CONSENSUS BETWEEN NONLINEARLY COUPLED DELAYED AGENTS.

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Abstract
We address the problem of synchronization (consensus) in multi-agent networks with switching topology and nonlinear couplings. The agents are assumed to obey linear stationary delay equations without strictly unstable poles, however, they may be heterogeneous and have arbitrary order. The couplings may be uncertain, assumed only to satisfy conventional sector inequalities. We offer easily verifiable synchronization criteria, based on the Popov method from absolute stability theory and close in spirit to circle and Popov stability criteria.

Key words
Complex networks, nonlinear systems, synchronization, networked systems

1 Introduction.

The phenomenon of synchronization or *consensus* among subsystems of a complex system, achieved via local interactions between them, lies at the heart of numerous natural phenomena and engineering solutions. Examples include, but are not limited to, flocking, swarming, and other forms or regular motion of biological or technical systems [Olfati-Saber *et al.*, 2007; Ren and Beard, 2008; Ren and Cao, 2011], synchronization in oscillator ensembles etc. This abundance of applications gave rise to enormous interest in consensus algorithms from various research communities.

The most investigated and widely used consensus protocols are those with linear couplings, see e.g. [Olfati-Saber *et al.*, 2007; Ren and Beard, 2008; Ren and Cao, 2011] and references therein. Stability of first and second order consensus algorithms is typically proven using either theory of positive matrices or various Lyapunov and LMI techniques, in some cases, high-order agents may be reduced to single integrators via decomposition of Laplacian matrix [Olfati-Saber *et al.*, 2007] or using special dynamic controllers [Scardovi and Sepulchre, 2009; Wieland *et al.*, 2011].

In the same time, many applications involve synchronization via nonlinear couplings. For example, this holds for networks of various oscillators, e.g., Kuramoto networks [Chopra and M.W.Spong, 2009], where agents are typically coupled by means of periodic functions. Nonlinear couplings naturally arise in motion coordination under range-restricted communication in order to maintain the group connectivity [Lin *et al.*, 2007*a*; Tanner *et al.*, 2007; Su *et al.*, 2009]. In a practical setting, linear algorithms may acquire nonlinearities because of distortions caused by saturations, imprecise measurements, analog-to-digital transformations, quantization effects etc. The mentioned applications motivated recent interest to nonlinear consensus theory which, however, was mainly focused on low-order agents such as first and second order integrators [Moreau, 2005; Lin *et al.*, 2007*b*; Abdessameud and Tayebi, 2010; Ren and Beard, 2008; Ren and Cao, 2011] and passive agents [Chopra and Spong, 2006; Arcak, 2007]. Those restrictions were mainly caused by using special types of Lyapunov functions (e.g. the diameter of the convex hull spanned by the agents states or total ”energy” of the system) that are applicable only for agents with special dynamics.

Another class of nonlinear networks was considered in recent papers [Proskurnikov, 2013*b*; Proskurnikov, 2014*a*] dealing with identical agents of arbitrary order, coupled by nonlinear and possibly uncertain mappings, satisfying however conventional *quadratic constraints* (e.g. sector inequalities with known slopes) [Gelig *et al.*, 2004] and also a symmetry condition, which resembles the Newton Third Law. The consensus criterion the mentioned papers employs the transfer function the agent and a quadratic form which defines the constraint, but not the coupling themselves. This criterion, close in spirit to the circle stability criterion for Lurie systems, in fact ensures that consensus is robust in the mentioned class of uncertain couplings. In the present paper we assume the same structure of couplings that in [Proskurnikov, 2013*b*; Proskurnikov, 2014*a*], however, the agents may be heterogeneous. Also the agents are modeled by delay equations (where delays may be both discrete and distributed), which makes techniques from [Proskurnikov, 2013*b*; Proskurnikov, 2014*a*] inapplicable. An important example of a network with delayed agents is given by a *microscopic traffic flow models* [Michiels *et al.*, 2009*b*; Sipahi *et al.*, 2007]. The topology of the heterogeneous network is assumed to be uniformly connected [Lin *et al.*, 2007*b*; Scardovi and Sepulchre, 2009], and the agents are assumed to have no exponentially unstable poles, which is a commonly adopted assumption in consensus problems [Seo *et al.*, 2009; Scardovi and Sepulchre, 2009]. For networks with exponentially unstable nodes we obtain a consensus criterion under more severe restrictions: the agents are to be identical and the topology should be constantly connected.

2 Preliminaries

Throughout the paper symbol $m:n$ (where $m \le n$ are naturals) denotes the set $\{m, m+1, \dots, n\}$.

Let $\mathbb{R}_+ := [0; +\infty)$ and $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re\,\lambda \ge 0\}$.

