DISTRIBUTION OF DIFFERENT NETWORK PARAMETERS IN SYNCHRONIZATION OPTIMIZED NETWORKS

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Abstract
Recently a new family of graphs, namely the entangled networks have been introduced which is shown to exhibit better synchronizability. These networks have extreme homogeneous structures, i.e. most of the nodes are linked with similar number of links. Other network topological parameters, such as betweenness centrality, shortest path length etc. are shown to be distributed in a relatively narrow interval. In this paper, we analyze different network topological parameters for synchronization optimized networks of coupled identical systems. The degree mixing patterns of these optimized networks shows that the network is disassortative in nature, i.e. the high degree nodes tend to connect with low degree nodes.

Key words
Synchronization, coupled systems, optimization

1 Introduction

Many complex systems can be considered as networks of interacting dynamical units. Depending on the topologies, the complex networks can be categorized as regular network, random network [Erdős, and Rényi, 1960], small world network [Watts and Strogatz, 1998], and scale free network [Barabási and Albert, 1999]. The regular network has well define structure and the nodes are connected with some fixed rule while in random networks two nodes are randomly connected with some probability. The small world network falls in between the regular network and the random network. In a small world network, along with an underlying regular structure, there exists few random connections between the nodes. The scale free or power law network is observed in many

The interacting dynamical systems are capable of exhibiting many rich behaviors which a single isolated unit can not show. List of these behaviors includes synchronization [Pecora and Carroll, 1990], amplitude death [Reddy, Sen and Johnston, 1998], chimera state [Abrams and Strogatz, 2004], multistability [Kim, Park and Ryu, 1997], phase flip [Prasad, Kurths, Dana, and Ramaswamy, 2006], etc. Synchronization is a fundamental nonlinear phenomenon which occurs between interacting dynamical systems when the interacting dynamical systems adjust some given properties of their trajectories to a common behavior due to coupling. Recently, synchronization processes of locally interacting dynamical systems has become the focus of intense research in physical, biological, chemical, technological and social sciences []. The simplest and the most studied form of the synchronization is the complete synchronization (CS) which is observed in coupled identical dynamical systems and their state variables become equal as they evolve with time.

In a recent paper, Barahona and Pecora [Barahona and Pecora, 2002] showed that the small world networks can exhibit better synchronizability because of existence of smaller path length between the nodes. Later, Nishikawa et. al. [Nishikawa, Motter, Lai and Hoppensteadt, 2003] observed that it is easier to synchronize homogeneous networks. Donnetti et. al. [Donnetti, Hurtado and Muñoz, 2005] introduces such homogeneous networks as entangled networks.

In this paper we study distribution of different network parameters, such as degree distribution, clustering, betweenness, and shortest path in synchronization optimized networks. We observed that the distribution in the optimized network become narrow indicating that the network has a homogeneous structure. The paper is organized as following, in Section II we discuss the master stability analysis of complete synchronization and discuss the method of constructing synchronization optimized networks from fixed. In Section III, we present our numerical results on chaotic Rössler systems and we conclude in Section IV.

2 Synchronization of coupled dynamical systems on networks

In this section we discuss the stability analysis of complete synchronization of coupled identical dynamical systems on networks by briefly discussing the Master Stability Function formalism [Pecora and Carroll, 1998]. The Master Stability Function (MSF) is one of the most important tools was introduced to simplify the stability analysis of complete synchronization

One very important tool for studying stability of complete synchronization of coupled identical dynamical systems. In 1998, the MSF was introduced by Pecora and Carroll for analyzing the stability of complete synchronization for coupled identical systems [Pecora and Carroll, 1998]. The MSF is defined as the largest Lyapunov exponent, calculated from a set of equations known as Master Stability Equation (MSE), as a function of coupling parameter. The negative MSF implies stable complete synchronization. The MSF simplifies the study of stability for complete synchronization by separating the effect of the network structure from that of the dynamics of individual system.

Let us consider a network of N coupled identical dynamical systems and the dynamics of the i-th systems is give as

$$\dot{x}^i = f(x^i) - \varepsilon \sum_{j=1}^{N} g_{ij} h(x^j);\ i = 1, \cdots, N \quad (1)$$

where, $x^i \in R^m$ is the m - dimensional state variable of the i - th system and $\dot{x} = f(x)$ gives the dynamics of an isolated system, ε and h are the coupling parameter and the coupling function respectively. The coupling matrix $G = [g_{ij}]$ is the Laplacian of the network and its elements are given as $g_{ii} = k_i$, where k_i is the degree of node i, when nodes i and j interact $g_{ij} = -1$, and otherwise $g_{ij} = 0$. The coupling matrix satisfies the condition $\sum_j g_{ij} = 0$, resulting the existence of an eigenvector $(1, \ldots, 1)^T$ corresponding to the eigenvalue $\gamma_1 = 0$ of the coupling matrix G. As we are interested in studying synchronization of the coupled dynamical systems, we consider the network to be connected. So, there exists only one zero eigenvalue of the coupling matrix G. Let, the rest of the eigenvalues of the coupling matrix G are $0 = \gamma_1 < \gamma_2 \leqslant \cdots \leqslant \gamma_N$.

