GENERALIZING GRÜNWALD-LETNIKOV'S FORMULAS FOR FRACTIONAL DERIVATIVES

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Abstract
Grünwald-Letnikov's formulas yield approximations to Marchaud's derivatives, in the form of discrete convolutions of mesh l, multiplied by $l^{-\alpha}$. A continuous variant of that formulas is presented, with integrals instead of series. It involves convolution kernels which mimic essential properties of Grünwald-Letnikov's weights, but are more general.

Key words
Integro-differential equations, Fractional derivatives, Random walks, Transport processes

1 Introduction

A variety of mathematical objects denoted as derivatives, and qualified by the word "fractional", were built in view of heterogeneous motivations. The subsequent confusion, which contrasts with what occured for derivatives of integer order, helped not making fractional calculus becoming popular. Yet, the many faces of the notion cannot be avoided, in so far as fractional derivatives have to interpolate between previously installed integer orders. Indeed, infinitely many paths join a given discrete set of points, except if we impose constraints which we have to choose carefully in order to prevent ill-posed problems.

Frequently used definitions [Samko, Kilbas and Marichev, 1993] [Rubin, 1996] [Kilbas, Srivastava and Trujillo, 2006] yield fractional derivatives, inverting fractional integrals, but even this point can take different forms. Here we focus on mappings inverting (at the left) fractional integrals, involving integrations over semi-infinite intervals, and we restrict to the one-dimensional case. Left inverses to such mappings can be given by explicit formulas, such as Riemann's and Liouville's, combining fractional integration and usual derivatives. Marchaud's method is more general, combines convolution and finite differences. It coincides with Riemann-Liouville's formulas for a broad class of functions, and also with Grünwald-Letnikov's definition, which is at the basis of numerical approximations to fractional derivatives.

We will show that the corresponding mappings also coincide with the limit, when l tends to zero, of

$$l^{-\alpha} \int_0^{+\infty} f(x \pm ly) F(y) dy. \qquad (1)$$

This point was proved in [Néel, Abdennadher and Joelson, 2007], for values of α belonging to a finite interval. Here we show that it holds for all positive values of the exponent.

After having stated conditions to be satisfied by F in order to ensure that the limit of (1) when l tends to zero is a fractional derivative of the order of α, we will prove the claim. Then, we will discuss an application.

2 A new setting for Fractional derivatives

Among the many mappings interpolating between derivatives of integer orders, Riemann-Liouville's and Marchaud's derivatives are intimately bound to fractional integrals over semi-infinite intervals, for which they play the role of left inverse mappings, in a definite setting. We will see that fractional derivatives also are limits of a broad set of convolutions.

2.1 Riemann-Liouville's and Marchaud's derivatives

For α being a positive real number, the left and right-sided fractional integrals of the order of α of function f are [Samko, Kilbas and Marichev, 1993] [Rubin, 1996] [Kilbas, Srivastava and Trujillo, 2006]

$$I_{\pm}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{I_{\pm}} (x-y)^{\alpha-1} f(y) dy, \qquad (2)$$

with $I_+ =]-\infty, x]$ and $I_- = [x, \infty[$.

Riemann Liouville's left and right-sided derivatives of the order of α are

$$\mathcal{D}^{\alpha}_{\pm}f(x) = \frac{1}{\Gamma(n-\alpha)}(\pm\frac{d}{dx})^n \int_{I_\pm} \frac{f(y)}{(x-y)^{\alpha-n+1}}dy, \tag{3}$$

with $n = [\alpha]+1$, provided α is not an integer. If function φ is locally integrable over R, provided also integrals $I^{[\alpha]+1}_{\pm}\varphi$ converge absolutely [Rubin, 1996], $(\mathcal{D}^{\alpha}_{\pm}I^{\alpha}_{\pm}\varphi)(x)$ and $\varphi(x)$ are equal almost everywhere. Hence, $\mathcal{D}^{\alpha}_{\pm}$ is a left inverse to $I^{\alpha}_{\pm}$ when the above conditions for φ are met.

