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STRATIFICATION OF APPROXIMATING SURFACES FOR THE LORENZ ATTRACTOR

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Abstract

We consider the approximation of the global Lorenz attractor by algebraic and semialgebraic sets and discuss the existence of a Whitney stratification for such sets. Analogous properties are investigated for the global attractors of differential equations on Riemannian manifolds.

Key words

Lorenz attractor, semialgebraic set, stratification, Riemannian manifolds

1 Introduction

In this paper we investigate some algebraic properties of sets which approximates the global attractor of a dynamical system. In Sections 2 and 3 we shortly describe some estimates of the global Lorenz attractor and introduce the concept of semialgebraic sets. Existence and realization of stratifications for some algebraic and analytic sets are discussed in Sections 4 and 5.

2 Algebraic approximation of the global Lorenz attractor

Consider the Lorenz equation

$$\begin{aligned} \dot{x} &= \sigma(y-x)\,, \\ \dot{y} &= rx - y - xz\,, \\ \dot{z} &= xy - bz\,, \end{aligned} \tag{1}$$

where $\sigma > 0,\ r > 0$ and $b > 0$ are positive parameters. Denote by $\{\varphi^t\}_{t\in\mathbb{R}}$ the global flow of (1), i.e. for any $u_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ the solution $u(t, u_0)$ of (1) starting in u_0 at $t = 0$ is given by $u(t, u_0) = \varphi^t(u_0),\ t \in \mathbb{R}$.

It is easy to show ([Boichenko, Leonov and Reitmann, 2005; Foias and Temam, 1988; Leonov, Bunin and Koksch, 1987; Leonov and Reitmann, 1986]) that (1) has a global $\mathcal{B}$-attractor $\mathcal{A}$, i.e. there exists a compact set $\mathcal{A} \subset \mathbb{R}^3$ such that $\varphi^t(\mathcal{A}) = \mathcal{A}, \forall\, t \in \mathbb{R}$, and $\mathrm{dist}_H(\varphi^t(\mathcal{B}), \mathcal{A}) \to 0$ for $t \to \infty$, where $\mathrm{dist}_H(\cdot,\cdot)$ denotes the Hausdorff semi-distance and $\mathcal{B} \subset \mathbb{R}^3$ is an arbitrary bounded set.

For certain positive parameters σ, r and b the attractor $\mathcal{A}$ is a fractal set ([Boichenko, Leonov and Reitmann, 2005; Leonov and Reitmann, 1986]), i.e. its Hausdorff dimension is greater than its topological dimension. For example, if $1 < b \le 2$, it is shown in [Boichenko and Leonov, 1990] (see also [Boichenko, Leonov and Reitmann, 2005]) that

$$\dim_H \mathcal{A} \le 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4\sigma r}}\,. \tag{2}$$

For the localization of $\mathcal{A}$ very often Lyapunov functions, defined by quadratic forms, are used ([Foias and Temam, 1988; Giacomini and Neukirch, 1997; Leonov and Reitmann, 1986; Shiota, 1997]). A typical Lyapunov function for (1) is given by ([Boichenko, Leonov and Reitmann, 2005])

$$V(x, y, z) := \frac{1}{2}[x^2 + y^2 + (z - \sigma - r)^2]\,. \tag{3}$$

It is easy to show ([Boichenko, Leonov and Reitmann, 2005]) that the global $\mathcal{B}$-attractor of (1) is included in the set

$$\mathcal{S} := \{(x,y,z) \,|\, \sigma x^2 + y^2 + \frac{b}{2}(z - \sigma - r)^2 \le \frac{b}{2}(\sigma + r)^2\}\,, \tag{4}$$

where $\lambda = \min\{\sigma, 1, \frac{b}{2}\}$. It turns out that in this case the approximation of the attractor is given by an algebraic or a semialgebraic set.