For suitable choice of coupling function h and coupling parameter ε, the coupled systems of Eq. (1) will synchronize to a state given by $x^1 = x^2 = \cdots = x^N = s(t)$, where $s(t)$ is the solution of an isolated system $\dot{s} = f(s)$. The linear stability analysis of the synchronization can be performed by expanding Eq. (1) in Taylor's series about the synchronized solution s and diagonalizing these equations in N blocks which can be written as

$$\dot{\phi}^k = [D_x f(s) - \varepsilon \gamma_k D_x h(s)]\phi^k;\ k = 1, \cdots N. \quad (2)$$

where, D_x is the differential operator and γ_k is the k - th eigenvalue of the coupling matrix. These diagonalized equations differs only in the parameter $\varepsilon\gamma_k$. Eq. (2) can be cast as the master stability equation by introducing a complex parameter $\alpha = \varepsilon\gamma_k$ and dropping the index k

$$\dot{\eta} = [D_x f - \alpha D_x h]\eta \quad (3)$$

The master stability function (MSF) λ_{max} is the largest Lyapunov exponent calculated from Eq. (3) as a function of α [Pecora and Carroll, 1998]. It has been observed that for many dynamical systems the master stability function λ_{max} is negative within a bounded interval (α_1, α_2) [Barahona and Pecora, 2002]. The synchronization of the coupled dynamical systems is stable when all the effective couplings lie within the interval, $\alpha_1 < \varepsilon\gamma_2 \leqslant \cdots \leqslant \varepsilon\gamma_N < \alpha_2$. The synchronization is stable for a network only when the eigenvalues of the coupling matrix satisfies the condition, $\gamma_N/\gamma_2 < \alpha_2/\alpha_1$, where the quantity on the left hand side comes from purely network structures and the quantity on the right hand side depends on the system dynamics f and coupling function h. The interval of the stable synchronization increases for a network when the ration $Q = \gamma_N/\gamma_2$ decreases. The synchronizability become maximum for a network when Q is as minimum as possible.

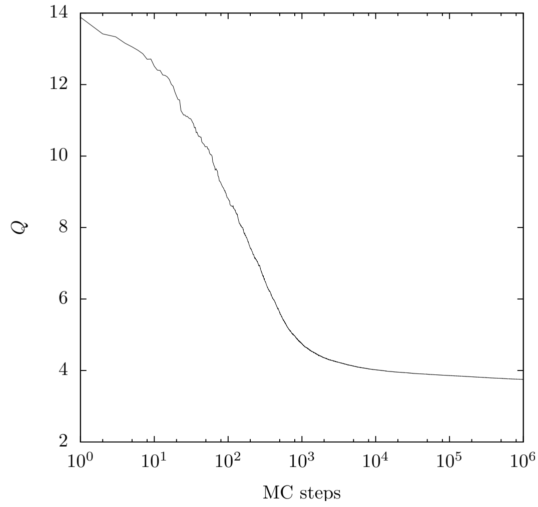

Figure 1. The eigen-ratio Q is plotted as a function of the Monte Carlo steps (averaged over 100 runs).

Starting with a connected network of N nodes and E links of arbitrary topology we rewire the links and use Metropolis algorithm to obtain the optimized networks. Let us consider that the coupling matrix of the initial starting network is G_{initial} and its eigenvalues are $0 = \gamma_1 < \gamma_2 \leqslant \cdots \leqslant \gamma_N$. The initial value of eigen-ratio Q_{initial} is obtained by taking the ratio of maximum eigenvalue with the minimum nonzero eigenvalue. We delete an existing link of this initial network and create a new link at a link vacancy. If the network become disconnected we reject it. Otherwise we determine the new value of Q_{final}. We accept the final network, when $\delta Q = Q_{\text{final}} - Q_{\text{initial}}$ is negative, otherwise we accept it with a probability $e^{(-\delta Q)/T}$, where T is a temperature like quantity. We start with high value of T and reduce T after $100N$ iterations or $10N$ accepted ones whichever occur first. The process is stopped when there are no more changes in Q for successive five temperature steps. At this we assume that a reasonably good approximation of the optimal networks has been achieved. In Fig. 1 the decrease in the eigenratio Q is shown as function of Monte Carlo iterations. The optimal networks achieved through this process is classified as a new family of graphs in Ref. [Donnetti, Hurtado and Muñoz, 2005], namely the *entangled networks* . These networks shows a very homogeneous structure, the degree distribution, betweenness centrality and the shortest path all have very narrow distribution. We briefly discuss these network parameters below.