For $0<\alpha<1$, when f is derivable with $\frac{d}{dx}f(x)$ tending to 0 at infinity as $|x|^{\alpha-1-\epsilon}$ with $\epsilon > 0$, [Samko, Kilbas and Marichev, 1993] $\mathcal{D}^{\alpha}_{+}f(x)$ is equal to $-\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}(f(x)-f(x+y))y^{-\alpha-1}dy$. More generally, for $\alpha>0$, provided function f is n times continuously derivable, we can define Marchaud's derivatives according to

$$D^{\alpha}_{\pm}f(x) = \frac{-1}{\Gamma(-\alpha)A_n(\alpha)}\int_0^{+\infty}\frac{(\Delta^n_{\pm t}f)(x)}{t^{\alpha+1}}dt \tag{4}$$

with n an integer satisfying $0<\alpha<n$, and $A_n(\alpha) = \Sigma^n_{k=1}(-1)^{k-1}\binom{n}{k}k^{\alpha}$. Finite differences of the order of n, with mesh t, are defined by $\Delta^n_t = (\Delta^1_t)^n$ and $\Delta^1_t = f(x)-f(x-t)$. Marchaud's method extends to a larger set of functions if we take for $D^{\alpha}_{\pm}f(x)$ the limit, when ϵ tends to 0, of $D^{\alpha}_{\pm,\epsilon}f(x)$, itself defined according to

$$D^{\alpha}_{\pm,\epsilon}f(x) =$$

$$\frac{-1}{\Gamma(-\alpha)A_n(\alpha)}\int_{\epsilon}^{+\infty}\frac{(\Delta^n_{\pm t}f)(x)}{t^{\alpha+1}}dt. \tag{5}$$

The limit can be understood in the sense of L^p or of uniform continuity, and $D^{\alpha}_{\pm}$ is a left inverse to $I^{\alpha}_{\pm}$, according to the following remark. But we have to take care that the integrals $I^{\alpha}_{\pm}\varphi$ of an element of $L^{\pm}_p = \{\varphi \in L^p_{loc}(R), \varphi \in L^p(R^{\pm})\}$ may not exist for $\alpha>1$.

Remark 1: The Theorem 10.21 of [Rubin, 1996] states that, for function f such that $f = I^{\alpha}_{\pm}\varphi$ where the integral exists (as a Lebesgue integral or as an improper one), the limit of $D^{\alpha}_{\pm,\epsilon}f(x)$ yields function $\varphi(x)$ according to what follows for $1 \le p \le +\infty$. If φ belongs to L^-_p with $f = I^{\alpha}_{+}\varphi$, then $D^{\alpha}_{+,\epsilon}f(x)$ tends to $\varphi(x)$ in $L^p]-\infty,a[$ for any real a, and also pointwise almost everywhere. For φ in L^+_p with $f = I^{\alpha}_{-}\varphi$, $D^{\alpha}_{-,\epsilon}f(x)$ tends to $\varphi(x)$ in $L^p]a,+\infty[$ for any real a, also pointwise a. e. If $f = I^{\alpha}_{+}\varphi$ or $f = I^{\alpha}_{-}\varphi$ holds while φ belongs to $L^p(R)$, then $D^{\alpha}_{\pm,\epsilon}f(x)$ tends to $\varphi(x)$ in L^p and pointwise a.e. in R. For φ in $\mathcal{C}(R)$ and tending to zero at $-\infty$, the limit of $D^{\alpha}_{+,\epsilon}f(x)$ is uniform on any $]-\infty,a[$. If φ tends to zero at $+\infty$, the limit of $D^{\alpha}_{-,\epsilon}f(x)$ is uniform on any $]a,+\infty[$. Hence, the Theorem 10.21 of [Rubin, 1996] allows us to compute $\varphi = D^{\alpha}_{\pm}f$, provided φ belongs to $L^{\pm}_p$, if also integral $f = I^{\alpha}_{\pm}\varphi$ exists.

Remark 2: Integrals $I^{\alpha}_{\pm}\varphi(x)$ exist for all functions φ in $L^{\pm}_p$ if p belongs to $]0,1/\alpha[$, but they may not exist for some φ in $L^{\pm}_p$ if p is larger than $1/\alpha$, a fortiori if α is larger than 1.