A *weighted graph* is a triple $G = (V, E, \mathscr{A})$ of two finite set V (the set of nodes), $E \subseteq V \times V$ (the set of arcs) and a weighting map $\mathscr{A} : V \times V \to [0; +\infty)$ where $\mathscr{A}(v, v') > 0$ if and only if $(v, v') \in E$. Any graph (V, E) may be considered as weighted by endowing it with a trivial weight $\mathscr{A}(v, v') := 1$ if $(v, v') \in E$ and $\mathscr{A}(v, v') := 0$ otherwise. Throughout the paper we deal with *undirected* graphs which means that $(v, v') \in E \Leftrightarrow (v', v) \in E$ and $\mathscr{A}(v, v') = \mathscr{A}(v', v)$. Any sequence of nodes $v_1, v_2, \dots, v_k$ with $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \dots, k-1$ is called *a path* between v_1 and v_k. An undirected graph is *connected* if a path between any two nodes exists. Throughout the paper $\mathbb{G}_N$ stands for the class of all undirected graphs $G = (V_N, E, \mathscr{A})$ with the node set $1:N$ and set of arcs E containing no self-loops, e.g. $(v, v) \notin E\,\forall v \in 1:N$. For such a graph we identify the mapping $\mathscr{A}$ with the graph adjacency matrix $\mathscr{A}(G) = (a_{jk}(G))$ where $a_{jk}(G) := \mathscr{A}(j, k)$. The number $d_j(G) := \sum_{k=1}^N a_{jk}(G)$ is referred to as the *degree* of the j-th node.

A time-variant graph $G(t) \in \mathbb{G}_N$ with locally summable weights $a_{jk}(t) := a_{jk}(G(t))$ is said to be *uniformly connected* if numbers $\varepsilon > 0, T > 0$ such that the graph $(V_N, \mathscr{E}_t)$ with the set of nodes $V_N = \{1, \dots, N\}$ and that of arcs $\mathscr{E}_t = \left\{(j,k) : \int_t^{t+T} a_{jk}(s)ds > \varepsilon\right\}$ is connected for all $t \ge 0$.

For a measurable space (X, Σ), a matrix-valued measure $M = (\mu_{jk}) \colon \Sigma \to \mathbb{R}^{m \times n}$ and a map $f \colon X \to \mathbb{R}^n$ let

$$\mathbb{R}^m \ni \int_X M(dx) f(x) := \left(\sum_{k=1}^n \int_X f_k(x) \mu_{jk}(dx) \right)_{j=1}^m .$$

The measure M is *finite* if $|\mu_{jk}|(X) < \infty$ for any j,k (where $|\mu|$ stands for the total variation of a signed measure μ). Given a finite measure M on Borel σ-algebra of $\mathbb{R}_+$, we denote with $\hat{M}$ its Laplace transform $\hat{M}(\lambda) := \int_0^\infty e^{-\lambda t} M(dt)$ which is defined whenever $Re\,\lambda \ge 0$.

3 Problem formulation.

Throughout the paper we deal with a team of $N \ge 2$ agents, indexed 1 through N and governed by a time-delay models

$$\begin{aligned} \dot{x}_j(t) &= \int_0^\infty \left[A_j(d\theta)x_j(t-\theta) + B_j(d\theta)u_j(t-\theta)\right] \\ y_j(t) &= \int_0^\infty C_j(d\theta)x_j(t-\theta), \quad j \in 1:N. \end{aligned} \tag{1}$$

Here A_j, B_j, C_j are finite matrix-valued measures on $\mathbb{R}_+ := [0; +\infty)$ and $x_j(t) \in \mathbb{R}^{n_j}$, $u_j(t) \in \mathbb{R}^m$, $y_j(t) \in \mathbb{R}^p$ stand for the state, control and output of the j-th agent respectively. We will always assume that initial functions $x_j(t), u_j(t)$ $(t < 0)$ are bounded thus all of integrals in (1) exist.

The controls are affected by interaction between the agents, via communication or otherwise. The interaction topology is time-varying and at time $t \ge 0$ is described by a weighted time-variant graph $G(t) \in \mathbb{G}_N$: the output $y_k(t)$ of k-th agent exerts influence on j-th one the if and only if $(k, j) \in E(t)$, the weight $a_{kj}(G(t))$ being a coupling gain.

Specifically, we examine a distributed control law

$$u_j(t) = \sum_{k=0}^N a_{jk}(t) \varphi_{jk}(y_k(t) - y_j(t)). \tag{2}$$

The maps $\varphi_{jk} \colon \mathbb{R}^p \to \mathbb{R}^m$, expressing the interaction law, are referred to as *couplings*. The aim of the paper is to disclose conditions under which the protocol (2) establishes consensus among the agents in the following sense.

Definition 1. *The protocol* (2) *establishes* the output consensus *if the following claim hold for all initial data:*

$$\lim_{t \to +\infty} |y_j(t) - y_k(t)| = 0 \quad \forall k, j; \tag{3}$$

To simplify matters, in the main body of the paper we consider SISO agents ($m = p = 1$), postponing more difficult MIMO case for Section 7

4 Agents without strictly unstable poles: consensus criteria

Throughout this section we deal with scalar agents (1): $m = p = 1$. We assume each of the agents to be stabilizable by means of arbitrarily weak static feedback.