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3 Semialgebraic and algebraic sets

Suppose that $\mathbb{R}[x_1,\ldots,x_n]$ is the ring of polynomials on $\mathbb{R}^n$. A set $\mathcal{S}\subset\mathbb{R}^n$ is called *semialgebraic* if there exist polynomials $\phi_i,\psi_j\in\mathbb{R}[x_1,\ldots,x_n]$, $i=1,2,\ldots,k,\ j=1,2,\ldots,s$ such that

$$\begin{aligned}\mathcal{S}=\{x\in\mathbb{R}^n\,|\,\phi_i(x)=0,\quad i=1,2,\ldots,k,\\ \psi_j(x)\geq 0\,,\quad j=1,2,\ldots,s\}\end{aligned}\tag{5}$$

and *algebraic* if

$$\mathcal{S}=\{x\in\mathbb{R}^n\,|\,\phi_i(x)=0,\ i=1,2,\ldots,k\}\,.\tag{6}$$

Properties of semialgebraic sets ([Shiota, 1997])

(P1) $\mathcal{S},\mathcal{T}\subset\mathbb{R}^n$ semialgebraic sets $\Rightarrow\mathcal{S}\times\mathcal{T},\mathcal{S}\cap\mathcal{T}$ and $\mathcal{S}\backslash\mathcal{T}$ semialgebraic sets.

(P2) $\mathcal{S}\subset\mathbb{R}^n$ a semialgebraic set $\Rightarrow\dim_{\rm top}(\bar{\mathcal{S}}\backslash\mathcal{S})<\dim_{\rm top}\mathcal{S}$ if $\mathcal{S}\neq\emptyset$ ($\dim_{\rm top}(\mathcal{S})$ denotes the topological dimension of a set $\mathcal{S}\subset\mathbb{R}^n$).

(P3) $\mathcal{S}\subset\mathbb{R}^n$ a semialgebraic and connected set $\Rightarrow$ The family of connected components of $\mathcal{S}$ is finite and each connected component is semialgebraic.

Let $r=1,2,\ldots$ or ω and let $\Sigma_r(\mathcal{S})$ denote the C^r singular point set of $\mathcal{S}$, i.e. the set of points where the germ of $\mathcal{S}$ is either of topological dimension $<\dim_{\rm top}\mathcal{S}$ or not $\mathcal{C}^r$ smooth. It is well-known that the set $\Sigma_r(\mathcal{S})$ is semialgebraic and of topological dimension $<\dim_{\rm top}\mathcal{S}$ ([Shiota, 1997]).

Example 1 Consider the Lorenz equation (1). It is shown in [Boichenko, Leonov and Reitmann, 2005] that the Lorenz attractor is contained in a semialgebraic set (5) characterized for $b\leq 2\sigma$ by the polynomials

$$\left.\begin{aligned}&\psi_1(x,y,z)=z-\tfrac{1}{2\sigma}\,x^2\,,\\ &\psi_2(x,y,z)=-(y^2+(z-r)^2-\ell^2r^2)\,,\\ &\text{where}\quad \ell\ =\left\{\begin{array}{ll}1\,, & \text{if}\quad b\leq 2\\ \frac{b}{2\sqrt{b-1}}\,, & \text{if}\quad b>2\,.\end{array}\right.\end{aligned}\right\}\tag{7}$$

Note that in the special case $b=2\sigma$ the algebraic set $\mathcal{S}_1=\{(x,y,z)\in\mathbb{R}^3\,|\,z-\frac{1}{2\sigma}x^2=0\}$ is invariant for system (1).

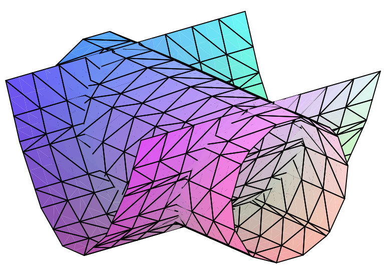

Fig.1: The semialgebraic set (5) with polynomials (7) ([Malykh, 2009])

4 Stratification of semialgebraic sets

A *stratification* of a set $\mathcal{S}\subset\mathbb{R}^n$ is a partition of $\mathcal{S}$ into submanifolds $\{\mathcal{S}_i\}$ of $\mathbb{R}^n$ such that the family $\{\mathcal{S}_i\}$ is locally finite at each point of $\mathcal{S}$. If each stratum $\mathcal{S}_i$ is an analytic submanifold of $\mathbb{R}^n$, we call the stratification *analytic*.

