ON TRACKING CONTROL PROBLEM FOR FRACTIONAL ORDER DESCRIPTOR SYSTEMS

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Abstract

In this paper, we discuss the tracking control problem for fractional order descriptor systems. The aim of this paper is to find the optimal tracking control-state pair satisfying the dynamic constraint of the form a fractional order descriptor system such that the performance index is minimized. The method of solving is to convert such the tracking control problem for a fractional order descriptor system into the standard fractional order system. Under some particular conditions, we find the explicit formulas of the optimal tracking control-state pair which minimize the performance index.

Key words

Tracking control problem, fractional order, descriptor system, Caputo fractional derivative, Mittag-Leffler function

1 Introduction

Recently, the issue on tracking control problem for fractional order systems and their applications has arisen in several literatures. The solved problem is to find a tracking control $\mathbf{u}$ that satisfy the fractional order system

$$\mathbf{y}^{(\delta)} = A\mathbf{y} + B\mathbf{u},\ \mathbf{y}(0) = \mathbf{y}_0 \tag{1}$$

such that the state $\mathbf{y} \to \mathbf{y}_\mathrm{d}$ if $t \to \infty$ and the performance index

$$\mathcal{J}_{\mathbf{u},\mathbf{y}} = \frac{1}{2}\int_0^\infty \Big(\langle \mathbf{y} - \mathbf{y}_\mathrm{d}, Q(\mathbf{y} - \mathbf{y}_\mathrm{d})\rangle + \langle \mathbf{u}, R\mathbf{u}\rangle \Big) dt, \tag{2}$$

is minimized. In the equation (1) and (2), $\mathbf{y} = \mathbf{y}(t) \in \mathbb{R}^n$ denotes the state, $\mathbf{y}_\mathrm{d} \in \mathbb{R}^n$ denotes the desired state, $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^r$ denotes the tracking control, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}$, $\langle \cdot\ ,\ \cdot\rangle$ denotes the inner product, Q and R are the symmetric positive definite matrices, and $\mathbf{y}^{(\delta)}$ denotes the fractional derivative of order δ of variable $\mathbf{y}$, with $\delta \in (m-1, m)$, $m \in \mathbb{N}$. It is clear that when $\delta = 1$, the problem (1) and (2) constitute a standard tracking control problem that have been discussed in some literatures such as [Gartemani et all, 2011], [Birgani et all, 2018], [Dul et all, 2020] and [Yulianti et all, 2019]. Solving of the problem (1)-(2) for $\delta \in (m-1, m)$ and several of its variant have been done by some researchers, such as [Ayas et all, 2018], [Liao and Xie, 2019], [Trivedi et all, 2020] and others. Some application of the fractional order system in identification of physical system has been discussed by several authors, see [Luongo, 2011] and [Jesus, 2008].

In this paper, we discuss the tracking control problem for fractional order descriptor system

$$E\mathbf{y}^{(\delta)} = A\mathbf{y} + B\mathbf{u},\ \mathbf{y}(0) = \mathbf{y}_0, \tag{3}$$

where $\mathrm{rank}(E) < n$, that is to find the pair tracking control $\mathbf{u}^*$ and the state $\mathbf{y}^*$, written by $(\mathbf{u}^*, \mathbf{y}^*)$, which satisfy the dynamic system (3) such that the state $\mathbf{y} \to \mathbf{y}_\mathrm{d}$ if $t \to \infty$ and the performance index (2) is minimized.

The fractional order descriptor systems have attracted the attention of many researcher in the past years due to the fact that, in some cases, they describe the behavior of physical systems better than standard systems. They can preserve the structure of physical systems and include a non dynamic constraint and an impulsive element. Systems of this kind have many important applications, e.g., in electrical circuit [Gomez et all, 2013], in mechanical system [Martinez et all, 2020]. Therefore, it is fair to say that the fractional order descriptor systems give a more complete class of dynamical models than conventional descriptor systems.

In contrast to the fractional order system (1), which always has a solution, the fractional order descriptor system (3) is possible to have no solution. It is well known that the solution of the fractional order descriptor system (3) exists and unique if $\det(s^\delta E - A) \neq 0$ for some scalar $s \in \mathbb{C}$ [Kaczorek and Rogowski, 2015], [Batiha et all, 2018], [Muhafzan et all, 2020] and [Nazra et all, 2020]. Therefore one may say that the fractional order descriptor system (3) constitutes an extension of the fractional order system (1).

We assume in this paper that $\det(s^\delta E - A) \neq 0$ for some scalar $s \in \mathbb{C}$ to guarantee the existence and uniqueness of the solution of (3), and $\mathbf{y}^{(\delta)}$ is the fractional derivative of Caputo type of $\mathbf{y}$ of order δ, with $\delta \in (0, 1)$, which is an extension of the derivative in usual means [Diethelm, 2010]. Moreover, we also assume that the system (3) is controllable and impulse controllable as well. To the best of the author's knowledge, this issue has not been solved yet to date. Therefore the results of this work constitute a new contribution in the field of optimization subject to fractional order descriptor system.

The rest of the paper is organized as follows. Section 2 discusses some useful materials related to the desired results, such as the information about the Caputo derivative, Mittag-Leffler function and fractional order differential equation systems. Section 3 presents the transformation process. The main result of this article and a numerical example illustrating the results is given in section 4. Section 5 concludes the paper.

