ANALYSIS OF RECTANGULAR PLATE VIBRATIONS IN A FRACTIONAL DERIVATIVE VISCOUS MEDIUM

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Abstract

An original method for solving the problem on transient vibrations of rectangular plates in viscous medium, when the viscoelastic features are described by fractional derivatives, has been presented in this article. It is based on the assumption that each mode of vibrations has its own damping coefficient and its own retardation time. The Laplace integral transform method is employed as a method of solution, which is followed by the expansion of the desired functions in series with respect to eigenfunctions of the problem. As this takes place, during the transition from image to pre–image, the nonrationalized characteristic equation with fractional powers is solved by the method suggested by the authors. The solution is obtained in the form of the sum of two terms, one of which governs the drift of the system's equilibrium position and is defined by the quasi–static processes of creep occurring in the system, and the other term describes damped vibrations around the equilibrium position and is determined by the systems's inertia and energy dissipation.

Key words

Fractional derivative viscoelasticity

1 Introduction

The notion of modal viscosity is often used for analyzing damped vibrations of structures, i.e., it is assumed that each mode of vibrations has its own viscous coefficient. Such an assumption is corroborated by experimental data obtained via ambient tests of various structures and its elements [Abdel–Ghaffar and Housner, 1978; Abdel–Ghaffar and Scanlan, 1985; Clough and Penzien, 1975]. For theoretical investigation of linear vibrations of mechanical systems, along with the modal viscosity the Rayleigh hypothesis is of frequent use [Clough and Penzien, 1975], which for ordinary Newtonian viscosity $\mu\dot{x}$, where x is the displacement, and an overdot denotes time-derivative, lies in the fact that the viscosity coefficient μ is a linear combination of the system's rigidity E and its mass m, i.e.,

$$\mu = \alpha E + \beta m, \tag{1}$$

or

$$\frac{\mu}{m} = \omega^2 \tau, \tag{2}$$

where α and β are coefficients of proportionality, $\omega^2 = Em^{-1}$, and $\tau = \alpha + \beta\omega^{-2}$ is the retardation time.

If Newtonian viscosity is defined by the Riemann–Liouville fractional derivative $\mu D^\gamma x$, where

$$D^\gamma x = \frac{d}{dt}\int\limits_0^t \frac{x(\tau)d\tau}{\Gamma(1-\gamma)(t-\tau)^\gamma}, \quad (0<\gamma\le 1), \tag{3}$$

and $\Gamma(1-\gamma)$ is the Gamma–function, then (2) takes the form

$$\frac{\mu}{m} = \omega^2 \tau^\gamma. \tag{4}$$

For a one-degree-of-freedom system, whose damping features are described by fractional derivative Kelvin-Voigt model [Rossikhin and Shitikova, 1997a]

$$F = E(x + \tau^\gamma D^\gamma x), \tag{5}$$

where F is the force, formula (4) is obtained automatically, since the equation of motion of such a system has the form

$$\ddot{x} + \omega^2\tau^\gamma D^\gamma x + \omega^2 x = f, \tag{6}$$

where f is the external force per unit mass. The analytical solution of equation (6) in the frequency domain

$$\left(p^2 + \omega^2(\tau p)^\gamma + \omega^2\right)\bar{x} = \bar{f}, \tag{7}$$

where p is the Laplace transform variable, with its inversion to the time domain is described in detail in [Rossikhin and Shitikova, 1997a].

If a system possesses an infinite number degrees-of-freedom, then the introduction of the modal viscosity and Rayleigh hypothesis allows one to obtain for each mode the characteristic equation similar to that of equation (7), i.e., the behaviour of each mode is modeled in terms of the behaviour of a mechanical oscillator, in so doing the oscillators corresponding to different modes are not depend on each other. Thus, the problem of oscillations of a viscoelastic rod with fractional derivative constitutive equations has been reduced to an infinite set of single-mass oscillators in [Rossikhin and Shitikova, 2004].

In the present paper, it will be shown that for rectangular plates, dynamic motion of which in a viscous medium is described by two coupled and one uncoupled linear equations involving fractional derivatives, that the behaviour of each mode of the coupled equations could be modeled already by the behaviour of a two-mass system [Rossikhin and Shitikova, 2001], in so doing the two-mass systems corresponding to different modes are separated from each other.

2 Governing Equations and the Method of Solution

Let us consider a rectangular isotropic plate, whose dynamic behaviour is described by three linear equations [Volmir, 1972]

$$\begin{gathered}\frac{Eh}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\,\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\,\frac{\partial^2 v}{\partial x \partial y}\right) = \rho h\,\frac{\partial^2 u}{\partial t^2} - q_1,\\ \frac{Eh}{1-\nu^2}\left(\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\,\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\,\frac{\partial^2 u}{\partial x \partial y}\right) = \rho h\,\frac{\partial^2 v}{\partial t^2} - q_2,\end{gathered} \tag{8}$$

$$D\nabla^4 w = -\rho h\,\frac{\partial^2 w}{\partial t^2} + q_3, \tag{9}$$

where $u(x,y,t)$, $v(x,y,t)$, and $w(x,y,t)$ are displacements of the points of the plate's median surface in three mutually orthogonal directions x, y, z, two of which, x and y, lie in the plate surface, and the third one, z, is out of the plate plane; q_1, q_2, and q_3 are the intensities of the given external loads applied in the $x-$, $y-$, and $z-$directions, respectively, ρ is the density, ν is Poisson's ratio, h is the plate thickness,

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad \nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