A stratification $\{\mathcal{S}_i\}$ of $\mathcal{S}$ is called a *Whitney stratification* if each pair of strata $\mathcal{S}_i$ and $\mathcal{S}_j, i\neq j$, satisfy the following *Whitney condition*: If $\{p_k\}_{k=1}^\infty$ and $\{q_k\}_{k=1}^\infty$ are sequences of points in $\mathcal{S}_i$ and $\mathcal{S}_j$, respectively, both converging to a point p of $\mathcal{S}_i$, if the sequence of the tangent spaces $\{T_{q_k}\mathcal{S}_j\}_{k=1}^\infty$ converges to a subspace $L\subset\mathbb{R}^n$ in $G_{n,m}$, where $m=\dim\mathcal{S}_j$, and if the sequence $\{\overrightarrow{p_kq_k}\}_{k=1}^\infty$ of lines containing 0 and q_k-p_k converges to a line $l\subset\mathbb{R}^n$ in $G_{n,1}$, then $l\subset L$.

If this is the case for a given point $p\in\mathcal{S}_i$ and for any sequences of points $\{p_k\}_{k=1}^\infty$ and $\{q_k\}_{k=1}^\infty$, we say that $\mathcal{S}_i$ and $\mathcal{S}_j$ satisfy the Whitney condition at p.
Recall that $G_{n,k}(n,k\in\mathbb{N},k\leq n)$ is the family of all k-dimensional subspaces of $\mathbb{R}^n$. It is well-known that $G_{n,k}$ has the structure of a real analytic manifold of dimension $n(n-k)$ which is called *Grassmann manifold*. In addition to this we assume the following condition:

If $\mathcal{S}_i\cap\overline{\mathcal{S}_j}\neq\emptyset$, then $\mathcal{S}_i\subset\overline{\mathcal{S}_j}$ (boundary condition).

A theorem by Whitney ([Whitney, 1934]) states that under rather general conditions such a stratification exists. For the description of algebraic sets and their singular points we use some representation which is based on ideals of polynomials. A well-known theorem by Hilbert ([Gatermann, 2000]) states that any such ideal of polynomials on $\mathbb{R}^n$ is finitely generated. It follows that an algebraic set can be written as $\mathcal{S}=\mathcal{V}_n(\mathcal{J})=\{x\in\mathbb{R}^n|\phi_i(x)=0,\ i=1,2,\ldots,k\}=:\mathcal{V}_n(\phi_1,\ldots,\phi_k)$. At each point x of $\mathcal{V}_n(\mathcal{J})$ we consider the $k\times n$ matrix $(\frac{\partial\phi_i}{\partial x_j})$. Assume that κ is the maximal rank of this matrix on $\mathcal{V}_n(\mathcal{J})$. A point $x\in\mathcal{V}_n(\mathcal{J})$ is regular if the rank of the matrix $(\frac{\partial\phi_i}{\partial x_j})$ at this point is κ. In other case the point is singular. In [Malykh, 2009] we have determined the sets of singular points and their topological dimension of some approximating algebraic and semialgebraic sets of the global $\mathcal{B}$-attractor of the Lorenz system (1). From the general theory ([Milner, 1968]) it follows that the set of regular points of an algebraic set is an analytic manifold of dimension $3-\kappa$ over $\mathbb{R}$. The set of singular points has the structure of an algebraic set. In particular, for this set exists a Whitney stratification. In [Foias and Temam, 1988; Foias and Temam, 1994] Foias and Temam have introduced algebraic and analytic sets that can approximate the global $\mathcal{B}$-attractor of (1) at an arbitrary high level of accuracy. It is shown in [Foias and Temam, 1988; Foias and Temam, 1994] that algebraic and analytic sets as approximation of possibly fractal sets exist under much wider conditions than inertial manifolds which are always smooth manifolds.

Example 2 Suppose

$$\begin{aligned}\mathcal{S} &= V_2(x_1(x_1 - x_2^2)) \\ &= \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1(x_1 - x_2^2) = 0\} \\ &= \{x_1 = 0\} \cup \{x_1 = x_2^2\}\end{aligned} \tag{8}$$

is an algebraic set (see also [Medved, 1992]).