Assumption 1. *For sufficiently small $\varepsilon > 0$ the feedback $u_j = -\varepsilon y_j$ exponentially stabilizes the agent* (1)*:*

$$\det(\lambda I_{n_j} - \hat{A}_j(\lambda) - \varepsilon \hat{B}_j(\lambda)\hat{C}_j(\lambda)) \ne 0\,\forall \lambda \in \mathbb{C}_+. \tag{4}$$

Assumption (1) implies that the agents have no exponentially unstable poles; for undelayed linear agents the latter condition is actually very close to Assumption 1 [Seo *et al.*, 2009]. The absence of unstable poles is considered to be a natural assumption in consensus problems [Scardovi and Sepulchre, 2009; Wieland *et al.*, 2011], which makes it possible to achieve consensus by weak coupling in the face of unpredictably changing network topology. If the agents are exponentially unstable, the consensus between them is impossible unless the interaction is sufficiently "strong", and to shape this reasonable restriction into conditions on the network topology and couplings is a problem that is far from being explored. Some sufficient conditions for identical exponentially unstable agents will be given in the next section.

Our main assumptions about the protocol (2) are the uniform connectivity of the network, symmetry condition and sector bounds for the couplings. We start with the assumption about the interaction graph.

Assumption 2. *The weighted graph $G(t) \in \mathbb{G}_N$ is uniformly connected (in particular, the weights $a_{jk}(G(t))$ are locally summable functions).*

Maintaining the network connectivity is clearly necessary to prevent the agents from dissemination into separate clusters that do not interact and thus cannot be synchronized. The uniform connectivity property is considered to be one of the weakest conditions under which consensus may be proved [Scardovi and Sepulchre, 2009] (becoming almost necessary for first-order systems [Lin *et al.*, 2007*b*]).

Assumption 3. *For any $j \neq k$ and y the following symmetry condition holds: $\varphi_{jk}(t,y) = -\varphi_{kj}(t,-y)$*

The latter assumption resembles the Newtons Third Law for couplings (since $\varphi_{jk}(t, y_k - y_j) = -\varphi_{kj}(t, y_j - y_k)$); in some applications, e.g. in oscillator networks [Strogatz, 2000] Assumption 3 holds due to exactly this law.

In the present paper we are concerned with the situation where the full information about couplings φ_{jk} may be unavailable, and the knowledge about them comes to a conventional *sector inequality* [Gelig *et al.*, 2004]. Specifically for a known constant $\gamma > 0$, they belong to the set $\mathfrak{S}(\gamma)$ of continuous maps $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) \equiv 0$ and for any $\sigma \neq 0$ one has

$$0 < \frac{\varphi(\sigma)}{\sigma} < \gamma; \tag{5}$$

The inequalities (9) mean that the graph of the function $\xi = \varphi(\sigma)$ lies strictly in the sector between the lines $\xi = 0$ and $\xi = \gamma\sigma$ everywhere except the origin, which explains the term "sector inequalities". The first pair of inequalities in (9) prohibit the decaying of the coupling to zero and restrict the "coupling gain"; and the second condition guarantees that the latter properties do not degrade as the argument grows.

Now we are in position to formulate our main results. The first one gives a frequency-domain consensus criterion for consensus under switching topology. To proceed, we introduce the transfer function of the j-th agent

$$W_j(\lambda) = \hat{C}_j(\lambda)(\lambda - \hat{A}_j(\lambda))^{-1}\hat{B}_j(\lambda)$$

We denote with $D_j := \sup_{t \geq 0} d_j(t)$ the maximal degree of the j-th node, $D_j > 0 \forall j$ due to Assumption 2.

Theorem 1. *Suppose the agents* (1) *and protocol* (2) *to satisfy Assumptions 1,2 and 3 and $\varphi_{jk} \in \mathfrak{S}(\gamma)$ for some $\gamma > 0$. Assume that for any $j \in 1:N$ and $\omega \in \mathbb{R}$ such that* $\det(\imath\omega - \hat{A}_j(\imath\omega)) \neq 0$ *the inequality holds:*

$$Re\, W_j(\imath\omega) + (2\gamma D_j)^{-1} > 0. \tag{6}$$

Then the protocol (2) *establishes the output consensus.*

The second result deals with time-invariant network topology, in which case the frequency-domain inequality (6) may be slackened.