Equations (8) and (9) are subjected to the initial conditions

$$u|_{t=0} = \dot{u}|_{t=0} = 0, \quad v|_{t=0} = \dot{v}|_{t=0} = 0, \quad w|_{t=0} = \dot{w}|_{t=0} = 0, \tag{10}$$

and the boundary conditions (of Navier type) for the simply supported edges free in the $x-$direction

$$\begin{gathered} w\Big|_{x=0} = w\Big|_{x=a} = 0, \qquad v\Big|_{x=0} = v\Big|_{x=a} = 0,\\ \frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=a} = 0, \quad \frac{\partial^2 w}{\partial x^2}\Big|_{x=0} = \frac{\partial^2 w}{\partial x^2}\Big|_{x=a} = 0,\end{gathered} \tag{11}$$

and for the simply supported edges free in the $y-$direction

$$\begin{gathered} w\Big|_{y=0} = w\Big|_{y=b} = 0, \quad u\Big|_{y=0} = u\Big|_{y=b} = 0,\\ \frac{\partial v}{\partial y}\Big|_{y=0} = \frac{\partial v}{\partial y}\Big|_{y=b} = 0, \quad \frac{\partial^2 w}{\partial y^2}\Big|_{y=0} = \frac{\partial^2 w}{\partial y^2}\Big|_{y=b} = 0,\end{gathered} \tag{12}$$

where a and b are the plate's dimensions along the $x-$ and $y-$axes, respectively.

Let us introduce the dimensionless values

$$\begin{gathered} u^* = \frac{u}{a}, \quad v^* = \frac{v}{a}, \quad w^* = \frac{w}{a},\\ x^* = \frac{x}{a}, \quad y^* = \frac{y}{b}, \quad t^* = \frac{t}{a}\sqrt{\frac{E}{\rho(1-\nu^2)}},\\ q_i^* = \frac{q_i(1-\nu^2)}{E\beta_2} \quad (i = 1,2,3),\end{gathered} \tag{13}$$

and rewrite the equations of motion (8) and (9) in the dimensionless form omitting asterisks near the dimensionless values

$$\begin{gathered} u_{xx} + \frac{1-\nu}{2}\,\beta_1^2 u_{yy} + \frac{1+\nu}{2}\,\beta_1 v_{xy} = \ddot{u} - q_1,\\ \beta_1 v_{yy} + \frac{1-\nu}{2}\,v_{xx} + \frac{1+\nu}{2}\,\beta_1 u_{xy} = \ddot{v} - q_2,\end{gathered} \tag{14}$$

$$\frac{\beta_2^2}{12}\left(w_{xxxx} + 2\beta_1^2 w_{xxyy} + \beta_1^4 w_{yyyy}\right) = -\ddot{w} + q_3, \tag{15}$$

as well as the boundary equations (11) for the simply supported edges free in the $x-$direction

$$w\Big|_{x=0} = w\Big|_{x=1} = 0, \qquad v\Big|_{x=0} = v\Big|_{x=1} = 0, \tag{16}$$
$$u_x\Big|_{x=0} = u_x\Big|_{x=1} = 0, \quad w_{xx}\Big|_{x=0} = w_{xx}\Big|_{x=1} = 0,$$

and the boundary conditions (12) for the simply supported edges free in the $y-$direction

$$w\Big|_{y=0} = w\Big|_{y=1} = 0, \quad u\Big|_{y=0} = u\Big|_{y=1} = 0, \tag{17}$$
$$v_y\Big|_{y=0} = v_y\Big|_{y=1} = 0, \quad w_{yy}\Big|_{y=0} = w_{yy}\Big|_{y=1} = 0,$$

where $\beta_1 = a/b$ and $\beta_2 = h/a$ are the parameters depending on the plate's dimensions, the lower indices x and y denote differentiation with respect to the corresponding coordinate, and overdots refer to the time-derivative.

Represent the functions of the dimensionless displacements in terms of the following expansions:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{1\,mn}(t)\eta_{1\,mn}(x,y),$$
$$v(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{2\,mn}(t)\eta_{2\,mn}(x,y), \tag{18}$$
$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{3\,mn}(t)\eta_{3\,mn}(x,y),$$

where the eigenfunctions have the form

$$\eta_{1\,mn}(x,y) = \cos\pi mx \sin\pi ny,$$
$$\eta_{2\,mn}(x,y) = \sin\pi mx \cos\pi ny, \tag{19}$$
$$\eta_{3\,mn}(x,y) = \sin\pi mx \sin\pi ny,$$

and m and n are integers, and $x_{1\,mn}$, $x_{2\,mn}$, and $x_{3\,mn}$ are the generalized displacements.

For solving the problem, let us apply the Laplace transform method. The displacements of the points lying in the median surface (18) in the Laplace domain have the form

$$\bar{u}(x,y,p) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{x}_{1\,mn}(p)\eta_{1\,mn}(x,y),$$
$$\bar{v}(x,y,p) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{x}_{2\,mn}(p)\eta_{2\,mn}(x,y), \tag{20}$$
$$\bar{w}(x,y,p) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{x}_{3\,mn}(p)\eta_{3\,mn}(x,y),$$

where p is the Laplace variable, and an overbar denotes the Laplace transform.

Rewrite Eqs. (14) and (15) in the Laplace domain, substitute formulae (20) in the net equations and consider that

$$\bar{q}_i(x,y,p) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{q}_{i\,mn}(p)\eta_{i\,mn}(x,y) \quad (i = 1,2,3). \tag{21}$$

Using condition of orthogonality of eigenfunctions within the domain of x and y, as a result we obtain

$$(p^2 + P_{1\,mn})\bar{x}_{1\,mn} + P_{2\,mn}\bar{x}_{2\,mn} = \bar{q}_{1\,mn}, \tag{22}$$
$$P_{2\,mn}\bar{x}_{1\,mn} + (p^2 + P_{3\,mn})\bar{x}_{2\,mn} = \bar{q}_{2\,mn},$$
$$(p^2 + P_{4\,mn})\bar{x}_{3\,mn} = \bar{q}_{3\,mn}, \tag{23}$$

where

$$P_{1\,mn} = \pi^2\left(m^2 + \frac{1-\nu}{2}\,\beta_1^2 n^2\right)$$
$$P_{2\,mn} = \pi^2\,\frac{1+\nu}{2}\,\beta_1 mn$$
$$P_{3\,mn} = \pi^2\left(\frac{1-\nu}{2}\,m^2 + \beta_1^2 n^2\right)$$
$$P_{4\,mn} = \pi^4\,\frac{\beta_2^2}{12}\,\left(m^2 + \beta_1^2 n^2\right)^2$$