2 Some Useful Results

There are several mathematical tools used in this study. Suppose that $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$ is an integrable function and derivative order m of the function $\mathbf{x}$, that is $\mathbf{x}^{(m)}$, exists for $m \in \mathbb{N}$. The fractional derivative of Caputo type of order δ with $\delta \in (m-1, m)$, $m \in \mathbb{N}$, is defined by:

$$\mathbf{x}^{(\delta)}(t) = \frac{1}{\Gamma(m-\delta)} \int_0^t \frac{\mathbf{x}^{(m)}(\tau)}{(t-\tau)^{\delta-m+1}} d\tau \qquad (4)$$

where $\Gamma(.)$ is the Gamma function [Batiha et all, 2018]. The physical aspect of the fractional derivative of Caputo type has been already discussed by several researchers, see [Aguilar et all, 2014] and [Traore and Sene, 2020]. In [Traore and Sene, 2020] is mentioned that the fractional order derivatives of Caputo type have many advantages in comparison with the ordinary order derivative. One of the most simple examples in which the fractional derivative of Caputo type has a significant impact can be noted in stability analysis. There exist differential equations that are not stable with the first-order derivative, but their fractional versions are stable with fractional order derivatives. Therefore, the solutions coming from fractional order differential equations appear to describe real-life data better compared to the solutions of the corresponding ordinary order differential equations.

The one parameter Mittag-Leffler function with parameter $\eta > 0$ is defined as the following infinite series [Batiha et all, 2018]:

$$\mathcal{E}_\eta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\eta + 1)}, \; z \in \mathbb{C}. \qquad (5)$$

Meanwhile, the two parameters Mittag-Leffler function with parameters $\eta, \gamma > 0$ are defined by [Batiha et all, 2018]:

$$\mathcal{E}_{\eta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\eta + \gamma)}, \; z \in \mathbb{C}. \qquad (6)$$

The Mittag-Leffler functions (5) and (6) are convergent series [Batiha et all, 2018]. One can replace variable z in (6) by Az for an arbitrary square matrix A, such that

$$\mathcal{E}_{\eta,\gamma}(Az) = \sum_{k=0}^{\infty} \frac{(Az)^k}{\Gamma(k\eta + \gamma)}. \qquad (7)$$

It is easy to see that $\mathcal{E}_{\eta,1}(Az) = \mathcal{E}_\eta(Az)$ and

$$\mathcal{E}_1(Az) = \sum_{k=0}^{\infty} \frac{(Az)^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(Az)^k}{k!} = \exp(Az). \qquad (8)$$

Using the definition of the Laplace transformation, one can observe that

$$\mathcal{L}[\mathbf{x}^{(\delta)}(t)] = s^\delta X(s) - \sum_{k=0}^{m-1} s^{\delta-k-1}\mathbf{x}^{(k)}(0), \qquad (9)$$

where $\mathcal{L}$ denotes the Laplace operator, $X(s) = \mathcal{L}[\mathbf{x}(t)]$ and $\delta \in (m-1, m)$ [Milici et all, 2019].

The Laplace transformation (9) and the Mittag-Leffler function play an important role in solving of the fractional order system (1). Using the equation (9), the solution of the fractional order system (1) for $\delta \in (0, 1)$ is given by the following equation:

$$\mathbf{y}(t) = \mathcal{E}_\delta(At^\delta)\mathbf{y}_0 + t^\delta \mathcal{E}_{\delta,1+\delta}(At^\delta) \circledast B\mathbf{u}, \qquad (10)$$

where $\circledast$ denotes the convolution on $[0, t]$.

It is well known that under the assumption $\det(s^\delta - A) \neq 0$, the fractional order descriptor system (3) can be decomposed into a slow sub-system and a fast subsystem where the slow subsystem constitutes a standard fractional order system [Kaczorek and Rogowski, 2015].

Similar to the conventional controllability concept, the controllability of the fractional order system (1) is defined as follows.

Definition 1 [Monje et all, 2010] *The fractional order system* (1) *is called to be controllable on* $[0, t_f]$ *if for any initial state* $\mathbf{y}_0$ *there exists a control* $\mathbf{u} : [0, t_f] \to \mathbb{R}^r$ *which drive the initial state* $\mathbf{y}_0$ *to the final state* $\mathbf{y}(t_f)$.

The controllability definition of the fractional order descriptor system (3) is similar to Definition 1.

Definition 2 [Xu and Yang, 2016] *The fractional order system* (3) *is called to be impulse controllable on* $[0, t_f]$ *if for any initial state* $\mathbf{y}_0$ *there exists a control* $\mathbf{u} : [0, t_f] \to \mathbb{R}^r$ *such that impulse part in the response of the fast subsystem of the fractional order descriptor system* (3) *is identically zero.*

The following theorem is useful to check the controllability of the fractional order system (1).