Marchaud's derivative coincides with $\mathcal{D}^{\alpha}_{\pm}f(x)$ when f is n times continuously derivable and falls off rapidly enough at infinity, or when f is the $I^{\alpha}_{\pm}$ image of some integrable function φ, with, moreover, $I^{\alpha}_{\pm}\varphi$ absolutely converging. While Riemann-Liouville's definition needs functions, tending to zero rapidly at infinity, Marchaud's method gives the left inverse to $I^{\alpha}_{\pm}$ in L^p spaces.

2.2 Grünwald-Letnikov's formulas

Grünwald-Letnikov's method yields approximations to the inverse of a fractional integral. For non integer values of α, it is based upon fractional finite differences $\Delta^{\alpha}_{\pm l}$, defined according to [Samko, Kilbas and Marichev, 1993]

$$\Delta^{\alpha}_{l}f(x) =$$

$$\Sigma^{\infty}_{k=0}(-1)^k\binom{\alpha}{k}f(x-kl) = \Sigma^{\infty}_{k=0}w^{\alpha}_k f(x-kl). \tag{6}$$

The limit when l tends to zero, of $l^{-\alpha}\Delta^{\alpha}_{\pm l}f(x)$, is called a Grünwald-Letnikov's derivative. It also yields a left inverse to $I^{\alpha}_{\pm}$, hence it coincides with Marchaud's derivative $D^{\alpha}_{\pm}f(x)$ when f is of the form of $I^{\alpha}_{\pm}\varphi$ with φ in $L^{\pm}_p$, due to Theorem A of [Samko, 1992]. The result was retrieved in [Meerschaert and Scheffler, 2002] with the help of Lévy representation formulas for infinitely divisible probability laws. Efficient numerical schemes are based upon Grünwald-Letnikov's approximation $l^{-\alpha}\Delta^{\alpha}_{\pm l}f(x)$ to $D^{\alpha}_{\pm}f(x)$.

The $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$ in (6) behave as $k^{-\alpha-1}$ when k is large, provided α is not an integer [Gorenflo and Mainardi, 1999]. Moreover, $\Sigma^{\infty}_{k=0}(-1)^k\binom{\alpha}{k} = 0$ holds for $\alpha>0$, and implies $\Sigma^{\infty}_{k=0}(-1)^k\binom{\alpha}{k}k^r = 0$ when r is an integer satisfying $0 \le r < \alpha$. We will see that convolutions, with integrals instead of series, combined with dilatations and contractions of dependent and independent variables (akin to the multiplication of k by l in the argument of $f(x-kl)$ in (6), combined with the factor $l^{-\alpha}$ in front of the series), also provide approximations to the inverse of a fractional integrals.

2.3 A new approach to the left inverse of a fractional integrals

In (6), weights $w^{\alpha}_k = (-1)^k\binom{\alpha}{k}$ match the discrete variant of asymptotic behaviour and oscillation described by the following hypotheses.

Hypothesis $H^1(\alpha)$: function F satisfies $H^1(\alpha)$ if, for any integer r such that $0 \le r < \alpha$, $y^r F(y)$ is integrable in R and satisfies $\int_0^{+\infty} F(y) y^r dy = 0$.

Hypothesis $H^2(\alpha)$: function F satisfies $H^2(\alpha)$ if F is of the form $F(y) = F_1(y) + Cy^{-\alpha-1}$ in a neighbourhood of $+\infty$, with $F_1(y)y^\alpha$ being integrable near $+\infty$.

When α is an integer, we use the following stronger version.

Hypothesis $H'^2(\alpha)$: function F satisfies $H'^2(\alpha)$ if $F(y)y^\alpha$ is integrable near $+\infty$.

We claim that combining appropriate dilatations of independent and dependent variables and convolution whose kernel satisfies $H^1(\alpha)$ and $H^2(\alpha)$ yields approximations to the inverse of the fractional integral I_-^α, according to the following theorem.

Theorem : Let α be a positive real number, and let function F satisfy $H^1(\alpha)$. Then, points *(i)-(iii)* hold if α is not an integer while F satisfies $H^2(\alpha)$. They also hold when α is an integer, provided F satisfies the stronger assumption $H'^2(\alpha)$ instead of $H^2(\alpha)$.