The characteristic equation for the set of Eqs. (22) is written in the form

$$f_{mn}^0(p) = p^4 + (P_{1\,mn} + P_{3\,mn})p^2 + P_{1\,mn}P_{3\,mn} - P_{2\,mn}^2 = 0 \tag{24}$$

and possesses two roots

$$p_{1\,mn}^2 = -\pi^2(m^2 + \beta_1^2 n^2),$$
$$p_{2\,mn}^2 = -\pi^2\,\frac{1-\nu}{2}\,(m^2 + \beta_1^2 n^2), \tag{25}$$

which correspond to the natural frequencies

$$\omega_{1\,mn} = \pi\sqrt{m^2 + \beta_1^2 n^2},$$
$$\omega_{2\,mn} = \pi\sqrt{\frac{1-\nu}{2}\,(m^2 + \beta_1^2 n^2)} \tag{26}$$

of the in-plane horizontal vibrations of the plate, in so doing $\omega_{2\,mn}^2\,\frac{2}{1-\nu} = \omega_{1\,mn}^2 = \omega_{mn}^2$.

The natural frequency of the out-of-plane vertical vibrations

$$\Omega_{mn} = \frac{\pi^2\beta_2}{2\sqrt{3}}\left(m^2 + \beta_1^2 n^2\right) = \frac{\beta_2}{2\sqrt{3}}\,\omega_{mn}^2 \tag{27}$$

corresponds to the root of the characteristic equation for Eq. (23).

Let us introduce the modal viscosity μ in Eqs. (22) by formula $\mu = \omega_{mn}^2 \tau_{mn}^\gamma p^\gamma$, but in Eq. (23) by formula $\mu = \Omega_{mn}^2 \tau_{3\ mn}^\gamma p^\gamma$, where τ_{mn} and $\tau_{3\ mn}$ are the retardation times of the mnth mode of the in-plane and out-of-plane vibrations, respectively.

As a result we obtain

$$\begin{aligned}(p^2 + \omega_{mn}^2 \tau_{mn}^\gamma p^\gamma + P_{1\ mn})\bar{x}_{1\ mn} \\ +P_{2\ mn}\bar{x}_{2\ mn} = \bar{q}_{1\ mn}, \\ P_{2\ mn}\bar{x}_{1\ mn} + (p^2 + \omega_{mn}^2 \tau_{mn}^\gamma p^\gamma \\ +P_{3\ mn})\bar{x}_{2\ mn} = \bar{q}_{2\ mn},\end{aligned} \quad (28)$$

$$(p^2 + \Omega_{mn}^2 \tau_{3\ mn}^\gamma p^\gamma + P_{4\ mn})\bar{x}_{3\ mn} = \bar{q}_{3\ mn}. \quad (29)$$

3 Analysis of the Characteristic Equations

The characteristic equation for the set of Eqs. (28) has the form

$$\begin{aligned}f_{mn}(p) = p^4 + 2\omega_{mn}^2 \tau_{mn}^\gamma p^{2+\gamma} + \omega_{mn}^4 \tau_{mn}^{2\gamma} p^{2\gamma} \\ +(p^2 + \omega_{mn}^2 \tau_{mn}^\gamma p^\gamma)(P_{1\ mn} + P_{3\ mn}) \\ +P_{1\ mn}P_{3\ mn} - P_{2\ mn}^2 = 0,\end{aligned} \quad (30)$$

but the characteristic equation for Eq. (29) is written in the form

$$f_{3mn}(p) = p^2 + \Omega_{mn}^2 \tau_{3mn}^\gamma p^\gamma + \Omega_{mn}^2 = 0. \quad (31)$$

Considering that

$$\begin{aligned}P_{1\ mn} + P_{3\ mn} = \omega_{1\ mn}^2 + \omega_{2\ mn}^2 = \frac{3-\nu}{2}\,\omega_{mn}^2, \\ P_{1\ mn}P_{3\ mn} - P_{2\ mn}^2 = \frac{1-\nu}{2}\,\omega_{mn}^4,\end{aligned} \quad (32)$$

rewrite Eq. (30) in the following form

$$\begin{aligned}f_{mn}(p) = p^4 + 2\omega_{mn}^2 \tau_{mn}^\gamma p^{2+\gamma} + \omega_{mn}^4 \tau_{mn}^{2\gamma} p^{2\gamma} \\ +(p^2 + \omega_{mn}^2 \tau_{mn}^\gamma p^\gamma) a\omega_{mn}^2 + b\omega_{mn}^4 = 0,\end{aligned} \quad (33)$$

where

$$a = \frac{3-\nu}{2}, \qquad b = \frac{1-\nu}{2}, \qquad b = a - 1.$$

Let us change the variables in the characteristic equation (33) using the formulas

$$p = p^* \omega_{mn}, \qquad \tau_{mn} = \tau^* \omega_{mn}^{-1}. \quad (34)$$

As a result we obtain the basic equation

$$\begin{aligned}f^*(p^*) = p^{*4} + 2\tau^{*\ \gamma} p^{*\ 2+\gamma} + \tau^{*\ 2\gamma} p^{*\ 2\gamma} \\ +a\left(p^{*\ 2} + \tau^{*\gamma} p^{*\gamma}\right) + b = 0.\end{aligned} \quad (35)$$