Theorem 2. *Suppose that the graph $G(t) \equiv const$ is connected, the agents* (1) *and protocol* (2) *satisfy Assumptions 1 and 3 with $\varphi_{jk} \in \mathfrak{S}(\gamma)$. Assume that there exists $\theta \in \mathbb{R}$ such that for any $j \in 1:N$ and any $\omega \in \mathbb{R}$ with* $\det(\imath\omega - \hat{A}_j(\imath\omega)) \neq 0$ *the inequality holds:*

$$Re[W_j(\imath\omega) + \theta\imath\omega W_j(\imath\omega)] + (2\gamma D_j)^{-1} \geq 0. \tag{7}$$

Then the protocol (2) *establishes the output consensus.*

It can be easily noticed that conditions (6), (7) (under technical Assumption 1) coincide with the *circle* and *Popov* criteria [Popov, 1973; Gelig *et al.*, 2004; Yakubovich, 2000; Yakubovich, 2002] for stability of the Lurie systems family

$$\begin{aligned} \dot{x}_j(t) &= \int_0^\infty \left[A_j(d\theta)x_j(t-\theta) + B_j(d\theta)u_j(t-\theta)\right] \\ y_j(t) &= \int_0^\infty C_j(d\theta)x_j(t-\theta),\, u_j = -\varphi(y_j). \end{aligned} \tag{8}$$

where $\varphi \in \mathfrak{S}(2\gamma D_j)$ may arbitrary uncertain nonlinearity. As was shown in [Proskurnikov, 2013*b*, Appendix A], for identical agents the result of Theorem 1 is a direct extension of the circle criterion for the Lurie system with one nonlinearity, and the same parallel may be drawn for the Popov criterion and Theorem 2.

The proofs of Theorems 1 and 2 will be published in extended version of this paper [Proskurnikov, 2014*b*] and are available upon request. They are based on the criteria of semiboundedness for quadratic functionals on Hardy spaces, see [Arov and Yakubovich, 1981, Theorem 2],[Likhtarnikov and Yakubovich, 1983, Theorem 2] and are close in spirit to proofs from [Proskurnikov, 2012, Appendix A] and [Proskurnikov, 2011].

5 Exponentially unstable agents.

In this section we discuss the situation where Assumption 1 may be violated, for instance, the agents (1) are exponentially unstable. Synchronization in such a network requires sufficiently strong couplings hence one can not expect robust consensus in the class of nonlinear couplings $\mathfrak{S}(\gamma)$ (e.g. the couplings $\varphi_{jk}(x) = \varepsilon x$ with $\varepsilon > 0$ small certainly do not provide the consensus). Unlike Theorems 1 and 2, the consensus criterion below which addresses this case is applicable only for identical agents, and its extension to heterogeneous agents remains an open issue.

Throughout this section the agents are identical so that $A_j \equiv A, B_j \equiv B, C_j \equiv C$. We introduce the class of sector nonlinearities $S[\alpha;\beta]$ where $\infty \geq \beta > \alpha > 0$ which consists of all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) \equiv 0$ and for any $\sigma \neq 0$ one has

$$\alpha < \frac{\varphi(\sigma)}{\sigma} < \beta. \tag{9}$$

The inequalities (9) mean that the graph of the function $\xi = \varphi(\sigma)$ lies strictly in the sector between the lines $\xi = \alpha\sigma$ and $\xi = \beta\sigma$. Thus $\mathfrak{S}(\gamma) = S[0;\gamma]$.

Our goal is disclose conditions under which any protocol (2) with couplings $\varphi_{jk} \in S[\alpha;\beta]$ (satisfying Assumption 3) establishes the output consensus, in this sense the consensus is *robust* against the uncertainty in couplings. Since the case of time-variant graph was examined in [Proskurnikov, 2013*b*], we bound ourselves with the case of fixed topology $G(t) \equiv G$. Evidently the robustness in question implies that the consensus is reached with linear couplings $\varphi_{jk}(y) := \mu y$ where $\mu \in (\alpha;\beta)$, or, equivalently [Fax and Murray, 2004], the feedback $u_j = -\mu\lambda y_j$ stabilizes the agent (1) whenever $\lambda \neq 0$ is an eigenvalue of the graph Laplacian

$$L(G) := \begin{bmatrix} \sum_{k=1}^N a_{1k} & -a_{12} & \dots & -a_{1N} \\ -a_{21} & \sum_{k=1}^N a_{2k} & \dots & -a_{2N} \\ \vdots & & \ddots & \\ \vdots - a_{N1} & -a_{N2} & \dots & \sum_{k=1}^N a_{Nk} \end{bmatrix} \tag{10}$$

In the following, we will assume formally weaker condition to be valid:

Assumption 4. *There exists $\mu \in (\alpha;\beta)$ such that $\det(zI_n - \hat{A}(z) - \mu\lambda_k(G)\hat{B}(z)\hat{C}(z)) \neq 0 \forall z \in \mathbb{C}_+, k = 2,\dots,N$. Here $0 = \lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_N(G)$ are eigenvalues of the matrix $L(G) = L(G)^T$ in the increasing order.*

We also introduce two constants

$$\delta := \frac{\alpha\beta}{\alpha+\beta}, \gamma := \frac{1}{\alpha+\beta}. \tag{11}$$

The following theorem gives a sufficient condition for consensus among identical agents.