Let us write $\mathcal{S}$ in the form $\mathcal{S} = \bigcup_{i=1}^{5} \mathcal{S}_i$, where

$$\begin{aligned}\mathcal{S}_1 &= \{(0, x_2) \in \mathbb{R}^2 \,|\, x_2 > 0\}\,, \\ \mathcal{S}_2 &= \{(0, x_2) \in \mathbb{R}^2 \,|\, x_2 < 0\}\,, \\ \mathcal{S}_3 &= \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 = x_1^{1/2}, x_1 > 0\}\,, \\ \mathcal{S}_4 &= \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 = -(x_1)^{1/2}, x_1 > 0\}\,, \\ \mathcal{S}_5 &= \{(0, 0)\}\,.\end{aligned}$$

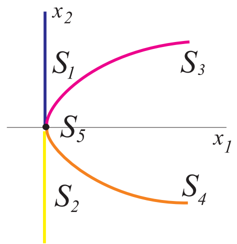

Fig.2: Whitney stratification of the set (8) ([Malykh, 2009])

It is clear that the sets $\{\mathcal{S}_i\}$ are disjunct and the following properties are true:
$\overline{\mathcal{S}_1} = \mathcal{S}_1 \cup \mathcal{S}_5$, $\overline{\mathcal{S}_2} = \mathcal{S}_2 \cup \mathcal{S}_5$, $\overline{\mathcal{S}_3} = \mathcal{S}_3 \cup \mathcal{S}_5$, $\overline{\mathcal{S}_4} = \mathcal{S}_4 \cup \mathcal{S}_5$, $\overline{\mathcal{S}_5} = \mathcal{S}_5$.

All pairs $(\mathcal{S}_i, \mathcal{S}_j)$ $(i \neq j)$ satisfy the Whitney condition. Let us show this for the pair $(\mathcal{S}_5, \mathcal{S}_1)$. Since $\mathcal{S}_5 \subset \overline{\mathcal{S}_1}$ we have to prove that $(\mathcal{S}_5, \mathcal{S}_1)$ satisfies the Whitney condition. Suppose that $q_k \in \mathcal{S}_1, k = 1, 2, \ldots,$ is an arbitrary sequence of points. We can assume that $q_k = (0, q_k') \neq (0, 0)$. Let $p_k = (0, 0), k = 1, 2, \ldots.$ Thus we have

$$\{(\overrightarrow{p_k q_k})\} = \{(0, x_2) \in \mathbb{R}^2 \,|\, x_2 \in \mathbb{R}\} =: l\,.$$

Furthermore it is easy to see that $T_{q_k}\mathcal{S}_1 =: L$ and $l \subset L$. The boundary condition is also satisfied. Consider, for example, $\mathcal{S}_5 \cap \overline{\mathcal{S}_3} \neq \emptyset$. Clearly, that $\mathcal{S}_5 \subset \overline{\mathcal{S}_3}$.

5 Analytic approximation of global attractors on manifolds

In this part we discuss some properties of analytic approximating sets for global $\mathcal{B}$-attractors of differential equations on Riemannian manifolds. One reason for this is that many systems from synchronization theory ([Leonov, Reitmann and Smirnova, 1992]) have such global compact $\mathcal{B}$-attractors on a cylinder, which is a special type of a smooth manifold.

Let us consider on the n-dimensional analytic manifold $(\mathcal{M}, g)$ the vector field

$$\dot{u} = F(u)\,, \tag{9}$$

where $F : \mathcal{M} \to T\mathcal{M}$ is analytic. We assume that $\mathcal{M}$ is countable at infinity, i.e. $\mathcal{M}$ is the union of at most a countable family of compact manifolds. This property is necessary if we want to use on the noncompact manifold $\mathcal{M}$ a partition of unity. Note that "almost all" natural n-dimensional manifolds such as $\mathbb{R}^n$ are countable at infinity.

Assume that for each $p \in \mathcal{M}$ the maximal integral curve $u(\cdot, p)$ of (9) satisfying $u(\cdot, p) = p$ exists on $\mathbb{R}$. Define $\varphi^{(\cdot)}(p) := u(\cdot, p)$ and denote by ρ the metric generated by the metric tensor g. Suppose also that there exists a scalar valued function $V : \mathcal{M} \to \mathbb{R}$ which is at least C^1. In order to get the existence of a global $\mathcal{B}$-attractor for (9) we use the $\mathcal{B}$-dissipativity property of system (9). Recall ([Boichenko, Leonov and Reitmann, 2005]) that (9) is called *$\mathcal{B}$-dissipative* if there exists a bounded set $\mathcal{D} \subset \mathcal{M}$ that attracts under (9) all bounded sets $\mathcal{B}$ from $\mathcal{M}$. The following theorem is a slight modification of a result from [Boichenko, Leonov and Reitmann, 2005].