Equation (35) can be represented in the form

$$f^*(p^*) = (p^{*\ 2} + \tau^{*\ \gamma} p^{*\ \gamma} + 1)(p^{*\ 2} + \tau^{*\ \gamma} p^{*\ \gamma} + b) = 0. \quad (36)$$

Equating to zero each expression in parenthesis of (36) yields

$$p^{*\ 2} + \tau^{*\ \gamma} p^{*\ \gamma} + 1 = 0, \quad (37)$$

$$p^{*\ 2} + \tau^{*\ \gamma} p^{*\ \gamma} + b = 0, \quad (38)$$

whence at τ^* it follows that

$$p_{10} = \pm i, \qquad p_{20} = \pm i\sqrt{b}. \quad (39)$$

Putting in Eqs. (37) and (38) $p^* = r^* e^{i\psi^*}$ and separating the real and imaginary parts, respectively, yields

$$\begin{aligned}r^{*2} R^{-1} \cos(2\psi^* - \Phi) + 1 = 0, \\ r^{*2} R^{-1} \sin(2\psi^* - \Phi) = 0,\end{aligned} \quad (40)$$

$$\begin{aligned}r^{*2} R_1^{-1} \cos(2\psi^* - \Phi_1) + 1 = 0, \\ r^{*2} R_1^{-1} \sin(2\psi^* - \Phi_1) = 0,\end{aligned} \quad (41)$$

where

$$\begin{aligned}R = \sqrt{1 + 2x\cos\gamma\psi^* + x^2}, \\ R_1 = \sqrt{b^2 + 2bx\cos\gamma\psi^* + x^2}, \\ \tan\Phi = \frac{x\sin\gamma\psi^*}{1 + x\cos\gamma\psi^*}, \\ \tan\Phi_1 = \frac{x\sin\gamma\psi^*}{b + x\cos\gamma\psi^*}, \quad x = (r^*\tau^*)^\gamma.\end{aligned}$$

From Eqs. (40) and (41) we obtain, respectively,

$$2\psi^* - \Phi = \pm\pi, \quad r^{*2} R^{-1} = 1, \quad (42)$$

$$2\psi^* - \Phi_1 = \pm\pi, \quad r^{*2} R_1^{-1} = 1. \quad (43)$$

Tending x to ∞ in (42) and (43), respectively, yields

$$\psi_{1\infty}^* = \pm\frac{\pi}{2-\gamma}, \quad r_{1\infty} = \infty, \quad (44)$$

$$\psi_{2\infty}^* = \pm\frac{\pi}{2-\gamma}, \quad r_{2\infty} = \infty. \quad (45)$$

Behaviour of the roots of Eqs. (37) and (38) as the function of the parameter τ^* is well understood [Rossikhin and Shitikova, 1997b], and therefore the investigation of the roots of the characteristic Eq. (36) is not a particular problem.

The behaviour of the roots (indicated by figures 1 and 2, respectively) in the complex plane as function of the

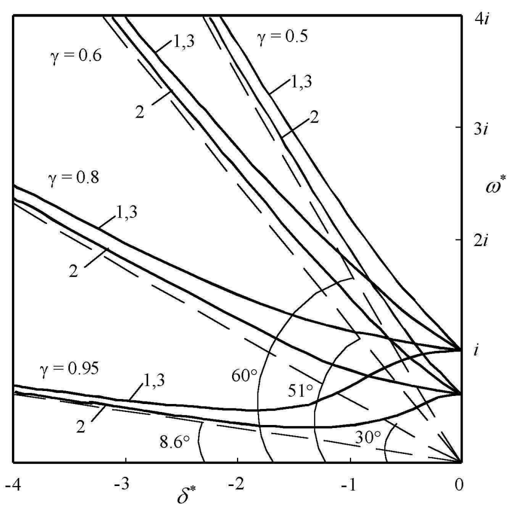

Figure 1. Behaviour of the complex conjugate roots of the basic Eqs. (35) and (48).

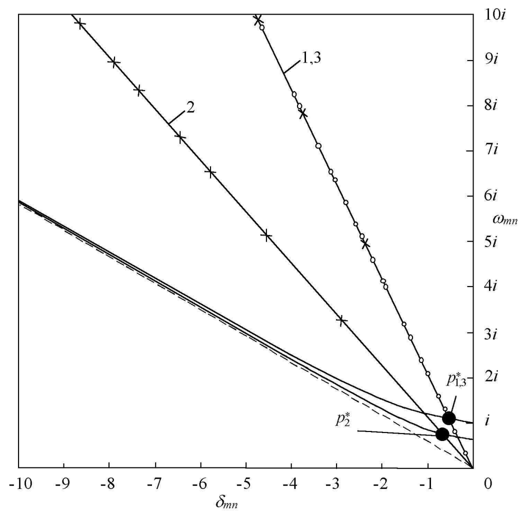

Figure 2. Behaviour of the roots of the characteristic Eqs. (30) and (31).

parameter τ^* is presented in Fig. 1, where the magnitudes of the value γ are indicated near the corresponding curves, and only the upper part of the complex half-plane is shown. Reference to Fig. 1 shows that two curves for the τ^*-dependence of the roots $p_{1,2}$ issue out of two different points defined by Eq. (39) and, as τ^* tends to ∞, come close to one and the same asymptote leaving the origin of the coordinates under the angle $\psi = \pm\pi/(2-\gamma)$ (see Eqs. (44) and (45)).

Note that the roots of a two-mass oscillator behave similarly to those of the basic Eq. (36) [Rossikhin and Shitikova, 2001].