Theorem 3. *Suppose that the agents are identical, $\varphi_{jk} \in \mathfrak{S}[\alpha;\beta]$ with $\alpha > 0$, Assumptions 3,4 hold, the graph $G \in \mathbb{G}_N$ is constant connected and there exists number $\theta \in \mathbb{R}$ such that for any $\omega \in \mathbb{R}$ with $\det(\imath\omega - \hat{A}(\imath\omega)) \neq 0$ the inequality holds:*

$$\begin{aligned} Re[W(\imath\omega) + \theta\imath\omega W(\imath\omega)] + \delta\lambda_2(G)|W(\imath\omega)|^2 + \\ +(2\gamma D_{max})^{-1} \geq 0. \end{aligned} \tag{12}$$

where $D_{max} := \max d_j$ stands for the maximal node degree. Then protocol (2) *establishes output consensus.*

The proof of Theorem 3 will be published in extended version of this paper [Proskurnikov, 2014*b*] and is available upon request.

Remark 1. *Unlike the inequalities* (7) *that are fully "decentralized", employing only local properties of the agents, the inequality* (12) *involves some global information about the network, namely, the algebraic connectivity $\lambda_2(G)$ and the maximal degree D_{max}. Notice that the inequality remains sufficient for consensus, replacing $\lambda_2(G)$ with its lower and D_{max} with its upper bound (e.g. $D_{max} \leq (N-1)\max_{j,k} w_{jk}$) respectively. A lot of non-conservative estimates for λ_2 coming from the algebraic graph theory may be put in use [Fiedler, 1973; Merris, 1994] in the case when its precise computation is troublesome.*

6 Illustrative examples

We now illustrate the potential of Theorems 1, 2 and 3 by considering several special types of agents.

6.1 Passive SISO agents

A conventional undelayed SISO model

$$\dot{x}_j(t) = \mathscr{A}_j x(t) + \mathscr{B}_j u_j(t), y_j(t) = \mathscr{C}_j x_j(t), \tag{13}$$

where $\mathscr{A}_j, \mathscr{B}_j, \mathscr{C}_j$ are constant matrices, is an important special case of the plant (1) (with $A = \delta\mathscr{A}_j, B = \delta\mathscr{B}_j, C = \delta\mathscr{C}_j$ and δ stands for the Dirac δ-measure). It is easily shown that Assumption 1 combined with the positive realness

$$Re W_j(i\omega) \geq 0 \tag{14}$$

is equivalent to *passivity* of the plant (13) [Chopra and Spong, 2006; Khalil, 1996]. Consensus and synchronization of passive systems earned a substantial interest recently; see e.g., [Chopra and Spong, 2006; Arcak, 2007] for a survey. However dealing with nonlinear agents, most of the concerned results assume time-invariant or switching with nonzero dwell time constantly connected interaction topology. The following proposition extends the result of [Chopra and Spong, 2006, Theorem 7] to the case of topology which may have zero dwell-time and lose its connectivity at some time intervals, being only uniformly connected.

Corollary 1. *Suppose that agents* (13) *satisfy* (14) *and Assumptions 1,2,3 holds. If $\varphi_{jk} \in \mathfrak{S}(\gamma)$ for some $\gamma > 0$, then the protocol* (2) *establishes output consensus.*

Proof is immediate from Theorem 1 since (14) implies (6) for any $\gamma > 0$.

6.2 Delayed first-order agents

Despite the consensus problems for first-order agents have been deeply investigated during recent decade, the effects caused by input delays seem to be far from fully explored. Unlike purely communication delays that affect only data from neighbors and do not violate the consensus, provided they remain bounded [U.Muñz *et al.*, 2011], large self-delay may cause instability of the system, which gives rise to the problem of estimating maximal tolerable delay margin. The most significant progress in this area was achieved by using the Lyapunov-Krasovskii method, which comes to non-trivial and high-dimensional LMIs [Sun and Wang, 2009; Lin and Jia, 2011] and various frequency-domain techniques [Tian and Liu, 2008; Bliman and Ferrari-Trecate, 2008; U.Muñz *et al.*, 2010; Lestas and Vinnicombe, 2010]. However, all of those results address the case of linear networks, while the effects of self-actuation delays in nonlinear consensus protocols remain almost unexplored. In the case when this delays are equal the results from the recent papers [Proskurnikov, 2012; Proskurnikov, 2013*a*] may be put in use which assume the delays to be incorporated in the couplings and satisfy some symmetry conditions. However, the latter symmetry is broken if the input delays are heterogeneous, which makes the results from [Proskurnikov, 2012; Proskurnikov, 2013*a*] inapplicable. Most of results existing in the literature deal with discrete delays, however, a number of applications (e.g. microscopic traffic flow models with delayed driver reaction [Sipahi *et al.*, 2007]) actually involve distributed delay.