(i) For $f = I_-^\alpha \varphi$ with φ in L_p^+ and $p \ge 1$, the limit of $l^{-\alpha} \int_0^\infty I_-^\alpha \varphi(x+ly)F(y)dy$ exists in L_p^+ and is equal to a constant Λ, times φ, in L_p^+ and also pointwise where φ is right-continuous.

(ii) For $f = I_+^\alpha \varphi$ with φ in L_p^- and $p \ge 1$, the limit of $l^{-\alpha} \int_0^\infty I_+^\alpha \varphi(x-ly)F(y)dy$ exists in L_p^- and is equal to $\Lambda\varphi$, in L_p^- and also pointwise where φ is left-continuous.

(iii) The constant Λ in *(i)* and *(ii)* is equal to $\int_0^\infty I_+^\alpha (HF)(y)dy$, with H representing Heaviside's function. Hence it does not depend on p. Suppose now that α is not an integer. If $F(y)$ is equal to $y^{-\alpha-1}$ in a neighbourhood of $+\infty$, we have $\Lambda = \Gamma(-\alpha)$. If $F(y)$ is equal to $y^{-\alpha-1-\varepsilon}B(y)$ in a neighbourhood of $+\infty$, with $\varepsilon > 0$ while B is bounded, we have $\Lambda = 0$.

Remark 3: $l^{-\alpha-1} \int_0^\infty f(.+y)F(\frac{y}{l})dy$ is equal to $l^{-\alpha} \int_0^\infty f(.+ly)F(y)dy$.

Remark 4: Due to the Theorem 10.21 of [Rubin, 1996], recalled in Remark 1, we have $\varphi(x) = D_\pm^\alpha f(x)$ in $L_p^\mp$ for $f = I_\mp^\alpha \varphi$ in φ in $L_p^\pm$. For such f, the present theorem states that the limit of $l^{-\alpha} \int_0^\infty f(x+ly)F(y)dy$ exists and is equal, in $L_p^\pm$, to Marchaud's derivative $D_\pm^\alpha f(x)$.

The limiting case $\alpha = 0$ is not included in the above claim, and the definition (2) of $I_\pm^\alpha$ does not make sense for this value of the exponent. Nevertheless, it is not difficult to see that a similar ansatz then holds.

3 Proof of the Theorem

Non-integer values of α will be considered first.

3.1 Proof of the Theorem, for α not an integer

Proving *(i)* will be enough for *(i)* and *(ii)*, and will be achieved by checking that φ equals the limit of $l^{-\alpha} \int_0^\infty f(.+ly)F(y)dy$ under hypotheses $H^1(\alpha)$-$H^2(\alpha)$. Setting $y = x + ls$ in

$$\int_0^\infty (I_-^\alpha \varphi)(x+lt)F(t)dt =$$

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} F(t) \int_{x+lt}^{+\infty} \varphi(y)(y-x-lt)^{\alpha-1} dy dt,$$

the latter expression yields

$$l^{-\alpha} \int_0^\infty f(x+lt)F(t)dt =$$

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} F(t) \int_t^{+\infty} \varphi(x+ls)(s-t)^{\alpha-1} ds dt.$$

Fubini's Theorem then implies

$$l^{-\alpha} \int_0^\infty (I_-^\alpha \varphi)(x+lt)F(t)dt =$$

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \varphi(x+ls) \int_0^s F(T)(s-t)^{\alpha-1} dt ds \quad (7)$$

a.e. in R if $I_+^\alpha(HF)(s) = \frac{1}{\Gamma(\alpha)} \int_0^s F(T)(s-T)^{\alpha-1} dT$ is in $L^1(R^+)$. Indeed, due to Young's inequality, the latter implies that $\int_0^{+\infty} |\varphi(x+ls)||I_+^\alpha(HF)(s)|ds$ belongs to $L^p(R)$, hence is finite a.e. in R. Moreover, the convolution $\int_0^{+\infty} \varphi(x+ls) I_+^\alpha(HF)(s)ds$ is $\int_0^\infty I_+^\alpha(HF)(s)ds$ times an approximation to Identity, due to Theorem 1.3 of [Samko, Kilbas and Marichev, 1993] which states that the right hand-side of (7) converges to $\int_0^\infty I_+^\alpha(HF)(s)ds \times \varphi(x)$ in L^p. The convergence is, moreover, pointwise where φ is right-continuous. Hence points *(i)* and *(ii)* of the Theorem will be a consequence of the following Lemma.