Having determined the roots of the basic Eq. (35), all roots of the characteristic Eq. (33) could be found by the formulas

$$p_{mn} = \omega_{mn} r^* e^{\pm i\psi^*}, \quad \tau_{mn} = \omega_{mn}^{-1}\tau^*. \tag{46}$$

Reference to Eqs. (46) shows that the roots of the characteristic Eq. (33) for each fixed magnitude of τ^* locate on two straight lines issuing from two points of the basic lines under the angles $\pm\psi^*_\alpha$ $(\alpha = 1, 2)$ at the distances $r_{\alpha\ mn} = r^*_\alpha \omega_{mn}$, in so doing the magnitudes of τ_{mn} corresponding to these roots decrease by the law $\tau_{mn} = \tau^* \omega_{mn}^{-1}$ (Fig. 2).

Now let us change the variables in the characteristic equation (31) using the formulas

$$p = p^*_3 \Omega_{mn}, \quad \tau_{3\ mn} = \tau^*_3 \Omega_{mn}^{-1}. \tag{47}$$

As a result we obtain the basic equation

$$p_3^{*2} + (p^*_3\tau^*_3)^\gamma + 1 = 0, \tag{48}$$

which coincides with Eq. (37).

With the variation of τ^*_3 from 0 to ∞ the roots of the basic Eq. (48) on the complex plane p are depicted by the basic curves which are indicated by figure 3 in Fig. 1.

Having determined the roots of basic Eq. (48), one can find all roots of characteristic Eq. (31) by the formulas

$$p_{mn} = \Omega_{mn} r^*_3 e^{\pm i\psi^*_3}, \quad \tau_{3\ mn} = \Omega_{mn}^{-1}\tau^*_3. \tag{49}$$

Reference to Eq. (49) shows that all roots of the characteristic Eq. (31) locate on the complex plane on one straight line intersecting the origin and a certain initial point p^* (Fig. 2) which is determined from the basic Eq. (48) at the initial retardation time τ^*_3. The initial retardation time, in its turn, is defined by the initial temperature of the plate by Arrhenius formula

$$\tau^*_3 = \text{const} \exp(-VR^{-1}T^{-1}), \tag{50}$$

where T is the absolute temperature, R is the gas constant, and V is the energy of activation.

With the variation in the plate's initial temperature and, hence, the initial point p^*, the location of this line on the plane p changes.

The magnitudes of the roots for the characteristic Eqs. (31) and (33) are presented in Table 1 at the following magnitudes of the parameters: $\gamma = 0.8$, $\beta_1 = 1$, $\beta_2 = 0.05$, $1 \le m, n \le 6$ for the initial retardation times $\tau^* = 1$ and $\tau^*_3 = 1$, what corresponds to the basic roots $p^*_{1,3} = -0.5314 \pm 1.1097i$ and $p^*_2 = -0.6467 \pm 0.7326i$. For specificity, the roots given in Table 1 are depicted in Fig. 2 by crosses and light circles for Eqs. (33) and (31), respectively. The basic roots are marked by the dark circles.

Table 1. The magnitudes of the roots for the characteristic Eqs. (31) and (33)

m,n	ω_{mn}	Ω_{mn}	τ_{mn}	τ^3_{mn}	$r^{(1)}_{mn}$	$r^{(2)}_{mn}$	$r^{(3)}_{mn}$
1,1	4.443	0.285	0.225	3.510	5.466	4.342	0.351
1,2	7.025	0.712	0.142	1.404	8.643	6.865	0.876
2,2	8.886	1.139	0.112	0.877	10.933	8.684	1.402
1,3	9.935	1.425	0.100	0.702	12.223	9.708	1.723
2,3	11.327	1.852	0.088	0.540	13.937	11.070	2.279
1,4	12.167	2.137	0.082	0.467	14.97	11.89	2.629
3,3	13.329	2.564	0.075	0.390	16.400	13.025	1.402
2,4	14.049	2.849	0.071	0.351	17.286	13.73	3.506
3,4	15.708	3.561	0.064	0.281	19.327	15.350	4.382
1,5	16.019	3.704	0.062	0.270	19.709	15.654	4.557
2,5	16.978	4.131	0.059	0.242	20.816	16.533	5.083
4,4	17.772	4.559	0.056	0.219	21.866	17.367	5.609
3,5	18.318	4.844	0.055	0.207	22.539	17.901	5.959
1,6	19.109	5.271	0.052	0.189	23.512	18.674	6.485
2,6	19.869	5.698	0.050	0.176	24.447	19.417	7.011
4,5	20.116	5.841	0.049	0.171	24.750	19.658	7.186
3,6	21.074	6.411	0.047	0.156	25.929	20.595	7.887
5,5	22.214	7.123	0.045	0.140	27.330	21.709	8.764
4,6	22.654	7.408	0.044	0.135	27.873	22.139	9.114
5,6	24.540	8.690	0.041	0.115	30.189	23.978	10.692
6,6	26.657	10.257	0.037	0.098	32.800	26.050	12.620

4 Construction of the Solution

From Eqs.(28) and (29) we can find

$$\bar{x}_{1\ mn} = \left[\bar{q}_{1\ mn}(p^2 + \omega^2_{mn}\tau^{\gamma}_{mn}p^{\gamma} + P_{3\ mn}) - \bar{q}_{2\ mn}P_{2\ mn}\right] f^{-1}_{mn}(p),$$
$$\bar{x}_{2\ mn} = \left[\bar{q}_{2\ mn}(p^2 + \omega^2_{mn}\tau^{\gamma}_{mn}p^{\gamma} + P_{1\ mn}) - \bar{q}_{1\ mn}P_{2\ mn}\right] f^{-1}_{mn}(p), \quad (51)$$

$$\bar{x}_{3\ mn} = \bar{q}_{3\ mn} f^{-1}_{3mn}(p), \quad (52)$$

where $f_{mn}(p)$ and $f_{3mn}(p)$ are defined by (30) and (31), respectively.

From Eqs. (51) and (52) it is seen that the functions $\bar{x}_{i\ mn}$ $(i = 1,2,3)$ on the complex plane p are multivalued functions with the branch points $p = 0$ and $p = -\infty$ and possess the poles at the magnitudes $p = p_k$ which vanish the denominators of (51) and (52), i.e., they are the roots of the characteristic equations (30) and (31).

