ be a subspace and $\mathcal{H}$ its fiber in the fiber bundle $\pi' : Gl(n; \mathbb{C}) \longrightarrow G_{\mathbb{C}}(k, n)$. Consider now, the subgroup $\mathcal{G}_1 \subset \mathcal{G}$ consisting in elements $g = (P, Q, R, F_E, F_A) \in \mathcal{G}$ with $P \in \mathcal{H}$ acting over $\mathcal{M}$ in the form

$$\begin{aligned} \alpha_{|\mathcal{G}_1} : \mathcal{G}_1 \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (g, (E, A, B)) &\longrightarrow \alpha(g, (E, A, B)) \end{aligned} \tag{8}$$

we have the following theorem.

Theorem 3.1. *Let G be a (E, A, B)-invariant subspace. Then G is an invariant subspace of all triples in its orbit $\mathcal{O}_1$ under the equivalence relation defined in* (8).

Remark 3.1. *In the particular case where $G = [e_1, \ldots, e_k]$ we have that $\mathcal{G}_1$ is formed for the elements $g = (P, Q, R, F_E, F_A) \in \mathcal{G}$ with $P = \begin{pmatrix} P_1 & C \\ 0 & P_2 \end{pmatrix}$ $P_1 \in Gl(k; \mathbb{C})$, $P_2 \in Gl(n - k; \mathbb{C})$.*

4 Controllability subspaces

In this section we are going to study a particular case of invariant subspaces. First of all we observe the following result.

Proposition 4.1. *Let (I_n, A, B) be a standard triple. Then*

$$G = [B, AB, \dots, A^{n-1}B]$$

is a (I_n, A, B)-invariant subspace.

Proof.

$$AG = A[B, AB, \dots, A^{n-1}B] = [AB, A^2B, \dots, A^nB]$$

Now, it suffices to apply the Cayley-Hamilton theorem.

Theorem 4.1. *Let*

$$C_r = \begin{pmatrix} E & & & B & & & \\ A & E & & & B & & \\ & \ddots & \ddots & & & \ddots & \\ & & E & & & B & \\ & & A & & & & B \end{pmatrix} \in M_{nr\times(n(r-1)+mr)}(\mathbb{C})$$

be the r-controllability matrix. Suppose r being the least such that $\operatorname{rank} C_r < (n(r-1)+mr)$*, and let* $\begin{pmatrix} v_1 \dots v_r & w_1 \dots w_{r+1}\end{pmatrix} \in \operatorname{Ker} C_r$ *(v_i are vectors in $\mathbb{C}^n$ and w_i vectors in $\mathbb{C}^m$). Then $G = [v_1, \dots, v_r]$ is a (E, A, B)-invariant subspace.*

Proof. We consider $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_{r-1}v_{r-1} + \lambda_r v_r$, $Av = \lambda_1 Av_1 + \lambda_2 Av_2 + \dots + \lambda_{r-1}Av_{r-1} + \lambda_r Av_r = \lambda_1(-Ev_2 - Bw_2) + \lambda_2(-Ev_3 - Bw_3) + \dots + \lambda_{r-1}(-Ev_r - Bw_r) - \lambda_r Bw_{r+1} = E(\lambda_1 v_2 - \lambda_2 v_3 - \dots - \lambda_{r-1}v_r) + B(-\lambda_1 w_2 - \lambda_2 w_3 - \dots - \lambda_{r-1}w_r - \lambda_r w_{r+1}) \in EG + \operatorname{Im} B$.

Definition 4.1. *The space sum of all spaces G in theorem before is a invariant subspace that we will call controllability subspace and we will denote it by $\mathcal{C}(E, A, B)$.*

Notice that $\mathcal{C}(E, A, B)$ is the set of states in which the system is controllable.

Corollary 4.1. *Let (E, A, B) be a triple with $E = I_n$. In this case the invariant subspace G obtained in the above theorem, coincides with the controllability (A, B)-invariant subspaces $[B, AB, \dots, A_{r-1}B]$.*

Proof. Making block-row elemental transformations to the matrix C_r we obtain the equivalent matrix

$$\begin{pmatrix} I_n & & & B & & & \\ 0 & I_n & & -AB & B & & \\ & \ddots & \ddots & & & \ddots & \\ & & I_n & (-1)^{r-2}A_{r-2}B & & -AB & B \\ & & 0 & (-1)^{r-1}A^{r-1}B & & & -AB\ B \end{pmatrix}.$$

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