Theorem 1. *Suppose that there exists a Lyapunov function V for (9) such that the following conditions are satisfied:*

1. *V is proper for $\mathcal{M}$, i.e. for any compact set $\mathcal{K} \subset \mathbb{R}$ the set $V^{-1}(\mathcal{K}) \subset \mathcal{M}$ is compact and V is bounded from below on $\mathcal{M}$;*
2. *There exists an $r > 0$ such that the derivative of V with respect to (9) satisfies the inequality $\dot{V}(p) := (F(p), \mathrm{grad}V(p)) \le 0$ for $p \in \overline{\mathcal{B}_r(0)}$;*
3. *The dynamical system (9) does not have a motion $\varphi^{(\cdot)}(q)$ with $\varphi^t(q) \notin \overline{\mathcal{B}_r(0)}$ and $\dot{V}(\varphi^t(q)) \equiv 0$ for $t \ge t_0$.*

Then the dynamical system $(\{\varphi^t\}_{t\in\mathbb{R}}, \mathcal{M}, \rho)$ is $\mathcal{B}$-dissipative.

From the general theory of attractors it follows that if system (9) is $\mathcal{B}$-dissipative then there exists a global $\mathcal{B}$-attractor for (9).

Let us introduce analytic functions on our manifold $\mathcal{M}$. Assume for this that $x : \mathcal{D}(x) \to \mathcal{R}(x)$ is an analytic chart on $\mathcal{M}$ and $\mathcal{O}_{x(p)}$ is a ring of real-valued analytic functions near $x(p)$. Then the ring of analytic functions near p on $\mathcal{M}$ can be defined with the help of the pull-back map x^* by $\mathcal{R} := (x^*)(\mathcal{O}_{x(p)})$. A semianalytic subset of an analytic manifold $\mathcal{M}$ is a set $\mathcal{S} \subset \mathcal{M}$ with the property that if p is an arbitrary point of $\mathcal{M}$ then there exists a neighborhood $\mathcal{U}$ of p and a finite collection $\mathcal{F}$ of real-valued analytic functions on $\mathcal{U}$, such that $\mathcal{S} \cap \mathcal{U}$ belongs to the

Boolean algebra of subsets $\mathcal{M}$ generated by the sets $\{p \in \mathcal{U} \mid \phi(p) = 0\}, \{p \in \mathcal{U} \mid \phi(p) > 0\}$ for all $\phi \in \mathcal{F}$.

Now we introduce the concept of a Whitney stratification on an analytic manifold. We follow here in some details the presentation in [Gauthier and Kupka, 2000].

Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are smooth submanifolds of $\mathcal{M}, \dim \mathcal{Q} = m$.

The pair $(\mathcal{P}, \mathcal{Q})$ satisfies *condition* (b) *of Whitney* at the point $p \in \mathcal{P} \cap \overline{\mathcal{Q}}$ if there exists a chart $x : \mathcal{D}(x) \to \mathcal{R}(x)$ of the manifold $\mathcal{M}$ near p with the following properties:

Suppose $\{p_k\}$ and $\{q_k\}$ are sequences of points on $\mathcal{M}$ such that

1. $p_k \in \mathcal{D}(x) \cap \mathcal{P}, \quad q_k \in \mathcal{D}(x) \cap \mathcal{Q},$ $p_k \neq q_k, \quad p_k \to p, \quad q_k \to p$ as $k \to \infty$;
2. $\{(\overrightarrow{x(p_k), x(q_k)})\}$, i.e. the 1-dimensional linear subspace containing the points 0 and $x(q_k) - x(p_k)$, converges to an 1-dimensional subspace l ;
3. The vector spaces $dx(T_{q_k}\mathcal{Q})$ (here dx denotes the differential of x and $T_{q_k}\mathcal{Q}$ is the tangent space at q_k) converge in the topology of the Grassmannian Gr (m, n) to the linear subspace $L \subset \mathbb{R}^m$.
   Then $l \subset L$.