Lemma 1: If F satisfies $H^1(\alpha)$ and $H^2(\alpha)$ with α not an integer, $I_+^\alpha(HF)$ is integrable over R^+.

Lemma 4.12 of [Rubin, 1996] (point *ii*) implies Lemma 1 when $F(x)x^\alpha$ is integrable, with α possibly being an integer. Hence, it suffices to prove Lemma 1 for functions F satisfying the hypotheses of Lemma 2 below, while for point *(iii)* we have to compute Λ when F behaves as a power of the argument near $+\infty$.

To this end, take $\varphi(x) = (1-x)^{m-\alpha}$ for $0 \le x < 1$ and $\varphi(x) = 0$ elsewhere, with m an integer satisfying $0 \le m < \alpha < m+1$. Function φ belongs to $L^p(R)$ for $p(\alpha - m) < 1$, and $f(x) = I_-^\alpha \varphi(x)$ satisfies $f(x) = 0$ for $x > 1$, $f(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 (1-t)^{m-\alpha}(t-x)^{\alpha-1} dt$ for x in $[0, 1[$ and $f(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{m-\alpha}(t-x)^{\alpha-1} dt$

for $x < 0$, where integrals converge absolutely. In view of Remark 1 we have $\varphi(x) = D^\alpha_- f(x)$ in $L^p(R)$ and also pointwise almost everywhere. Since $I^\alpha_-\varphi(x)$ is equal to $I^\alpha_{1,-}((1-t)^{m-\alpha})(x) = \frac{\Gamma(m+1-\alpha)}{\Gamma(m+1)}(1-x)^m$ for $0 \leq x < 1$ [Kilbas, Srivastava and Trujillo, 2006], function f satisfies $f(x) = 0$ for $x > 1$ and $f(x) = \frac{\Gamma(m+1-\alpha)}{\Gamma(m+1)}(1-x)^m$ for $0 \leq x < 1$. Due to point *(i)*, the limit of $J(\alpha, l, x) \equiv l^{-\alpha}\int_0^{+\infty} f(x+ly)F(y)dy$ is $\Lambda\varphi(x)$ in L^+_p, and pointwise where φ is right continuous, hence in any interval included in $]0,1[$.

To compute the pointwise limit for x in $]0,1[$, we use $H^1(\alpha)$ which implies $\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}J(\alpha,l,x) = -l^{-\alpha-1}\int_{1-x}^{+\infty}(1-x-Y)^m F(Y/l)dY$ with $ly = Y$. The latter expression is equal to $-\int_{1-x}^{+\infty}(1-x-Y)^m Y^{-\alpha-1}dY$ provided $F(y) = y^{-\alpha-1}$ holds near infinity, when l is small enough. From this we obtain $\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}J(\alpha,l,x) = -(1-x)^{m-\alpha}(-1)^m\int_0^1(1-T)^m T^{\alpha-m-1}dT$, where we recognize a Bernoulli Beta function [Abramowitz and I. Stegun, 1965], and we deduce

$$J(\alpha,l,x) = \Gamma(-\alpha)(1-x)^{m-\alpha}$$

when l is small. Hence we have $\Lambda = \Gamma(-\alpha)$ if $F(y)$ is equal to $y^{-\alpha-1}$ near infinity, in agreement with partial results presented in [Néel, Abdennadher and Joelson, 2007]. If, instead, $F(y)$ is equal to $\mathcal{B}(y)y^{-\alpha-1-\varepsilon}$ there, we have $\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}J(\alpha,l,x) = l^{-\varepsilon}\int_{1-x}^{+\infty}(1-x-Y)^m Y^{-\alpha-1-\varepsilon}\mathcal{B}(Y/l)dY$, which tends to zero when l does.

Hence point *(iii)* is proved. It remains to prove Lemma 1 for $F(x)$ being proportional to $x^{-\alpha-1}$ near infinity.

3.2 Proof of Lemma 1

It is enough to prove the following lemma.