In this subsection we consider a team of delayed first-order integrators

$$\dot{x}_j(t) = \int_0^\infty u_j(t-\theta)\mu_j(d\theta),\, j \in 1:N, \qquad (15)$$

where μ_j is a positive finite measure on $\mathbb{R}_+$ such that the "expectation" of the delay $E_j := \int_0^\infty \theta\mu_j(d\theta) < \infty$ (as usual, $|\mu|$ stands for the total variation of the signed measure μ). The following theorem gives a sufficient condition for consensus in such a network.

Theorem 4. *Suppose that Assumptions 2 and 3 hold, $\varphi_{jk} \in \mathfrak{S}(\gamma)$ and $2\gamma D_j E_j < 1\,\forall j$. Then the protocol* (2) *establishes output consensus between agents* (15).

Proof. It is easy to show that the agents (15) satisfy Assumption 1. Indeed, $\lambda + \varepsilon\hat{\mu}(\lambda) = \varepsilon f_\varepsilon(\lambda)$, where $\lambda' := \varepsilon^{-1}\lambda$ and $f_\varepsilon(\lambda') := \lambda' + \hat{\mu}_j(\varepsilon\lambda')$. We have $\hat{\mu}_j(\varepsilon\lambda') := \int_0^\infty e^{-\varepsilon\lambda' t}\mu_j(dt) \to \mu_j(\mathbb{R}_+) > 0$ as $\varepsilon \downarrow 0$, and the convergence is uniform over the ball $\{\lambda' : |\lambda'| \leq \mu(\mathbb{R}_+)\}$ to which all of the zeros of $f_\varepsilon(\lambda')$ with $Re\,\lambda' \geq 0$ evidently belong. Thus taking $\varepsilon > 0$ sufficiently small, one has $f_\varepsilon(\lambda') \neq 0$ whenever $Re\,\lambda' \geq 0$, i.e. $\lambda + \varepsilon\hat{\mu}(\lambda) \neq 0$ when $Re\,\lambda \geq 0$. Notice that

$$Re\,W_j(i\omega) = -\int_0^\infty \frac{\sin\omega\theta}{\omega}\mu_j(d\theta) \geq -\int_0^\infty \theta\mu_j(d\theta) = -E_j,$$

hence $E_j\gamma D_j < 1$ implies the inequality (6).

6.3 Application to microscopic traffic flow models

Investigation of the vehicular traffic dynamics as a result of interactions between vehicles, drivers and road infrastructure has been recognized as a problem of ultimate importance, since traffic accidents and congestions lead to considerable economic and ecological losses. A *microscopic* traffic flow models, closely related to models of self-propelled particle ensembles [Helbing, 2001], are commonly adopted as rather simple but instructive tool for multi-vehicle traffic investigation. Those models consider the traffic system as a chain of individual vehicles, whose drivers apply some strategy, aiming typically at maintaining the uniform flow with constant vehicle velocities. Since pioneering works on microscopic traffic flow models [Chandler *et al.*, 1958], the *delay* in drivers reaction has been recognized as a factor, playing crucial role in the overall flow dynamics, see e.g. [Sipahi *et al.*, 2007; Michiels *et al.*, 2009*a*] and references therein. Possibly the simplest model taking the reaction delay into account [Chandler *et al.*, 1958; Helbing, 2001; Sipahi *et al.*, 2007] deals with a line of N vehicles indexed 1 through N, following along a straight single lane road. The first vehicle travels with constant speed, and any other vehicle aligns its velocity with one in front:

$$\dot{v}_j(t) = K(v_{j-1}(t-\tau) - v_j(t-\tau)),$$

where $v_j(t)$ is the velocity of the j-th vehicle, τ is the delay in the driver's action and K stands for the driver "sensitivity" to the change of relative velocity of the vehicle in front of him. This model represents the dynamics of the velocity perturbations around constant velocity solutions, and the key problem is stability of such solutions. In analogous problem for circular road the first vehicle is considered to be a follower of the last one, moreover, each driver is able to trace not only one but several leading and following vehicles (up to the whole formation in ideal situation), as was suggested in [Michiels *et al.*, 2009*a*]. The latter paper considers the network with fixed topology and fixed homogeneous delay, which, however may be not only discrete but also distributed which, as discussed in [Sipahi *et al.*, 2007], allows to take into account effects of human memory and different behavior of individual drivers.