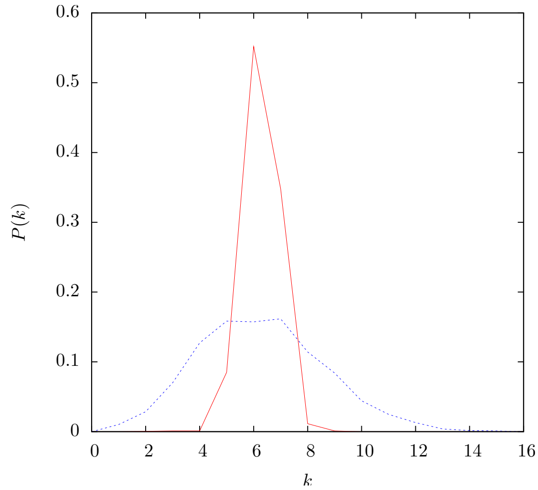

Figure 2. The degree distribution of the initial network (dashed blue line) and the optimal network (solid red line) is shown. For the optimal network the degree distribution is more narrow.

3 Numerical Results

In this section we discuss the numerical results on chaotic Rössler systems. We consider a connected network of $N = 64$ chaotic Rössler systems with coupling in x-component and total number of links $E = 202$.

3.1 Degree Distribution

The degree distribution $P(k)$ of a given network gives the probability that a randomly chosen node on the network will have k degree. In Fig. 2 the degree distribution $P(k)$ of the initial networks (dashed blue line) and the same of the optimal networks are shown. The degree distribution of the optimal network shows a sharp peak and narrow band width than that of the initial random network. The optimal networks show more homogeneous structure where most of the nodes are connected to approximately same number of links.

3.2 Clustering coefficients

Clustering coefficients gives the probability that two distinct immediate neighbors of a common node on a given network are connected. Let us consider the degree of a node i on a given network is k_i and let G' is the subgraph defined by the node i and its immediate k_i neighbors. The maximum possible number of links that can exist among the neighbors of the node i is $k_i(k_i - 1)/2$. The clustering coefficient c_i of node i is defined as the ratio of actual number of existing links e_i to the maximum number of possible links among the neighbors of node i

$$c_i = \frac{2e_i}{k_i(k_i - 1)} \tag{4}$$

We define the distribution of the clustering coefficient $P(c)$ as the probability that a randomly chosen node on a given network will have clustering coefficient c.

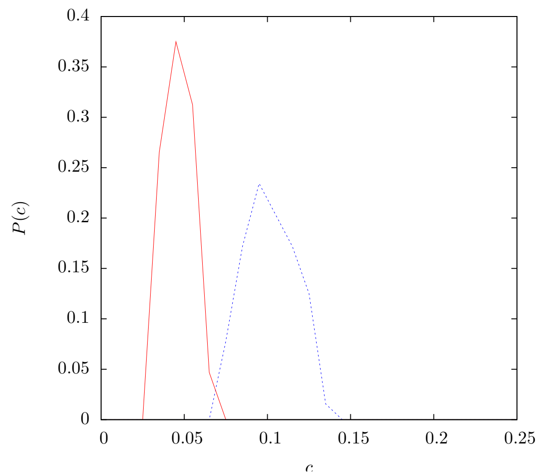

Figure 3. The distributions of the clustering coefficient are shown for initial random network (blue dashed line) and the optimized network (red solid line). There is an overall decrease in the clustering coefficient of the optimized networks and the distribution become narrower.

In Fig. 3 this distributions are shown for initial random networks (blue dashed line) and the optimized networks (red solid line). We can find that the clustering coefficients of most of the nodes in the optimized networks has become smaller than that of the initial random networks and also the distribution is narrow.