Theorem 1 [Monje et all, 2010] *The fractional order system* (1) *is controllable if and only if the controllability matrix* $[B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B]$ *has a full rank.*

3 Transformation Process

Reconsider the tracking control problem for fractional order descriptor systems (2) and (3) under the assumption $\det(s^\delta E - A) \neq 0$ and $\delta \in (0, 1)$. A pair $(\mathbf{u}, \mathbf{y})$ is called admissible for the tracking control problem (2) and (3) if it satisfies the fractional order descriptor system (3) and $\mathcal{J}_{\mathbf{u},\mathbf{y}} < \infty$ for an initial state $\mathbf{y}_0 \in \mathbb{R}^n$. A pair $(\mathbf{u}^*, \mathbf{y}^*)$ is called an optimal pair for the tracking control problem (2) and (3) if it is an admissible and $\mathcal{J}_{\mathbf{u}^*,\mathbf{y}^*} = \min\ \mathcal{J}_{\mathbf{u},\mathbf{y}}$. Let us define the admissible pairs set for the tracking control problem (2) and (3) by

$$\mathcal{X} \triangleq \{(\mathbf{u}, \mathbf{y}) | \mathbf{u} \text{ and } \mathbf{y} \text{ are the continuous vector functions that satisfy (3) and } \mathcal{J}_{\mathbf{u},\mathbf{y}} < \infty\}.$$

It is clear that $\mathcal{X} \neq \emptyset$. We will find the explicit formulation of the optimal pairs $(\mathbf{u}^*, \mathbf{y}^*) \in \mathcal{X}$ such that $\mathcal{J}_{\mathbf{u}^*,\mathbf{y}^*} = \min\ \mathcal{J}_{\mathbf{u},\mathbf{y}}$.

First of all, let us transform the tracking control problem (2) and (3) into the tracking control problem for the fractional order systems (2) and (1). For this purpose, we adopt Definition 1 in [Fang et all, 2014] and the Singular Value Decomposition(SVD) Theorem in [Klema and Laub, 1980] to find a restricted system equivalent (r.s.e.) to the system (3).

Definition 3 *A fractional order descriptor system*

$$\breve{E}\breve{\mathbf{y}}^{(\delta)} = \breve{A}\breve{\mathbf{y}} + \breve{B}\mathbf{u},\ \breve{\mathbf{y}}(0) = \breve{\mathbf{y}}_\mathbf{0}$$

is said to be a restricted system equivalent (r.s.e.) to the system (3) *if there exists two nonsingular matrices* $M, N \in \mathbb{R}^{n\times n}$ *such that* $MEN = \breve{E}$, $MAN = \breve{A}$, $MB = \breve{B}$ *and* $\mathbf{y} = N\breve{\mathbf{y}}$.

Obviously, the restricted system equivalence is an equivalent relationship and it is consistent with Definition 1 in [Fang et all, 2014] for the standard descriptor systems.

Let $\text{rank}(E) = p < n$. Based on the Singular Value Decomposition(SVD) Theorem [Klema and Laub, 1980], there exists nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$ such that

$$MEN = \begin{bmatrix} I_p & O \\ O & O \end{bmatrix} \tag{11}$$

where I_p is an identity matrix of size $p\times p$ and O is a zero matrix of suitable size. Using these M and N matrices, we have

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and

$$N^{-1}\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \triangleq \breve{\mathbf{y}}, \tag{12}$$

with $A_{11} \in \mathbb{R}^{p\times p}$, $B_1 \in \mathbb{R}^{p\times r}$, $\mathbf{y}_1 \in \mathbb{R}^p$ and $\mathbf{y}_{10} = \begin{bmatrix} I_p & O \end{bmatrix} M\mathbf{y}_0$. Thus we have the following fractional order descriptor system

$$\begin{bmatrix} I_p & O \\ O & O \end{bmatrix} \begin{bmatrix} \mathbf{y}_1^{(\delta)} \\ \mathbf{y}_2^{(\delta)} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u}, \tag{13}$$

with $\mathbf{y}_1(0) = \mathbf{y}_{10}$, which is r.s.e. to the fractional order descriptor system (3). The equation (13) can be written as

$$\mathbf{y}_1^{(\delta)} = A_{11}\mathbf{y}_1 + A_{12}\mathbf{y}_2 + B_1\mathbf{u} \tag{14}$$

$$\mathbf{0} = A_{21}\mathbf{y}_1 + A_{22}\mathbf{y}_2 + B_2\mathbf{u}. \tag{15}$$

Using the transformation (12), the performance index (2) can be replaced by

$$\mathcal{J}_{\mathbf{u},\breve{\mathbf{y}}} = \frac{1}{2}\int_0^\infty (\langle \breve{\mathbf{y}} - \breve{\mathbf{y}}_\mathrm{d}, \bar{Q}(\breve{\mathbf{y}} - \breve{\mathbf{y}}_\mathrm{d}) \rangle + \langle \mathbf{u}, R\mathbf{u} \rangle)dt, \tag{16}$$

where $\bar{Q} = N^\top QN \triangleq \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{11} \end{bmatrix}$ with $Q_{11} \in \mathbb{R}^{p\times p}$, $Q_{12} \in \mathbb{R}^{p\times(n-p)}$ and $Q_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $\breve{\mathbf{y}}_\mathrm{d} = N^{-1}\mathbf{y}_\mathrm{d} \triangleq \begin{bmatrix} \mathbf{y}_{1\mathrm{d}} \\ \mathbf{y}_{2\mathrm{d}} \end{bmatrix}$. Under the assumption that the system (3) is impulse controllable, one can observe that the transformations (11) and (12) imply that the fractional order descriptor system (13) is impulse controllable as well, see [Nazra et all, 2020], and this implies rank $\left[A_{22}\ B_2\right] = n-p$. Therefore, the solution of equation (15) is

$$\begin{bmatrix} \mathbf{y}_2 \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} -\widehat{A}^{\dagger} A_{21}\ \Upsilon \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{v} \end{bmatrix}, \tag{17}$$

for some full rank matrix $\Upsilon \in \mathbb{R}^{(n-p+r)\times r}$ with $\Upsilon \in \ker \left[A_{22}\ B_2\right]$, and for some $\mathbf{v} \in \mathbb{R}^r$ with

$$\widehat{A}^{\dagger} = \left[A_{22}\ B_2\right]^{\top} \left(A_{22}A_{22}^{\top} + B_2 B_2^{\top}\right)^{-1}$$

is the generalized inverse of the matrix $\left[A_{22}\ B_2\right]$. Using the expression (17), the following transformation is created:

$$\begin{bmatrix} \mathbf{y}_1 \\ \hline \mathbf{y}_2 \\ \mathbf{u} \end{bmatrix} = \left[\begin{array}{c|c} I_p & O \\ \hline -\widehat{A}^{\dagger} A_{21} & \Upsilon \end{array}\right] \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{v} \end{bmatrix}. \tag{18}$$