Below we consider a microscopic traffic flow model for circular road, analogous to that from [Michiels *et al.*, 2009*a*]. The delays may be distributed and, unlike the mentioned paper, heterogeneous (the reaction times of individual drivers may differ). Also the driver reaction may be nonlinear function of relative velocities of the neighboring vehicles. We do not assume the topology to be fixed (some drivers may lose sight of some of their predecessors and followers for a while due to e.g. the relief specifics or weather conditions), also the drivers action are saturated, since in practice rapid change of the velocity is impossible. Precisely, consider a circular formation of $N \geq 2$ vehicles numbered 1 through N. We denote with $\oplus$ addition modulus N, so that $N \oplus 1 = 1$. Suppose that $v_j(t)$ is a velocity of the j-th vehicle, and its driver controls the acceleration $u_j(t)$ but his/her reaction is delayed, so that the vehicle speed is governed by equation analogous to (15):

$$\dot{v}_j = \int_0^\infty u_j(t-\theta)\mu_j(d\theta),\ j \in 1:N.$$

We assume that the control $u_j(t)$ is affected by the velocities of $p \leq N-1$ leading and p following vehicles as follows

$$u_j(t) = \sum_{m=-p}^{p} a^m \varphi^m(v_{j\oplus m}(t) - v_j(t)), \qquad (16)$$

We suppose only that reactions to the m-th leader and the m-th follower are the same: $a^m = a^{-m}$ and $\varphi^m = \varphi^{-m}$ for any $m = 1,2,\dots,p$. Applying Theorem 4, one obtains the following.

Theorem 5. *Let* $a^m = a^{-m} \geq 0$ *and let* $\varphi^m(v) = \varphi^{-m}(v)$ *be odd functions from* $\mathfrak{S}(\gamma)$*. Suppose that* $a^1 > 0$*. If the "expected" delays* $E_j := \int_0^\infty \theta\mu_j(d\theta)$ *satisfy*

$$E_j \leq \left(2\gamma \sum_{m=-p}^{p} a^m\right)^{-1},$$

the protocol (16) *achieves the velocity consensus, e.g.* $v_j(t) - v_k(t) \to 0$ *as* $t \to \infty$ *for any* j,k.

Introduce the coupling weights a_{jk} and corresponding couplings φ_{jk} by $a_{j,j\oplus m} := a^m$, $\varphi_{j,j\oplus m}(v) = \varphi^m(v)$ for $m = -p,\dots,p$ and $a_{jk} := 0, \varphi_{jk} := 0$ for any other pair j,k. It is easy to show that Assumptions 2,3 hold. Now the proof of Theorem 5 is immediate from Theorem 4 since $D_j = \sum_{m=-p}^{p} a^m$ for any j.□

Theorem 5 shows that under sufficiently small reaction delays the traffic flow becomes asymptotically "uniform": the vehicles travel with equal velocities, and traffic jams are impossible.

6.4 Consensus among unstable first-order delayed agents

In this subsection we demonstrate the use of Theorem 3 for unstable agents. We consider a team of identical agents

$$\dot{y}_j - ay_j = u_j(t-\tau) \in \mathbb{R}, \qquad (17)$$

where $a > 0$ and $\tau \geq 0$. We are interested in finding a criteria for consensus in the networked system (17),2 where $\varphi_{jk} \in \mathfrak{S}[\alpha;\beta]$, $\beta > \alpha > 0$.

To start with, we determine conditions under which Assumption 4 holds. We need the following lemma, which may be derived from [Hale, 1977, Appendix, Theorem A.5] so its proof is omitted here.

Lemma 1. *The feedback* $u_j = -Ky_j$ *stabilizes the agent* (17) *if and only if* $K > a$ *and*

$$f(a,K) := \frac{\arccos\frac{a}{K}}{\sqrt{K^2-a^2}} < \tau. \qquad (18)$$

It may be shown that the left-hand side of (18) is decreasing when $K > a$ so (18) is satisfied in and only if $K > K_*(a,\tau)$, where $f(a,K_*(a,\tau)) = \tau$ and $K_*(a,\tau) > a$. Assumption 4 may be formulated as follows: $\lambda_2(G) > K_*(a,\tau)/\alpha$.

The following result gives a sufficient condition for consensus among the agents (17).

Theorem 6. *Suppose that* $\varphi_{jk} \in \mathfrak{S}[\alpha;\beta]$ *with* $\alpha > 0$*, Assumption 3 holds, the graph* $G \in \mathbb{G}_N$ *is connected,* $\lambda_2(G) > K_*(a,\tau)/\alpha$ *and* $2(\alpha+\beta)D_{max} < a$*. Then the protocol* (2) *establishes the output consensus.*

Remark 2. *The conditions of Theorem 6 require the coupling weights* a_{jk} *to be sufficiently large (lower bound for* $\lambda_2(G)$*) to meet Assumption 4 and sufficiently small to satisfy the frequency-domain inequality (upper bound for* D_{max}*). It is evidently possible to satisfy both requirements for* $\tau = 0$ *and thus for all sufficiently small* τ*, however, despite it is a non-trivial problem to get analytic bound for maximal possible* τ *in terms of* a*,* α *and* β*.*