For multivalued functions possessing the branch points, the Mellin-Fourier inversion formula

$$x_{i\ mn}(t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \bar{x}_{i\ mn}(p)e^{pt}dp \ \ (i = 1,2,3) \quad (53)$$

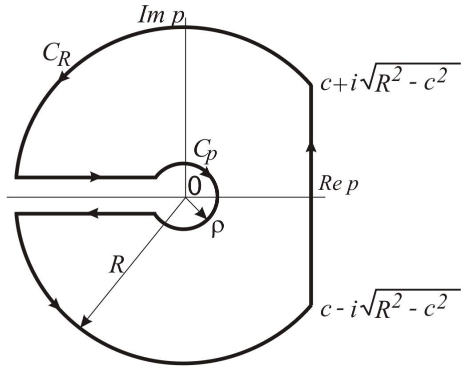

Figure 3. Contour used to calculate the complex inversion integral in the Laplace method.

is valid only for the first sheet of the Riemannian surface, i.e., when $-\pi < \arg p < \pi$. Thus, the integration contour should be chosen in the form presented in Fig. 3.

According to Jordan lemma, curvilinear integrals taken along the arcs c_R tend to zero when $R \to \infty$, and the integral calculated along c_ρ also tends to zero as $\rho \to 0$.

Using the main theorem of the theory of residues, the solution of Eqs. (53) could be written in the form

$$x_{i\ mn}(t) = x^{\rm drift}_{i\ mn}(t) + x^{\rm vibr}_{i\ mn}(t), \ \ (i = 1,2,3) \quad (54)$$

$$x^{\rm drift}_{i\ mn}(t) = \frac{1}{2\pi i}\int\limits_0^{\infty}\left[\bar{x}_{i\ mn}(se^{-i\pi}) - \bar{x}_{i\ mn}(se^{i\pi})\right] e^{-st}ds, \quad (55)$$

$$x^{\rm vibr}_{i\ mn}(t) = \sum_k \ {\rm res}\left[\bar{x}_{i\ mn}(p_k)e^{p_k t}\right], \quad (56)$$

where summation is carried out over all isolated singular points (poles).

In other words, the solution (54) is obtained in the form of the sum of two terms, where the first one (55) governs the drift of the system's equilibrium position and is defined by the quasi–static processes of creep occurring in the system, and the other term (56) describes damped vibrations around the equilibrium position and is determined by the systems's inertia and energy dissipation.

In order to obtain the solution in an explicit form, let us put in Eqs. (51) and (52) $\bar{q}_{1\ mn}(p) = 1$ $(i = 1,2,3)$ for all m and n. Such an assumption corresponds to the input signal in a form of Dirac pulse.

Knowing the roots of the characteristic Eqs. (30) and (31), and substituting Eqs.(51) and (52) in (54)-(56)

yields

$$x_{1\,mn} = \sum_{\alpha=1}^{2} d_{1\,mn}^{(\alpha)} e^{-\delta_{mn}^{(\alpha)} t} \sin(\omega_{mn}^{(\alpha)} t - \varphi_{1\,mn}^{(\alpha)}) + \frac{1}{\pi} \int_0^\infty \chi_{1\,mn}(s) e^{-st} ds, \tag{57}$$

$$x_{2\,mn} = \sum_{\alpha=1}^{2} d_{2\,mn}^{(\alpha)} e^{-\delta_{mn}^{(\alpha)} t} \sin(\omega_{mn}^{(\alpha)} t - \varphi_{2\,mn}^{(\alpha)}) + \frac{1}{\pi} \int_0^\infty \chi_{2\,mn}(s) e^{-st} ds, \tag{58}$$

$$x_{3\,mn} = d_{3\,mn} e^{-\delta_{mn}^{(3)} t} \sin(\omega_{mn}^{(3)} t - \varphi_{3\,mn}) + \frac{1}{\pi} \int_0^\infty \chi_{3\,mn}(s) e^{-st} ds, \tag{59}$$

where for $i = 1, 2$ and $\alpha = 1, 2$

$$d_{i\,mn}^{(\alpha)} = \frac{2}{\left(H_{mn}^{(\alpha)}\right)^2 + \left(Q_{mn}^{(\alpha)}\right)^2} \times \sqrt{\left(q_{mn}^{(\alpha)} H_{mn}^{(\alpha)} - h_{i\,mn}^{(\alpha)} Q_{mn}^{(\alpha)}\right)^2 + \left(h_{i\,mn}^{(\alpha)} H_{mn}^{(\alpha)} + q_{mn}^{(\alpha)} Q_{mn}^{(\alpha)}\right)^2},$$

$$\tan \varphi_{i\,mn}^{(\alpha)} = \frac{h_{i\,mn}^{(\alpha)} H_{mn}^{(\alpha)} + q_{mn}^{(\alpha)} Q_{mn}^{(\alpha)}}{q_{mn}^{(\alpha)} H_{mn}^{(\alpha)} - h_{i\,mn}^{(\alpha)} Q_{mn}^{(\alpha)}}$$

$$p_\alpha = r_{mn}^{(\alpha)} e^{\pm i \psi_{mn}^{(\alpha)}} = -\,\delta_{mn}^{(\alpha)} \pm i \omega_{mn}^{(\alpha)},$$