Lemma 2 : For $0 \leq m < \alpha < m+1$, with m in N, $g_1^*(x) = x^{-\alpha-1}\chi_{[A,+\infty[}$ and $g_2^*(x) = \Sigma_{i=0}^m b_i\chi_{[i,i+1[}$, the fractional integral $I^\alpha_+(Hg^*)$ of $g^* = g_1^* - g_2^*$ is integrable in R^+ if and only if g^* satisfies

$$\int_0^{+\infty} y^n g^*(y)dy = 0, n = 0...m. \quad (8)$$

Proof of Lemma 2 : Since g^* is locally integrable, it suffices to check whether $I^\alpha_+(Hg^*)$ is integrable in a neighbourhood of $+\infty$.

First, notice that (8) is equivalent to $\Sigma_{i=0}^m b_i\frac{(i+1)^{r+1}-i^{r+1}}{r+1} = \frac{A^{-\alpha+r}}{\alpha-r}$ for $r = 0,...,m$. Moreover, for $x > A$, in

$$\Gamma(\alpha)I^\alpha_+(Hg^*)(x) =$$

$$\int_A^x (x-y)^{\alpha-1}y^{-\alpha-1}dy - \Sigma_{i=0}^m b_i\Gamma(\alpha)I^\alpha_+\chi_{[i,i+1[}, \quad (9)$$

the $\Gamma(\alpha)I^\alpha_+\chi_{[i,i+1[}(x)$ expand as $\frac{x^\alpha}{\alpha}[\Sigma_{k=1}^{m+1}\frac{(\alpha)...(\alpha+1-k)}{k!x^k}(-1)^k(i^k - (i+1)^k) + B_i(x)x^{-m-2}]$, with the B_i being bounded, so that the $x^\alpha B_i(x)x^{-m-2}$ are integrable near $+\infty$.

Setting $G(X) = \int_0^X[(1-t)^{\alpha-1} - (1+$

$$\Sigma_{k=1}^m\frac{(-1)^k}{k!}(\alpha-1)...(\alpha-k)t^k)]t^{-\alpha-1}dt \quad (10)$$

allows us to write the first integral on the right hand-side of (9) as

$$\int_A^x (x-y)^{\alpha-1}y^{-\alpha-1}dy = x^{-1}[G(1)$$

$$-\frac{1}{\alpha}[1+\Sigma_{k=1}^m\frac{(-1)^k}{k!}\alpha...(\alpha-k+1)]] + x^{-1}[$$

$$-G(A/x) + \frac{1}{\alpha}[(\frac{x}{A})^\alpha + \Sigma_{k=1}^m\frac{(-1)^k}{k!}\alpha...(\alpha-k+1)(\frac{x}{A})^{\alpha-k}]].$$

Since $G(A/x)$ is the integral of a continuous function dominated by $\frac{|\alpha...(\alpha-m-1)|}{(m+1)!}t^{m+1-\alpha}$, $x^{-1}G(A/x)$ is integrable in a neighbourhood of $+\infty$.

Now, we will see that $G(1) - \frac{1}{\alpha}[1 + \Sigma_{k=1}^m\frac{(-1)^k}{k!}(\alpha)...(\alpha-k+1)]$ is equal to zero. Indeed, setting

$$g(p,q) = \int_0^1((1-t)^{q-1} -$$

$$[1+\Sigma_{k=1}^m\frac{(-t)^k}{k!}(q-1)...(q-k)])t^{p-1}dt, \quad (11)$$

we have $G(1) = g(-\alpha,\alpha)$, while, on the right hand-side of (11), we recognize the Bernoulli beta function ([Abramowitz and I. Stegun, 1965] $B(p,q)$, equal to $\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ for complex valued p and q satisfying $Re(p) > 0$ and $Re(q) > 0$. For such (p,q), we have $g(p,q) = B(p,q) - \frac{1}{p} - \Sigma_{k=1}^m\frac{(-1)^k}{k!(p+k)}(q-1)...(q-k)$. With q being fixed, equal to α, this equality extends to complex valued p which are not negative integers and satisfy $Re(p) \geq p_0 > -m-1$. Indeed, the right hand-side $\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} - [\frac{1}{p} + \Sigma_{k=1}^m(-1)^k\frac{(q-1)...(q-k)}{k!(p+q)}]$ is analytic for $Re(p) \geq p_0 > -m-1$, except at poles $0,-1,...,-m$ of Γ. The left hand-side $g(p,q)$ is also analytic in $\{p \in C/Re(p) \geq p_0\}$, due to dominated convergence theorem. Hence, analytic continuation in $Rep \geq p_0 - \{0,...,-m\}$ yields

