Successive approximation algorithm for center manifolds and its applications

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Abstract A successive approximation method to calculate center manifolds is proposed. The proposed algorithm consists of function sequences that uniformly converge to a center manifold. The algorithm is more suitable for computer implementation than the standard Taylor expansion method. The properties and applications of the method are discussed.

1 Introduction

Center manifolds play important roles in dynamical system theory. A center manifold appears when the linear part of the differential equation describing the dynamical system has eigenvalues on the imaginary axis. In such a case, one cannot conclude the stability of the system from the linear part and the stability of the system is determined by that of the dynamics on the center manifold. Also, in bifurcation theory and singular perturbation theory, center manifold theory is employed by augmenting describing equations with the perturbation parameters as system states. In control theory, center manifold theory is important as well when designing a feedback controller for asymptotically tracking a reference signal or rejecting undesired disturbances, called *the output regulation problem* (see, e.g., [2],[3],[4]).

For calculation of center manifolds, however, the analytic method widely known is only by the Taylor expansion. Two of the authors in the present paper proposed an analytic calculation method of center manifolds for the output regulation problem, which is totally different from the Taylor expansion method[5]. The method is created by defining successive function sequences that converge to the center manifold. In this paper, we would like to re-introduce the algorithm and further examine its properties and potential applications for control theory and dynamical system theory.

2 Successive approximation method of center manifolds

Let us consider the following set of differential equations

$$\begin{cases} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y), \end{cases} \tag{1}$$

where $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$.

Assumption 1 *A is an $n\times n$ constant real matrix whose eigenvalues have zero real parts. B is an $m\times m$ constant real matrix and its eigenvalues have negative real part.*

From Assumption 1, it follows that for any constant $a > 0$, there exists a constant $C_1(a) > 0$ such that

$$\left|e^{At}x\right| \leqslant C_1(a)e^{a|t|}|x|, \ (^\forall t\in\mathbb{R}, \ ^\forall x\in\mathbb{R}^n).$$

Also, it follows that there exist constant $b > 0$ and $C_2 > 0$ such that

$$\left|e^{-Bt}y\right| \leqslant C_2 e^{bt}|y|, \ (0\geqslant{}^\forall t\in\mathbb{R}, \ ^\forall y\in\mathbb{R}^m).$$

Assumption 2 *$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are C^r functions ($r \geqslant 2$) and for all $|x| \leqslant \varepsilon$, $|x'| \leqslant \varepsilon$, $|y| \leqslant \varepsilon$, $|y'| \leqslant \varepsilon$, there exist continuous scalar functions $K_1(\varepsilon)$, $K_2(\varepsilon)$ such that*

$$\begin{cases} |f(x,y)| \leqslant \varepsilon K_1(\varepsilon) \\ |g(x,y)| \leqslant \varepsilon K_2(\varepsilon) \\ |f(x,y) - f(x',y')| \leqslant K_1(\varepsilon)(|x-x'| + |y-y'|) \\ |g(x,y) - g(x',y')| \leqslant K_2(\varepsilon)(|x-x'| + |y-y'|) \end{cases}$$

where, $f(0,0) = 0$, $g(0,0) = 0$, $(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = 0$, $(\frac{\partial g}{\partial x}(0,0), \frac{\partial g}{\partial y}(0,0)) = 0$, $K_1(0) = 0$, $K_2(0) = 0$.

We define a set of sequences $\{x_k(t,\xi)\}$, $\{h_k(\xi)\}$, $(k=0,1,2,\cdots)$ by the following.

$$\begin{cases} x_0(t,\xi) = e^{At}\xi \\ h_0(\xi) = 0 \\ x_{k+1}(t,\xi) = e^{At}\xi + \int_0^t e^{A(t-s)} f(x_k(s,\xi), h_k(x_k(s,\xi)))ds \\ h_{k+1}(\xi) = \int_{-\infty}^0 e^{-Bs} g(x_k(s,\xi), h_k(x_k(s,\xi)))ds. \end{cases} \tag{2}$$

Theorem 3 *Under Assumptions* 1 *and* 2*, system* (1) *possesses a local center manifold $y = h(x)$ around the origin and $h_k(x)$ in (2) converges to the local center manifold when $k \to \infty$.*

Example 2.1 Let us calculate the center manifold $y = h(x)$ for the following system.

$$\begin{cases} \dot{x} = -x^3, \ x_0 = x(0) \\ \dot{y} = -y + x^2 \end{cases}$$

Using Theorem 3, the algorithm is applied 12 times and we obtained

$$\begin{aligned} x(t, x_0) &= x_0 - {x_0}^3 t + \frac{3{x_0}^5}{2}t^2 - \frac{5{x_0}^7}{2}t^3 + \frac{35{x_0}^9}{8}t^4 \\ &\quad - \frac{63{x_0}^{11}}{8}t^5 + \frac{231{x_0}^{13}}{16}t^6 \\ y = h(x_0) &= {x_0}^2 + 2{x_0}^4 + 8{x_0}^6 + 48{x_0}^8 + 384{x_0}^{10} \\ &\quad + 3840{x_0}^{12} \end{aligned}$$

The standard approach to calculate $y = h(x)$ is based on the Taylor expansion, which can be applied setting $\phi(x) = \sum_{n=0}^{\infty} a_{2n+2} x^{2n+2}$ and calculate coefficients. In this case, we have $a_{2n+2} = 2na_{2n}$, $a_2 = 1$ and the same result is confirmed.