From (18), we also have

$$\begin{bmatrix} \mathbf{y}_1 - \mathbf{y}_{1d} \\ \hline \mathbf{y}_2 - \mathbf{y}_{2d} \\ \mathbf{u} \end{bmatrix} = \left[\begin{array}{c|c} I_p & O \\ \hline -\widehat{A}^{\dagger} A_{21} & \Upsilon \end{array}\right] \begin{bmatrix} \mathbf{y}_1 - \mathbf{y}_{1d} \\ \mathbf{v} \end{bmatrix}. \tag{19}$$

By substituting (19) into (16), the performance index (16) can be written as

$$\mathcal{J}_{\mathbf{v},\mathbf{y}_1} = \frac{1}{2}\int\limits_0^{\infty} \Big(\left\langle \mathbf{y}_1 - \mathbf{y}_{1d}, \bar{Q}_{11}(\mathbf{y}_1 - \mathbf{y}_{1d}) \right\rangle + \\ 2\left\langle \mathbf{y}_1 - \mathbf{y}_{1d}, \bar{Q}_{12}\mathbf{v} \right\rangle + \left\langle \mathbf{v}^{\top}, \bar{Q}_{22}\mathbf{v} \right\rangle \Big) dt, \tag{20}$$

where

$$\bar{Q}_{11} = Q_{11} - 2\left[Q_{12}\ O\right]\widehat{A}^{\dagger} A_{21} + \\ (\widehat{A}^{\dagger} A_{21})^{\top} \begin{bmatrix} Q_{22} & O \\ O & R \end{bmatrix} \widehat{A}^{\dagger} A_{21},$$

$$\bar{Q}_{12} = \left[Q_{12}\ O\right]\Upsilon - (\hat{A}^{\dagger} A_{21})^{\top} \begin{bmatrix} Q_{22} & O \\ O & R \end{bmatrix} \Upsilon,$$

$$\bar{Q}_{22} = \Upsilon^{\top} \begin{bmatrix} Q_{22} & O \\ O & R \end{bmatrix} \Upsilon.$$

Moreover, by substituting (17) into (14) we have the fractional order system as follows:

$$\mathbf{y}_1^{(\delta)} = \bar{A}\mathbf{y}_1 + \bar{B}\mathbf{v},\ \mathbf{y}_1(0) = \mathbf{y}_{10} \tag{21}$$

with $0 < \delta < 1$, where $\bar{A} = A_{11} - \left[A_{12}\ B_1\right]\widehat{A}^{\dagger} A_{21}$ and $\bar{B} = \left[A_{12}\ B_1\right]\Upsilon$.

One can see now that the performance index (20) and the fractional order system (21) constitute a tracking control problem for the fractional order standard system with the state $\mathbf{y}_1$ and control $\mathbf{v}$. Likewise, the solving of the tracking control problem for fractional order descriptor system (3) with the performance index (2) is reduced to the solving of the tracking control problem for fractional order standard system (21) with the performance index (20). Therefore, to find the solving of the tracking control problem for the fractional order descriptor system (3) with the performance index (2), one can solve the tracking control problem for fractional order standard system (21) with the performance index (20). The problem to solve here is to find the pair $(\mathbf{v}^*, \mathbf{y}_1^*)$, which satisfy the fractional system (21) such that the state $\mathbf{y}_1 \to \mathbf{y}_{1d}$ if $t \to \infty$ and the performance index (20) is minimized.

4 Main Result

In order to find the pair $(\mathbf{v}^*, \mathbf{y}_1^*)$, we use the theory of the standard fractional linear quadratic control problem as introduced in [Li and Chen, 2008] and [Matychyn and Onyshchenko, 2018] by making some modifications. First of all, based on the performance index (20) and the fractional order system (21), let us consider the following augmented performance index:

$$\mathcal{J}^a_{\mathbf{v},\mathbf{y}_1} = \int_0^{\infty} \Big(\frac{1}{2}\big[(\mathbf{y}_1 - \mathbf{y}_{1d})^{\mathsf{T}}\bar{Q}_{11}(\mathbf{y}_1 - \mathbf{y}_{1d}) \\ + 2(\mathbf{y}_1 - \mathbf{y}_{1d})^{\mathsf{T}}\bar{Q}_{12}\mathbf{v} + \mathbf{v}^{\mathsf{T}}\bar{Q}_{22}\mathbf{v}\big] \\ + \mathbf{q}^{\mathsf{T}}(\bar{A}\mathbf{y}_1 + \bar{B}\mathbf{v} - \mathbf{y}_1^{(\delta)})\Big) dt \tag{22}$$