Next, we study the behavior of the clustering coefficients of a network as a function of iterations, as we approach the optimal topology. The clustering coefficient C of a network is define as the average value of the individual clustering coefficients of its nodes, $C = 1/N \sum_i^N c_i$. In Fig. 4, we have shown the changes in the clustering coefficient (C) of a network as a function of iterations while leading towards the synchronization optimized network. The clustering coeffi-

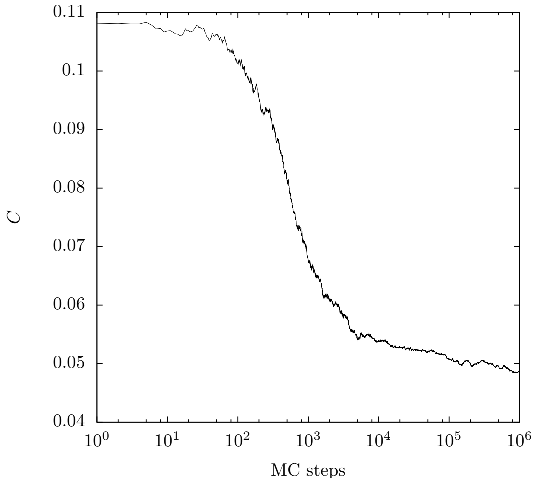

Figure 4. The clustering coefficient C of a network is plotted as a function of Monte - Carlo steps. The clustering coefficient reduces and saturates to a lower value.

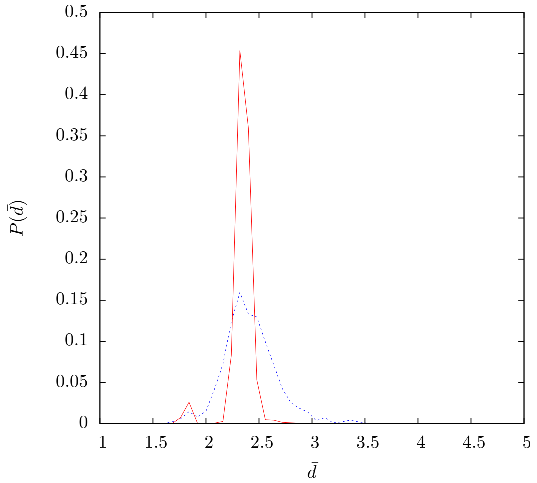

Figure 5. The distributions of the average shortest path length $P(\bar{d})$ of the initial random networks (blue dashed line) and the synchronization optimized networks (red solid line) are shown.

cient reduces and saturates to a lower value. Thus the number of short loops in the synchronization optimized networks decreases.

3.3 Average shortest path length

The shortest path length between a pair of nodes on a network plays an important role in information transfer and communication between the selected pair on nodes. The shortest path length gives an optimal pathway for fast transfer of information. Let, d_{ij} provides the shortest path length between nodes i and j. We calculate the average shortest path length of node i from the other nodes in the network as

$$\bar{d}_i = \frac{2}{N(N-1)} \sum_{j,j\neq i}^{N} d_{ij} \quad (5)$$

The distribution of the average shortest path $P(\bar{d})$ gives the probability that a randomly chosen nodes on the network will have average shortest path length $\bar{d}$. In Fig. 5 the distributions are plotted for initial random networks (blue dashed line) and the optimal networks (red solid line). The optimized networks shows a sharp peak in the distribution of average shortest path length while the same distribution is flatter for initial random networks.

3.4 Betweenness and closeness centrality

Betweenness centrality C_B and closeness centrality C_C are two important network parameters. These parameters provides the important nodes which are responsible for intra-network information transfer.

In Fig. 6 and Fig. 7 we have plotted the distribution of the betweenness centrality $P(C_B)$ and the closeness centrality $P(C_C)$ respectively. In both figures the distributions for the initial network are shown as blue dashed lines and those for the optimal network are as the red solid lines. The distributions for the initial network are flatter than the distributions of the optimal network. All nodes in the optimized networks have similar betweenness centrality.

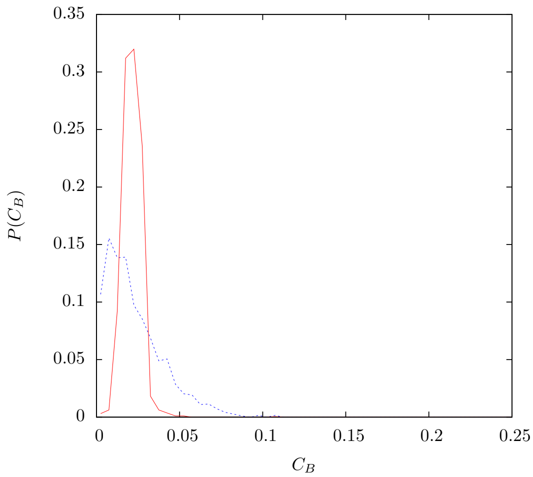

Figure 6. The distributions of betweenness centrality $P(C_B)$ are shown for the initial network (blue dashed line) and the optimal network (red solid line).