Proof We notice that $W(z) = e^{-z\tau}/(z-a)$ and thus

$$Re\,W(\imath\omega) = Re\,\frac{e^{-\imath\omega\tau_j}}{\imath\omega - a} = -\frac{a\cos\omega\tau + \omega\sin\omega\tau}{a^2+\omega^2}$$
$$Re\,\imath\omega W(\imath\omega) = \frac{\omega^2\cos\omega\tau - a\omega\sin\omega\tau}{a^2+\omega^2}$$
$$Re\,W(1+\mathfrak{P}\imath\omega) = \frac{\mathfrak{P}\omega^2 - a}{a^2+\omega^2}\cos\omega\tau - \frac{(1+a\mathfrak{P})\omega}{a^2+\omega^2}\sin\omega\tau$$

In particular, taking $\mathfrak{P} := -a^{-1}$, one obtains that if $2(\alpha+\beta)D_{max} < a$, then $Re\,W(1+\mathfrak{p}\imath\omega) + \gamma/(2D_{max}) = (2(\alpha+\beta)D_{max})^{-1} - a\cos\omega\tau \geq 0$ which implies (12).

7 Extensions to MIMO case.

In this section we discuss more complicated case when the agents may have vector inputs and outputs (either $m = \dim u_j > 1$ or $p = \dim y_j > 1$). We start with protocols of special type, the consensus criteria for which may be considered as a direct extension of Theorems 1,2.

Let $m = p$ and $y_j = col(y_j^1, \ldots, y_j^m)$ and $u_j = col(u_j^1, \ldots, u_j^m)$. The protocol has the form

$$u_j^q(t) = \sum_{k=1}^{N} a_{jk}(t)\varphi_{jk}^q(y_k^q(t) - y_j^q(t)),\ j \in 1:N, q \in 1:m. \tag{19}$$

Thus the interaction between the agents is "decoupled" in the sense that at each time the q-th scalar output affects only correspondent scalar input. Although the restriction of $m = p$ seems to be restrictive, it often can be provided by adding "virtual" inputs. Consider e.g. a second-order network

$$\ddot{z}_j = \sum_{k=1}^{N} a_{jk}(t)\left[\varphi_{jk}^1(z_k - z_j) + \varphi_{jk}^2(\dot{z}_k - \dot{z}_j)\right]. \tag{20}$$

Formally the agent $\ddot{z}_j = u_j$ is "underactuated" as it has one scalar input and two scalar outputs. However, one may introduce two inputs u_j^1, u_j^2, where u_j^q $(q = 1,2)$ are given by (19), so that $u_j = u_j^1 + u_j^2$.

Introducing a map $\varphi_{jk} : \mathbb{R}^m \to \mathbb{R}^m$ by $\varphi_{jk}(y) := col(\varphi_{jk}^1(y^1), \ldots, \varphi_{jk}^m(y^m))$, the equation (19) is rewritten in the form (2). We say that $\varphi_{jk} \in \mathfrak{S}(\gamma)$ if $\varphi_{jk}^q \in \mathfrak{S}(\gamma), \forall q$.

The following theorem gives an extension of Theorems 1 and 2 to the vector case.

Theorem 7. *Suppose that $\varphi_{jk} \in \mathfrak{S}(\gamma)$ and Assumptions 1, 3 hold. Let diagonal matrices $\mathfrak{P} = diag(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$, $\mathfrak{Q} = diag(\mathfrak{q}_1, \ldots, \mathfrak{q}_m)$ with $\mathfrak{p}_k \in \mathbb{R}, \mathfrak{q}_k > 0$ exist such that*

$$(\mathfrak{Q} + \imath\omega\mathfrak{P})W_j(\imath\omega) + W_j(\imath\omega)^*(\mathfrak{Q} - \imath\omega\mathfrak{P}) + \frac{1}{\gamma D_j}\mathfrak{Q} \geq 0 \tag{21}$$

for any $j \in 1;N$ and $\omega \in \mathbb{R}$, such that $W_j(\imath\omega)$ is well defined. If the graph $G(t) \equiv G$ is constant and connected, then the output consensus is achieved. If (21) *holds for $\mathfrak{P} = 0$, the same is true for any uniformly connected graph $G(t)$.*

For the case of $m = 1$ and $\mathfrak{Q} = 1$, $\mathfrak{P} = \theta$ the inequality (21) is equivalent to (7).

Analogous MIMO result extending Theorem 3 may be obtained for the case of identical agents and connected graph $G(t)$.

8 Conclusions

The paper addresses the consensus problem for network of heterogeneous linear agents which may have arbitrary order and are governed by delay equations. The couplings may be nonlinear uncertain, assumed only to satisfy a symmetry assumption which is close in spirit to the Newton Third Law and conventional sector inequalities. We offer easily verifiable synchronization criteria, similar in flavor to the celebrated circle and Popov stability criteria, and demonstrate their application for important special situations, such as teams of first and second-order delayed agents. We also discuss application of our results to simple microscopic traffic flow models. Analogous result may be obtained for more general problems of leader-following and reference-tracking consensus.

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