$$g(p,q) + \frac{1}{p} + \Sigma_{k=1}^m(-1)^k\frac{(q-1)...(q-k)}{k!(p+k)} = B(p,q)$$

for $p = -\alpha$. This proves $G(1) - \frac{1}{\alpha}[1 + \Sigma_{k=1}^{m} \frac{(-1)^k}{k!}\alpha...(\alpha-k)] = 0$, due to $B(-\alpha,\alpha) = 0$.

Now,

$$\Gamma(\alpha)I_+^\alpha(Hg_2^*)(x) - \frac{x^\alpha}{\alpha}[\frac{A^{-\alpha}}{x} +$$

$$\Sigma_{k=2}^{m+1} x^{-k}(-1)^{k+1}\frac{\alpha...(\alpha-k+2)}{(k-1)!}A^{-\alpha-1+k}]$$

is $x^{\alpha-m-1}$ times a polynomial, plus a function, integrable near $+\infty$. The polynomial is identically zero if and only if (8) holds. Hence Lemma 2 is proved.

For integer values of α the hypotheses of Lemma 2 and Theorem 1 have to be strenghted, excluding kernels F proportional to $x^{-\alpha-1}$ near ∞.

3.3 Integer values of α

Let now α be a positive integer. When $F(x)x^\alpha$ is integrable, the lemma 4.12 of [Rubin, 1996] implies the theorem.

It does not extend to kernels F decresasing exactly as $x^{-\alpha-1}$ at $+\infty$. Indeed, with the notations of Lemma 2, $\Gamma(\alpha)I_+^\alpha(Hg_2^*)(x)$ is now a polynomial. For $x > A$, $\Gamma(\alpha)I_+^\alpha(Hg_1^*)(x)$ is equal to $\Sigma_{k=0}^{\alpha-1} x^k \binom{\alpha-1}{k}\frac{(-1)^{\alpha-k}}{k+1}(x^{-1-k} - A^{-1-k})$, which splits into the sum of $\frac{1}{x}\Sigma_{k=0}^{\alpha-1}\binom{\alpha-1}{k}\frac{(-1)^{\alpha-k}}{k+1}$, plus a polynomial. The coefficient of $\frac{1}{x}$ is $\frac{-1}{\alpha}\Sigma_{k'=1}^{\alpha}\binom{\alpha}{k'}(-1)^{\alpha-k'} = \frac{1}{\alpha}$. Therefore, it is even not possible to find coefficients b_i such that $I_+^\alpha(Hg^*)$ be integrable near $+\infty$.

4 Application to random walks

For values of α between 0 and 1, formula (1) applies to random walks, whose successive independent jumps, identically distributed, have tail distributions satisfying $H^2(\alpha)$. The corresponding jump length distributions belong to the domain of stable attraction of $\alpha+1$ stable laws [Feller, 1970]. We will see that then, the flux of walkers splits into expressions, very similar to (1), and that the diffusive limit is a linear combination of derivatives of the order of α.

4.1 The flux of walkers

Here we assume that jumps are distributed according to lX, where the random variable X has density φ, equivalent to $x^{-2-\alpha}$ near infinity, while, moreover, pausing times have finite expectation τ. Parameter l is a length scale.