$$\begin{aligned} H_{mn}^{(\alpha)} = \Re f'_{mn}(p_\alpha) &= 4 {r_{mn}^{(\alpha)}}^3 \cos 3\psi_{mn}^{(\alpha)} \\ &+ 2a\omega_{mn}^2 r_{mn}^{(\alpha)} \cos \psi_{mn}^{(\alpha)} \\ &+ 2(2+\gamma)\omega_{mn}^2 \tau_{mn}^\gamma {r_{mn}^{(\alpha)}}^{1+\gamma} \cos[(1+\gamma)\psi_{mn}^{(\alpha)}] \\ &+ a\omega_{mn}^4 \tau_{mn}^\gamma \gamma {r_{mn}^{(\alpha)}}^{\gamma-1} \cos[(\gamma-1)\psi_{mn}^{(\alpha)}] \\ &+ \omega_{mn}^4 \tau_{mn}^{2\gamma} 2\gamma {r_{mn}^{(\alpha)}}^{2\gamma-1} \cos[(2\gamma-1)\psi_{mn}^{(\alpha)}], \end{aligned}$$

$$\begin{aligned} Q_{mn}^{(\alpha)} = \Im f'_{mn}(p_\alpha) &= 4 {r_{mn}^{(\alpha)}}^3 \sin 3\psi_{mn}^{(\alpha)} \\ &+ 2a\omega_{mn}^2 r_{mn}^{(\alpha)} \sin \psi_{mn}^{(\alpha)} \\ &+ 2(2+\gamma)\omega_{mn}^2 \tau_{mn}^\gamma {r_{mn}^{(\alpha)}}^{1+\gamma} \sin[(1+\gamma)\psi_{mn}^{(\alpha)}] \\ &+ a\omega_{mn}^4 \tau_{mn}^\gamma \gamma {r_{mn}^{(\alpha)}}^{\gamma-1} \sin[(\gamma-1)\psi_{mn}^{(\alpha)}] \\ &+ \omega_{mn}^4 \tau_{mn}^{2\gamma} 2\gamma {r_{mn}^{(\alpha)}}^{2\gamma-1} \sin[(2\gamma-1)\psi_{mn}^{(\alpha)}], \end{aligned}$$

$$q_{mn}^{(\alpha)} = {r_{mn}^{(\alpha)}}^2 \sin 2\psi_{mn}^{(\alpha)} + \omega_{mn} \tau_{mn}^\gamma {r_{mn}^{(\alpha)}}^\gamma \sin(\gamma \psi_{mn}^{(\alpha)}),$$

$$h_{1\,mn}^{(\alpha)} = {r_{mn}^{(\alpha)}}^2 \cos 2\psi_{mn}^{(\alpha)} + \omega_{mn} \tau_{mn}^\gamma {r_{mn}^{(\alpha)}}^\gamma \cos(\gamma \psi_{mn}^{(\alpha)}) + P_{3\,mn} - P_{2\,mn},$$

$$h_{2\,mn}^{(\alpha)} = {r_{mn}^{(\alpha)}}^2 \cos 2\psi_{mn}^{(\alpha)} + \omega_{mn} \tau_{mn}^\gamma {r_{mn}^{(\alpha)}}^\gamma \cos(\gamma \psi_{mn}^{(\alpha)}) + P_{1\,mn} - P_{2\,mn},$$

$$\chi_{i\,mn}(s) = \frac{B_{mn}(s) a_{i\,mn}(s) - A_{mn}(s) b_{mn}(s)}{[A_{mn}(s)]^2 + [B_{mn}(s)]^2},$$

$$a_{1\,mn}(s) = s^2 + \omega_{mn} \tau_{mn}^\gamma s^\gamma \cos(\gamma\pi) + P_{3\,mn} - P_{2\,mn},$$

$$a_{2\,mn}(s) = s^2 + \omega_{mn} \tau_{mn}^\gamma s^\gamma \cos(\gamma\pi) + P_{1\,mn} - P_{2\,mn},$$

$$b_{mn}(s) = \omega_{mn} \tau_{mn}^\gamma s^\gamma \sin(\gamma\pi),$$

$$\begin{aligned} A_{mn}(s) = s^4 &+ a\omega_{mn}^2 s^2 \\ &+ 2\omega_{mn}^2 \tau_{mn}^\gamma s^{2+\gamma} \cos(2+\gamma)\pi \\ &+ a\omega_{mn}^4 \tau_{mn}^\gamma s^\gamma \cos \gamma\pi \\ &+ \omega_{mn}^4 \tau_{mn}^{2\gamma} s^{2\gamma} \cos 2\gamma\pi + b\omega_{mn}^4, \end{aligned}$$

$$\begin{aligned} B_{mn}(s) = 2\omega_{mn}^2 \tau_{mn}^\gamma s^{2+\gamma} \sin(2+\gamma)\pi \\ + a\omega_{mn}^4 \tau_{mn}^\gamma s^\gamma \sin \gamma\pi \\ + \omega_{mn}^4 \tau_{mn}^{2\gamma} s^{2\gamma} \sin 2\gamma\pi, \end{aligned}$$

$$d_{3\,mn} = \frac{2}{\sqrt{H_{3\,mn}^2 + Q_{3\,mn}^2}},$$

$$\tan \varphi_{3\,mn} = -\,\frac{H_{3\,mn}}{Q_{3\,mn}},$$

$$H_{3\,mn} = \Re f'_{3\,mn}(p_3) = 2 r_{mn}^{(3)} \cos \psi_{mn}^{(3)}$$

$$+\gamma \Omega_{mn}^2 \tau_{3\,mn}^\gamma {r_{mn}^{(3)}}^{\gamma-1} \cos[(\gamma-1)\psi_{mn}^{(3)}],$$

$$Q_{3\,mn} = \Im f'_{3\,mn}(p_3) = 2 r_{mn}^{(3)} \sin \psi_{mn}^{(3)}$$

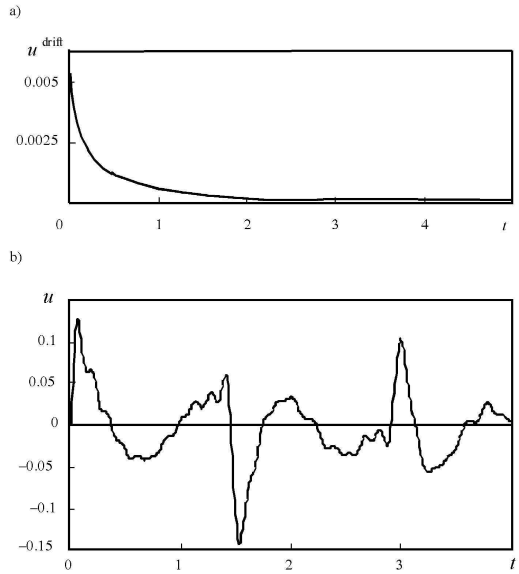

Figure 4. The time dependence of the functions (a) u^{drift}, and (b) $u(t)$.