where $\mathbf{q} = \mathbf{q}(t) \in \mathbb{R}^p$ is a costate variable. By defining the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\Big[(\mathbf{y}_1 - \mathbf{y}_{1d})^{\mathsf{T}}\bar{Q}_{11}(\mathbf{y}_1 - \mathbf{y}_{1d}) \\ + 2(\mathbf{y}_1 - \mathbf{y}_{1d})^{\mathsf{T}}\bar{Q}_{12}\mathbf{v} + \mathbf{v}^{\mathsf{T}}\bar{Q}_{22}\mathbf{v}\Big] \\ + \mathbf{q}^{\mathsf{T}}(\bar{A}\mathbf{y}_1 + \bar{B}\mathbf{v}), \tag{23}$$

the augmented performance index (22) can be written as:

$$\mathcal{J}^a_{\mathbf{v},\mathbf{y}_1} = \int_0^{\infty} f(\mathbf{y}, \mathbf{v}, \mathbf{q}, \mathbf{y}^{(\delta)}, \mathbf{v}^{(\delta)}, \mathbf{q}^{(\delta)}, t) dt, \tag{24}$$

where

$$f(\mathbf{y}, \mathbf{v}, \mathbf{q}, \mathbf{y}^{(\delta)}, \mathbf{v}^{(\delta)}, \mathbf{q}^{(\delta)}, t) = \mathcal{H} - \mathbf{q}^{\mathsf{T}}\mathbf{y}^{(\delta)}. \tag{25}$$

By using the procedure in [Li and Chen, 2008] and [Matychyn and Onyshchenko, 2018] we find the following Euler-Lagrange equation:

$$\frac{\partial f}{\partial \mathbf{v}} - \left[\frac{\partial f}{\partial \mathbf{v}^{(\delta)}}\right]^{(\delta)} = \mathbf{0}, \tag{26}$$

$$\frac{\partial f}{\partial \mathbf{q}} - \left[\frac{\partial f}{\partial \mathbf{q}^{(\delta)}}\right]^{(\delta)} = \mathbf{0}, \tag{27}$$

$$\frac{\partial f}{\partial \mathbf{y}} - \left[\frac{\partial f}{\partial \mathbf{y}^{(\delta)}}\right]^{(\delta)} = \mathbf{0}. \tag{28}$$

After some calculations, the equations (26), (27) and (28) result:

$$
\begin{aligned}
\mathbf{v} &= -Q_{22}^{-1}(\bar{B}^{\intercal}\mathbf{q} + \bar{Q}_{12}^{\top}(\mathbf{y}_1 - \mathbf{y}_{1\mathrm{d}})), && (29)\\
\mathbf{q}^{(\delta)} &= -Q_{11}(\mathbf{y} - \mathbf{y}_{\mathrm{d}}) - Q_{12}\mathbf{v} - \bar{A}^{\top}\mathbf{q}, && (30)\\
\mathbf{y}_1^{(\delta)} &= \bar{A}\mathbf{y}_1 + \bar{B}\mathbf{v}. && (31)
\end{aligned}
$$

By substituting equation (29) into (30) and (31) we have the following boundary problem:

$$
\begin{aligned}
\mathbf{y}_1^{(\delta)} &= (\bar{A} - \bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top})\mathbf{y}_1 - \bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\top}\mathbf{q}\\
&\quad - \bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\mathbf{y}_{1\mathrm{d}},\\
\mathbf{q}^{(\delta)} &= (-\bar{Q}_{11} + \bar{Q}_{12}R^{-1}\bar{Q}_{12}^{\top})\mathbf{y}_1 \qquad (32)\\
&\quad +(-\bar{A}^{\top} + \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{B}^{\top})\mathbf{q}\\
&\quad -\bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\mathbf{y}_{1\mathrm{d}}
\end{aligned}
$$

with boundary conditions $\mathbf{y}_1(0) = \mathbf{y}_{10}$ and

$$
\lim_{t\to\infty} \mathbf{q}(t) = \mathbf{0}. \qquad (33)
$$

In order to make $\mathbf{y}_1(t) \to \mathbf{y}_{1\mathrm{d}}$ and to satisfy (33), a state fedback is proposed such that the state $\mathbf{y}_1$ and costate $\mathbf{q}$ are related by a constant symmetric positive definite matrix $S \in \mathbb{R}^{p\times p}$ as follows:

$$
\mathbf{q} = S\mathbf{y}_1 + \mathbf{w}, \qquad (34)
$$

for a constant vectors $\mathbf{w} \in \mathbb{R}^p$. Substituting the Caputo fractional derivative of (34), i.e.

$$
\mathbf{q}^{(\delta)} = S\mathbf{y}_1^{(\delta)}, \qquad (35)
$$

into the equation (30), we have

$$
\begin{aligned}
S\mathbf{y}_1^{(\delta)} &= (-\bar{Q}_{11} + \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top})\mathbf{y}_1\\
&\quad + (-\bar{A}^{\top} + \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{B}^{\top})(S\mathbf{y}_1 + \mathbf{w})\times\\
&\quad \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\mathbf{y}_{1\mathrm{d}}. \qquad (36)
\end{aligned}
$$

By substituting (32) into (36) we get

$$
\begin{aligned}
&\Big[S(\bar{A} - \bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}) - S\bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\top}S\\
&\quad + (\bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}) + (\bar{A}^{\top} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{B}^{\top})S\Big]\mathbf{y}_1\\
&\quad = \Big[S\bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\top} + (-\bar{A}^{\top} + \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{B}^{\top})\Big]\mathbf{w}\\
&\qquad + \Big[S\bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\Big]\mathbf{y}_{1\mathrm{d}}. \qquad (37)
\end{aligned}
$$