Then, with $G_+(x)$ representing the tail distribution $\int_x^{+\infty}\varphi(y)dy$, the probability rate of a tagged walker to cross location x to the right is $\tau^{-1}\int_0^{+\infty} f(x-y)G_+(y/l)dy$, equal to $Kl^{-\alpha-1}\int_0^{+\infty} f(x-y)G_+(y/l)dy$ if τ and l satisfy the *scaling law* $l^{\alpha+1} = K\tau$. Hence the flux is the balance $Kl^{-\alpha-1}[\int_0^{+\infty} f(x-y)G_+(y/l)dy - \int_0^{+\infty} f(x+y)G_-(y/l)dy]$ with G_- being the tail distribution $G_-(x) = \int_{-\infty}^{-x}\varphi(y)dy$, if f represents the density of walkers. In view of Remark 3, the flux is equal to $Kl^{-\alpha}[\int_0^{+\infty} f(x-ly)G_+(y)dy - \int_0^{+\infty} f(x+ly)G_-(y)dy]$.

Of course kernels $G\pm$ satisfy $H^2(\alpha)$ but not $H^1(\alpha)$. Nevertheless, we can modify the $G_\pm$ so that $H^1(\alpha)$ be satisfied without changing the flux, provided we check that

$$\int_0^{+\infty} G_+(y)dy = \int_0^{+\infty} G_-(y)dy = I \qquad (12)$$

holds. Indeed, substracting from the $G_\pm$ a compactly supported function whose integral is I yields kernels satisfying the hypotheses of our theorem.

Then, letting l tend to zero yields that the diffusive limit of the flux is a fractional derivative of the order of α: this was proved by [Néel, Abdennadher and Joelson, 2007] in a slightly restricted context, thus retrieving a result obtained by [Paradisi, Cesari, Mainardi, and Tampieri, 2001] from space-fractional diffusion equations.

4.2 Proof of (12)

To prove (12), notice that the difference $\int_0^{+\infty}(G_+(y) - G_-(y))dy$ is also the integral, over $[0,+\infty[$ of the cumulated tail difference $D(x) = \int_x^{+\infty}(\varphi(y) - \varphi(-y))dy$. According to [Bingham, Goldie and Teugels, 1987], $\int_0^{+\infty} D(x)dx$ is the limit when k tends to 0, of $\frac{V(k)}{k}$, with $V(k)$ representing the imaginary part of the characteristic function $\Phi(k)$ of φ, hence the following remark holds.

Remark 5: All densities whose characteristic function has imaginary part $V(k)$ equivalent to $k^{1+\varepsilon}$ with $\varepsilon > 0$ for k near 0 are such that $\int_0^{+\infty} D(x)dx = 0$.

It remains to check that $V(k)$ behaves as $k^{1+\varepsilon}$ near 0. To this end, we use the Theorem 8 of [Pitman, 1968]. It states that densities such as φ whose cumulated tail difference D is integrable satisfy $V(k) = k\int_0^{+\infty} D(x)dx + V_1(k)$ with $V_1(k)$ equivalent to a constant times $D(1/k)$ when k is small, provided D is regularly varying [Bingham, Goldie and Teugels, 1987] of type $-m$ near infinity, with m strictly between 1 and 3. Continuous densities behaving as $x^{-\alpha-1}$ are in this case, and the following remark holds, which achieves proving (12).

Remark 6: For functions $G_\pm$ which are tail integrals of probability laws whose density is proportional to $x^{-\alpha-2}$ near infinity, integrals $\int_0^{+\infty} G_+(y)dy$ and $\int_0^{+\infty} G_-(y)dy$ are equal.

5 Conclusion

Formula (1), which can be used for all positive values of the order α of the derivation, generalizes Grünwald-

Letnikov's, with integrals instead of series. It combines convolution, contraction (multiplication by l) or dilatation (multiplication by $1/l$) of the argument of one among the two involved functions, and dilatation (multiplication by $l^{-\alpha}$) of the issue. Then, the limit "l tending to zero" yields a fractional derivative, according to our theorem. In order to satisfy $H^1(\alpha)$, the kernel F has to oscillate such that all moments of integer order smaller than α be equal to zero. Except when α is an integer, $\alpha + 1$ represents the first power of x^{-1} in the expansion, near infinity, of the kernel. Some improvements should now allow us to extend our theorem to complex orders and, more importantly, it remains to generalize to higher dimensions.

For α between 0 and 1, our theorem helps computing the flux of particles performing random walks. Here, we discussed this point for random walks in infinite domains. In fact, it adapts to cases with boundaries, sources and sinks, as sketched in [Néel, Abdennadher and Joelson, 2007].

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