$$+\gamma\Omega_{mn}^2\tau_{3\ mn}^{\gamma}r_{mn}^{(3)\ \gamma-1}\sin[(\gamma-1)\psi_{mn}^{(3)}],$$

$$p_3=r_{mn}^{(3)}e^{\pm i\psi_{mn}^{(3)}}=-\ \delta_{mn}^{(3)}\pm i\omega_{mn}^{(3)},$$

$$\chi_{3\ mn}(s)=\frac{B_{3\ mn}(s)}{[A_{3\ mn}(s)]^2+[B_{3\ mn}(s)]^2},$$

$$B_{3\ mn}(s)=\Omega_{mn}^2\tau_{3\ mn}^{\gamma}s^{\gamma}\sin\gamma\pi,$$

$$A_{3\ mn}(s)=s^2+\Omega_{mn}^2+\Omega_{mn}^2\tau_{3\ mn}^{\gamma}s^{\gamma}\cos\gamma\pi.$$

Substituting the generalized displacements $x_{i\ mn}$ $(i=1,2,3)$ defined by Eqs. (57)-(59) into relationships (18), we could obtain the desired displacements u, v and w of the viscoelastic plate under consideration.

The drifts of the equilibrium position $u^{\text{drift}}(t)$, $v^{\text{drift}}(t)$, and $w^{\text{drift}}(t)$, and the displacements $u(t)$, $v(t)$, and $w(t)$ are presented in Figs. (4)-(6), respectively, for the plate point $x=1/2$, $y=1/2$ with due account for 15 terms in the series (18). From Figs. (4a)-(6a) it is evident that in the case of the input signal in a form of Dirac pulse the drifts of the equilibrium position quickly decay with time, therefore, their influence on damped vibrations may be ignored when calculating the displacements.

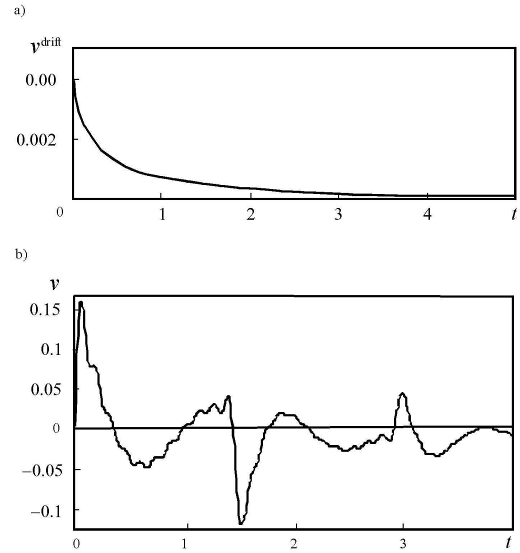

Figure 5. The time dependence of the functions (a) v^{drift}, and (b) $v(t)$.

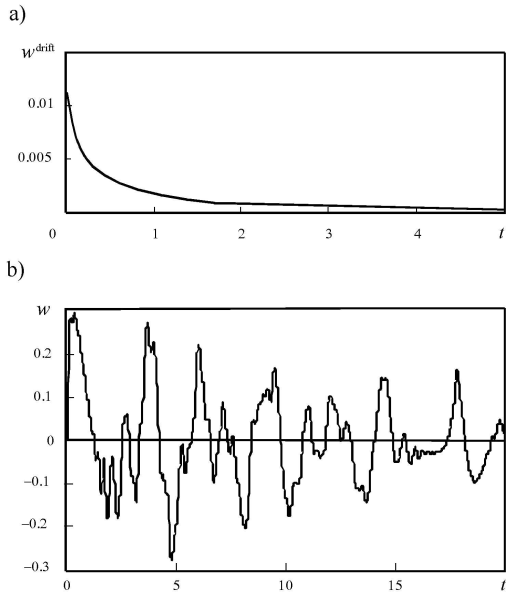

Figure 6. The time dependence of the functions (a) w^{drift}, and (b) $w(t)$.

5 Conclusion

An original method for solving the problem on transient vibrations of linear viscoelastic plates, whose viscoelastic features are described by fractional derivatives, has been presented in this article. It is based on the assumption that each mode of vibrations has its own damping coefficient and its own retardation time. This assumption considerably simplifies the solution of the problem under consideration, since all roots of the characteristic equations locate on two straight lines intersecting the origin of the coordinates and two basic points. The location of the basic points on the complex plane depends on the temperature of the plate and on the order of the fractional derivative.

The Laplace integral transform method has been employed as a method of solution, with further expansion of the desired functions in series with respect to eigenfunctions of the problem. However, unlike in the traditional approach, when rationalization of a characteristic equation with fractional powers is carried out during the transition from image to pre–image, here the nonrationalized characteristic equation has been solved by the method suggested by the authors. As a result of such an approach, the solution has been obtained in the form of the sum of two terms, one of which governs the drift of the system's equilibrium position and is defined by the quasi–static processes of creep occurring in the system, and the other term describes damped vibrations around the equilibrium position and is determined by the systems's inertia and energy dissipation.

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