Let us summarize the properties of the proposed algorithm based on Theorem 3.

- The integrals in (2) always exist.
- Unlike the Taylor expansion method, no equation needs to be solved.
- When nonlinearities f and g are polynomial, the integrations in (2) can be performed by the integration-by-part, which means that all the calculations in the algorithm are algebraic. The resulting approximation $h_k(x)$ is a polynomial function.
- With polynomial nonlinearities, the center manifold calculation can be done even with undetermined parameters. Also, unlike the Taylor expansion method, unnecessary terms do not appear during the process of calculation.

3 Application in aerospace engineering

In this section, we consider the attitude stabilization problem of a satellite with only two thrusters. This problem is solved in [6, 7, 8] using center manifold theory. Here, we show that by the proposed method, it is possible not necessarily to design a stabilizing controller but also trajectories in the phase space because exponentially stable modes, corresponding the second equation in (1), rapidly converge to the center manifold.

The equations of motion for a satellite with control inputs aligned only with two principal axes are given as follows.

$$\begin{cases} \dot\omega_1 = I_{23}\omega_2\omega_3 \\ \dot\omega_2 = I_{31}\omega_3\omega_1 + c_2u_1 \\ \dot\omega_3 = I_{12}\omega_1\omega_2 + c_3u_2 \end{cases}$$

where

$$\begin{cases} I_3 > I_2 > I_1 > 0 \\ I_{23} = (I_2 - I_3)/I_1 \\ I_{31} = (I_3 - I_1)/I_2 \\ I_{12} = (I_1 - I_2)/I_3 \\ c_2 = 1/I_2,\ c_3 = 1/I_3 \end{cases}$$

Choosing control inputs as

$$\begin{cases} u_1 = \frac{1}{c_2}(-\omega_2 + p_1\omega_1^2 + p_2\omega_1^3) \\ u_2 = \frac{1}{c_3}(-\omega_3 + q_1\omega_1^2 + q_2\omega_1^3), \end{cases}$$

we have the colsed loop system as follows.

$$\begin{cases} \dot\omega_1 = I_{23}\omega_2\omega_3 \\ \dot\omega_2 = -\omega_2 + I_{31}\omega_3\omega_1 + p_1\omega_1^2 + p_2\omega_1^3 \\ \dot\omega_3 = -\omega_3 + I_{12}\omega_1\omega_2 + q_1\omega_1^2 + q_2\omega_1^3 \end{cases}$$

From this equation, ω_2, ω_3 exponentially converges to the center manifold $\omega_2 = \tilde\varphi_1(\omega_1), \omega_3 = \tilde\varphi_2(\omega_1)$ and thus, the stability of equilibrium $(\omega_1, \omega_2, \omega_3) = (0, 0, 0)$ is determined by that of the dynamics on the center manifold (see, e.g., [1]). Therefore, our task is now to choose the parameters p_1, p_2, q_1, q_2 so that the dynamics on the stable manifold is asymptotically stable. By the 3rd order approximation of the center manifold, the condition for the asymptotic stability is $p_1q_1 = 0$ and $I_{23}q_1(I_{31}q_1 + p_2) < 0$ or $I_{23}p_1(I_{12}p_1 + q_2) < 0$ ([7]). We obtain higher order approximation by the proposed method, including undetermined control parameters p_1, p_2, q_1 and q_2 satisfying the above condition. The parameters can be used to design trajectories of the closed loop system. Figures 1, 2 show approximated center manifolds of 3rd and 10th order and the closed loop trajectories for different values of the parameters.

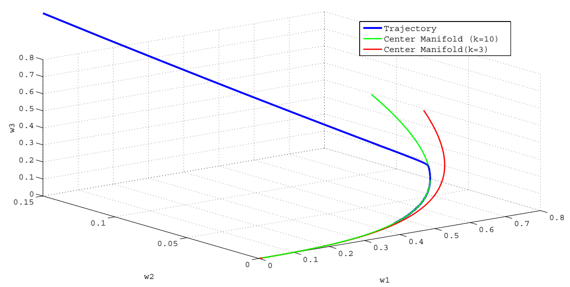

Figure 1: $(p1, p2, q1, q2) = (0, 0.1, 0.1, 0.1)$

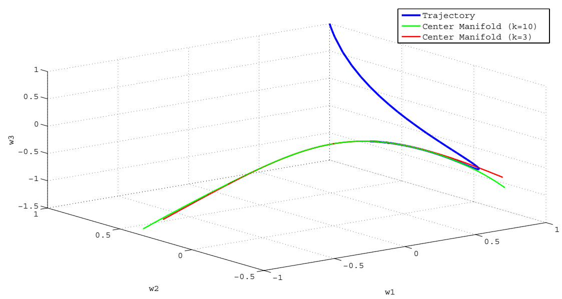

Figure 2: $(p1, p2, q1, q2) = (0, 0.1, -1, 0.1)$

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