In order to the relation (37) holds for each $\mathbf{y}_1$ on t, it must hold

$$
\begin{aligned}
O &= S(\bar{A} - \bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}) - S\bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\top}S\\
&\quad + (\bar{Q}_{11} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}) + (\bar{A}^{\top} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{B}^{\top})S \qquad (38)
\end{aligned}
$$

and

$$
\begin{aligned}
O &= \Big(S\bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\top} + (-\bar{A}^{\top} + \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{B}^{\top})\Big)\mathbf{w}\\
&\quad + \Big(S\bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top} - \bar{Q}_{12}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\Big)\mathbf{y}_{1\mathrm{d}}. \qquad (39)
\end{aligned}
$$

Using (29) and (31) one can see that the tracking control

$$
\mathbf{v}^{*} = -\bar{Q}_{22}^{-1}(\bar{B}^{\intercal}S + \bar{Q}_{12}^{\top})\mathbf{y}_1 - \bar{Q}_{22}^{-1}\bar{B}^{\intercal}\mathbf{w} + \bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\mathbf{y}_{1\mathrm{d}} \qquad (40)
$$

minimize $\mathcal{J}^{a}_{\mathbf{v},\mathbf{y}_1}$ if the matrix S is the solution of the equation (38), and $\mathbf{y}_1$ is the solution of the following fractional order differential equation:

$$
\begin{aligned}
\mathbf{y}_1^{(\delta)} &= \big(\bar{A} - \bar{B}\bar{Q}_{22}^{-1}(\bar{B}^{\intercal}S + \bar{Q}_{12}^{\top})\big)\,\mathbf{y}_1\\
&\quad - \bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\intercal}\mathbf{w} + \bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\mathbf{y}_{1\mathrm{d}}, \qquad (41)
\end{aligned}
$$

with the initial condition $\mathbf{y}_1(0) = \mathbf{y}_{10}$. Using the equation (10), the solution of equation (41) is given by

$$
\mathbf{y}_1^{*}(t) = \mathcal{E}_{\delta}(\mathcal{K}t^{\delta})\mathbf{y}_{10} + t^{\delta}\mathcal{E}_{\delta,1+\delta}(\mathcal{K}t^{\delta}) \circledast \mathcal{U}, \qquad (42)
$$

where

$$
\mathcal{K} = \bar{A} - \bar{B}\bar{Q}_{22}^{-1}(\bar{B}^{\intercal}S + \bar{Q}_{12}^{\top})
$$

and

$$
\mathcal{U} = -\bar{B}\bar{Q}_{22}^{-1}\bar{B}^{\intercal}\mathbf{w} + \bar{B}\bar{Q}_{22}^{-1}\bar{Q}_{12}^{\top}\mathbf{y}_{1\mathrm{d}}.
$$

The equations (40) and (42) constitute the pair $(\mathbf{v}^{*}, \mathbf{y}_1^{*})$ which satisfy the fractional order system (21) such that the state $\mathbf{y}_1 \to \mathbf{y}_{1\mathrm{d}}$ if $t \to \infty$ and the performance index (20) is minimized.

Furthermore, using the transformations (12) and (18), the optimal pair $(\mathbf{u}^{*}, \mathbf{y}^{*})$ which is the solution of the tracking control problem for fractional order descriptor systems (2) and (3) is given by

$$
\begin{aligned}
\begin{bmatrix}\mathbf{y}^{*}\\ \mathbf{u}^{*}\end{bmatrix} &= \begin{bmatrix}N & O\\ O & I_r\end{bmatrix}\begin{bmatrix}\mathbf{y}_1^{*}\\ \mathbf{y}_2^{*}\\ \mathbf{u}^{*}\end{bmatrix}\\
&= \begin{bmatrix}N & O\\ O & I_r\end{bmatrix}\begin{bmatrix}I_p & O\\ -\widehat{A}^{\dagger}A_{21} & \Upsilon\end{bmatrix}\begin{bmatrix}\mathbf{y}_1^{*}\\ \mathbf{v}^{*}\end{bmatrix}\\
&= \begin{bmatrix}N & O\\ O & I_r\end{bmatrix}\left[\begin{array}{c|c}I_p & O\\ \mathcal{A}_1 & \Upsilon_1\\ \hline \mathcal{A}_2 & \Upsilon_2\end{array}\right]\begin{bmatrix}\mathbf{y}_1^{*}\\ \mathbf{v}^{*}\end{bmatrix}\\
&= \left[\begin{array}{c|c}N\begin{bmatrix}I_p\\ \mathcal{A}_1\end{bmatrix} & N\begin{bmatrix}O\\ \Upsilon_1\end{bmatrix}\\ \hline \mathcal{A}_2 & \Upsilon_2\end{array}\right]\begin{bmatrix}\mathbf{y}_1^{*}\\ \mathbf{v}^{*}\end{bmatrix},
\end{aligned}
$$