3.5 Degree mixing in networks

In this section we study degree mixing in networks [Newman 2002; Newman 2003]. Degree mixing gives the tendency of nodes to be connected with similar nodes in a given network. The measure of the degree mixing in networks given by the correlation coefficients of the degrees at either ends of an edge. Let the degrees of nodes at the ends of the ith edge in a network is given by j_i and k_i, following Ref [Newman 2002] the degree mixing coefficient r can be calculated

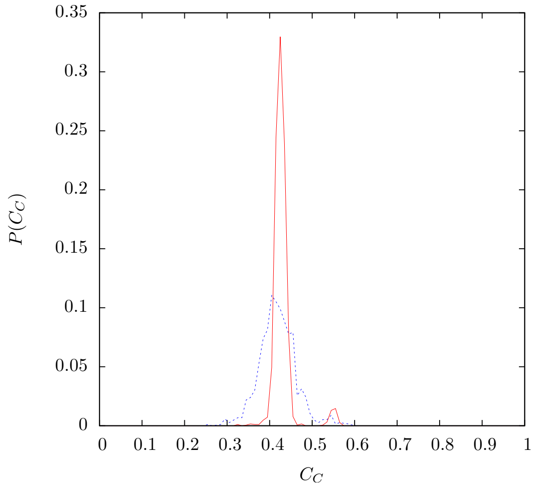

Figure 7. The distribution of closeness centrality $P(C_C)$ is shown for initial random network (blue dashed line) and synchronization optimized networks (red solid line).

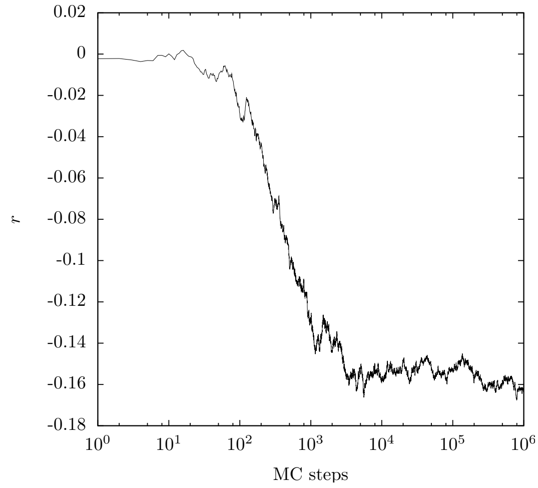

Figure 8. The degree mixing correlation coefficient r is shown as a function of Monte Carlo iterations. As we start with an initial random network, the assortative coefficient r remains zero for few the initial iterations after that it started decreasing and saturates to a negative value.

as

$$r = \frac{M^{-1}\sum_i j_i k_i - [M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2}{M^{-1}\sum_i \frac{1}{2}(j_i^2 + k_i^2) - [M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2}, \tag{6}$$

where, M is the total number of edges in the network. When like degrees nodes of the networks are got connected the correlation coefficient r is positive and the network is called assortative network. The network is called disassortative network when the coefficient r is negative. This happen when high degree nodes tend to connect with low degree nodes. For networks which show no assortative mixing the correlation coefficient r is zero. The random networks of Erdős and Rényi and the scale free network model of Barabási and Albert shows no assortative mixing. It has been observed that many naturally evolving networks, such as Internet, WWW, protein interaction, neural networks, etc. shows disarrortative mixing of degree [Newman 2002].

In Fig. 8 this degree mixing correlation coefficient r is shown as a function of Monte Carlo iterations. As we start with an initial random network, the coefficient r remains zero for few the initial iterations after that it started decreasing and saturates to a negative value. Thus, we conclude that the optimized networks are disassortative in nature.

In some earlier paper [Bernardo, Garofalo, and Sorrentino, 2005; Sorrentino, Bernardo, and Garofalo, 2007] it has been observed that disassortative mixing of degrees of nodes increases the synchronizability of the network. Our finding supports this earlier observation. In Ref. [Chavez, Hwang, Martinerie, and Boccaletti, 2006], it has been observed that there exists an threshold of the assortative coefficient r below which the network loses its synchronizability.

4 Conclusion

In conclusion we have studied synchronizability of complex undirected network. The synchronization can be enhanced by rewiring the links of a network. The optimal network which shows better synchronizability, is a very homogeneous network. The degree distribution, clustering coefficients, betweenness centrality, shortest path lengths of this optimal network are distributed in a narrow range. The degree mixing correlation coefficient of the optimal network is negative which implies that in the optimal networks the nodes with high degrees tend to connect with nodes with low degree. Many naturally evolving networks also show negative degree mixing correlation coefficient.

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