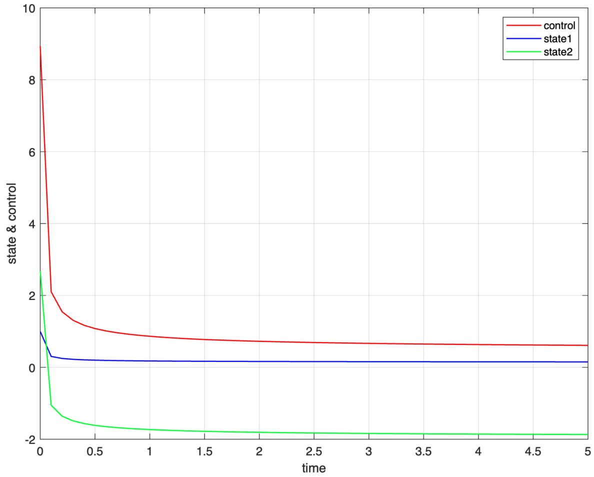

Figure 1. State and control for $\delta = 0.6$

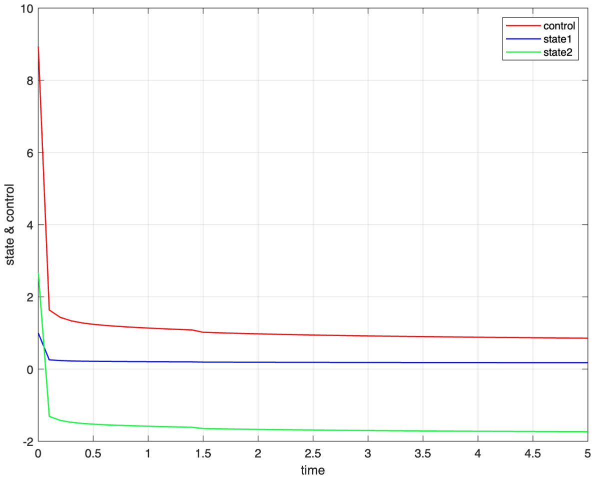

Figure 2. State and control for $\delta = 0.3$

where $\mathcal{A}_1 \in \mathbb{R}^{(n-p)\times p}, \mathcal{A}_2 \in \mathbb{R}^{r\times p}, \Upsilon_1 \in \mathbb{R}^{(n-p)\times r}$ and $\Upsilon_2 \in \mathbb{R}^{r\times r}$, $\begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{bmatrix} = -\hat{A}^{\dagger} A_{21}$, $\Upsilon = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}$, or separately given by

$$\mathbf{u}^* = \Big(\mathcal{A}_2 - \Upsilon_2 \bar{Q}_{22}^{-1}(\bar{B}^\intercal S + \bar{Q}_{12}^\top)\Big)\mathbf{y}_1^* - \Upsilon_2 \bar{Q}_{22}^{-1}\left(\bar{B}^\intercal \mathbf{w} + \bar{Q}_{12}^\top \mathbf{y}_{1\mathrm{d}}\right) \quad (43)$$

and

$$\mathbf{y}^* = N\left(\begin{bmatrix} I_p \\ \mathcal{A}_1 \end{bmatrix} - \begin{bmatrix} O \\ \Upsilon_1 \end{bmatrix} \bar{Q}_{22}^{-1}(\bar{B}^\intercal S + \bar{Q}_{12}^\top)\right)\mathbf{y}_1^* - N\begin{bmatrix} O \\ \Upsilon_1 \end{bmatrix} \bar{Q}_{22}^{-1}\left(\bar{B}^\intercal \mathbf{w} - \bar{Q}_{12}^\top \mathbf{y}_{1\mathrm{d}}\right). \quad (44)$$

One can see that $\mathbf{u}^*$ is stated in terms of Mittag-Leffler function. This due to $\mathbf{y}_1^*$ is stated in terms of Mittag-Leffler function. The equations (43) and (44) constitute the explicit formula of the optimal tracking control-state pair $(\mathbf{u}^*, \mathbf{y}^*)$ which satisfy the fractional order system (3) such that the state $\mathbf{y} \to \mathbf{y}_\mathrm{d}$ if $t \to \infty$ and the performance index (2) is minimized.

As an illustration of the above procedure, let us consider the tracking control problem (2) and (3) where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ Q = 2,$$
$$R = 1,\ \mathbf{y}_0 = \begin{bmatrix} 1 \\ 2.8 \end{bmatrix},\ \mathbf{y}_\mathrm{d} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Choosing $M = N = I_3$, the solution of the equation (38) is $S = 8.362$, and from (39) we have $\mathbf{w} = 2.078$. Moreover, the equation (41) becomes

$$y_1^{(\delta)} = -9.818 y_1 + 1.342,\ y_1(0) = 1,$$

which give the following solution:

$$y_1 = \mathcal{E}_\delta(-9.818t^\delta) + 1.342t^\delta \mathcal{E}_{\delta,\delta+1}(-9.818t^\delta).$$

Thus, using this y_1, the tracking optimal-state pair is given by

$$u^* = 9.818\Big(\mathcal{E}_\delta(-9.818t^\delta) + 1.342t^\delta \mathcal{E}_{\delta,\delta+1}(-9.818t^\delta)\Big) - 0.8802,$$

$$\mathbf{y}^* = \begin{bmatrix} \mathcal{E}_\delta(-9.818t^\delta) + 1.342t^\delta \mathcal{E}_{\delta,\delta+1}(-9.818t^\delta) \\ 5.364(\mathcal{E}_\delta(-9.818t^\delta) + 1.342t^\delta \mathcal{E}_{\delta,\delta+1}(-9.818t^\delta)) \end{bmatrix} - \begin{bmatrix} 0 \\ 2.684 \end{bmatrix}.$$

The trajectories of the state $\mathbf{y}^*$ and the control u^* for some δ is shown in Figure 1 and Figure 2. One can see that $\mathbf{y}^* \to \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ if $t \to \infty$.

5 Conclusion

We have found the explicit formula of the optimal tracking control-state pair of tracking control problem for fractional order descriptor systems. The optimal tracking control-state pair is stated in terms of Mittag-Leffler function. The results show that using the tracking optimal control $\mathbf{u}^*$ result the state $\mathbf{y} \to \mathbf{y}_\mathrm{d}$ if $t \to \infty$ and the performance index $\mathcal{J}_{\mathbf{u},\mathbf{y}}$ (2) is minimized. An example that illustrating the result has been presented.

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References

Aguilar, J.F. G., Hernandez, R. R. and Lieberman, D. G. (2014) A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response. In *Revista Mexicana de Fisica*, **60**, pp. 32-38.

Ayas, M. S., Altas, I. H. and Sahin, E. (2018) Fractional order based trajectory tracking control of an ankle rehabilitation robot. *Transactions of the Institute of Measurement and Control*, **40**(2), pp. 550-564.

Batiha, I. M., El-Khazali, R., Alsaedi, A. and Shaher, M. (2018) The general solution of singular fractional-order linear time-invariant continuous systems with regular pencils. *Entropy*, **20**(6), pp. 1-13.

Birgani, S. N., Moaveni, B. and Sedigh, K. (2018) Infinite horizon linear quadratic tracking problem: a discounted cost function approach. *Optimal Control: Applications and Methods*, **39**(4), pp. 1549-1572.

Diethelm, K. (2010) *The analysis of fractional differential equations*. Springer. Heidelberg.

Dul, F., Lichota, P. and Rusowics, A. (2020) Generalized linear quadratic control for full tracking problem in aviation *Sensors*, **20**(2955), 1-16.

Fang, Q., Zhang, B. and Feng, J. (2014) Singular LQ problem for irregular singular systems *Journal of Applied Mathematics*, **853415**, pp. 1-9.

Gartemani, M. K., Khajehoddin, S. A., Jain, P. and Bakhshai, A. (2011), Linear quadratic output tracking and disturbance rejection *International Journal of Control*, **84**(8), 1442-1449.

Gomez, F., Rosales, J. and Guia, M., (2013), RLC electrical circuit of non-integer order, *Central European Journal of Physics*, **11**, 1361-1365.

Jesus, I. S., Machado, J. A. T. and Barbosa, R. S., (2008) Fractional modelling of the electrical conduction in NaCl electrolyte. In *Proceeding of 6th EUROMECH Nonlinear Dynamics Conference*, Saint Petersburg, Russia.

Kaczorek, T. and Rogowski, K. (2015) *Fractional linear systems and electrical circuits*. Springer. London.

Klema, V.C. and Laub, A. J. (1980) The singular value decomposition: its computation and some applications. *IEEE Transactions on Automatic Control*, **25**(2), pp. 164-176.

Li, Y. and Chen, Y. Q. (2008) Fractional order linear quadratic regulator. In *Proceeding of IEEE/ASME International Conference on Mechtronic and Embedded Systems and Applications*, Beijing, China, Oct. 12-15, pp. 363-368.

Liao, F. and Xie, H. (2019) Preview tracking control for a class of fractional-order linear systems. *Advances in Difference Equations*, **472**, 1-19.

Luongo, A. and D'Annibale, F., (2011) Nonlinear bifurcation of damped visco-elastic planar beams under simultaneous gravitational and follower forces. In *Proceeding of 5th International Conference on Physics and Control*, Leon, Spain.

Martinez, J. E. E, Mendoza, L. J. M., Orduna, M. I. C., Achach, M. R., Behera, D., Camacho, J. R. L., Calderon, H. D. L. and Cruz, V. M. L., (2020), Fractional differential equation modeling of viscoelastic fluid in mass-spring-magnetorheological damper mechanical system, *The European Physical Journal Plus*, **135**(847), 1-14.

Matychyn, I. and Onyshchenko, V. (2018) Optimal control of linear systems with fractional derivatives. *Fractional Calculus & Applied Analysis*, **1**, pp. 134-150.

Milici, C., Draganescu, G. and Machado, J. T. (2019) *Introduction to fractional differential equations*. Springer. Switzerland.

Monje, C. A., Chen, Y. Q., Vinagre, B. M., Xue, D. and Feliu, V. (2010) *Fractional-orde Systems and Controls*. Springer. London.

Muhafzan, Nazra, A., Yulianti, L. Zulakmal and Revina R. (2020) On LQ optimization problem subject to fractional order irregular singular systems *Archives of Control Sciences*, **30**(4), 745-756.

Nazra, A., Zulakmal, Yulianti, L. and Muhafzan (2020) Linear quadratic optimization for fractional order differential algebraic system of Riemann-Liouville type. *Cybernetics and Physics*, **9**(4), 192-197.

Traore, A. and Sene, N. (2020) Model of economic growth in the context of fractional derivative *Alexandria Engineering Journal*, **59**(6), 4843-4850.

Trivedi, P., Vyawahare, V. and Patil, M. D. (2020) Tracking control for fractional order systems with high relative degree outputs. *IFAC Papers Online*, **53**, 170-175.

Xu, D. and Yang, X. (2016) Controllability of fractional descriptor system. *Advances in Theoretical and Applied Mathematics*, **11**(4), 373-382.

Yulianti, L., Nazra, A., Zulakmal, Bahar, A. and Muhafzan. (2019) On discounted LQR control problem for disturbanced singular system. *Archives of Control Sciences*, **29**(1), 147-156.