The pair $(\mathcal{P}, \mathcal{Q})$ satisfies the *condition* (b) *of Whitney* if $\mathcal{P} \subset \overline{\mathcal{Q}}$ and $(\mathcal{P}, \mathcal{Q})$ satisfies condition (b) of Whitney at any point $p \in \mathcal{P} \cap \overline{\mathcal{Q}}$.

It is easy to see that the stratification introduced in part 4 is the realization of the above definition for $\mathcal{M} = \mathbb{R}^n$.

As an example of a dynamical system on a manifold we have considered in [Malykh, 2009] the equation of the mathematical pendulum given by

$$\ddot{x} + \alpha\dot{x} + \sin x = 0\,, \tag{10}$$

where $\alpha > 0$ is a parameter. This equation is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -\alpha y - \sin x\,. \tag{11}$$

Since the right-hand side of (11) is globally Lipschitz we have the global existence and uniqueness of all solutions. The dynamical system $\{\varphi^t\}_{t \in \mathbb{R}}$, generated by (11), can be considered on the flat cylinder $\mathbb{R}^2/\Gamma$, where $\Gamma = \{ke_1, k \in \mathbb{Z}\}$ is the discrete subgroup of $\mathbb{R}^2$, generated by the element $e_1 = (2\pi, 0)$ of the canonical basis of $\mathbb{R}^2$. It is easy to show ([Boichenko, Leonov and Reitmann, 2005]) that the global $\mathcal{B}$-attractor of (11) is *quasiregular*, i.e. it is the union of the set of equilibria and the associated unstable manifolds to these equilibria. For the approximation of this attractor analytic Lyapunov function of the type $V(x, y) = \frac{y^2}{2} + (1 - \cos x), (x, y) \in \mathbb{R}^2$, are used in [Malykh, 2009]. It is evident that such Lyapunov functions define certain classes of analytic or semianalytic sets ont the cylinder.

References

Boichenko, V.A. and Leonov, G.A. (1990). On estimates of attractors dimension and global stability of generalized Lorenz equations. *Vestn. Leningr. Gos. Univers. Ser.,* 1, **1**, pp. 7–13 (in Russian). *English transl. Vestn. Leningrad Univ. Math.,* **23**, (2), pp. 6–12.

Boichenko, V.A., Leonov, G.A. and Reitmann, V. (2005). *Dimension Theory for Ordinary Differential Equations.* Teubner, Wiesbaden.

Davenport, J.,Siret, Y. and Tournier, E. (1988). *Computer Algebra: Systems and Algorithms for Algebra Computation.* Academic Press, London.

Foias, C. and Temam, R. (1988). The algebraic approximation of attractors; The finite dimensional case. *Phys. D*, **32**, pp. 163–182.

Foias, C. and Temam, R. (1994). Approximation of attractors by algebraic or analytic sets. *SIAM J. Math. Anal.*, **25**, (5), pp. 1269–1302.

Gatermann, K. (2000). *Computer Algebra Methods for Equivariant Dynamical Systems.* LNM 1728, Springer-Verlag, Berlin.

Gauthier, J.-P. and Kupka, I. (2000). *Deterministic Observation, Theory and Application.* Cambridge University Press, Cambridge.

Giacomini, H. and Neukirch, S. (1997). Integrals of motion and shape of the attractor for the Lorenz model. *Phys. Letters A*, **227**, pp. 309–318.

Krishchenko, A.P. and Starkov, K.E. (2006). Localization of compact invariant sets of the Lorenz system. *Phys. Letters A*, **353**, pp. 383–388.

Leonov, G.A., Bunin, A.I. and Koksch, N. (1987). Localization of attractors of the Lorenz system. *ZAMM 67*, **12**, pp. 649–656 (in German).

Leonov, G.A. and Reitmann, V. (1986). *Approximation of Attractors for Non-linear Systems.* Teubner-Verlag, Leipzig (in German).

Leonov, G.A., Reitmann, V. and Smirnova, V.B. (1992). *Non-local Methods for Pendulum-Like Feedback Systems.* Teubner-Verlag, Stuttgart-Leipzig.

Malykh, A.E. (2009). *Stratifications and Singular Points in the Approximation of Attractors.* Diploma thesis, Univ. St. Petersburg (in Russian).

Mather, J. (1971). *Stratifications and Mappings.* Havard University notes.

Medved, M. (1992). *Fundamentals of Dynamical Systems and Bifurcation Theory.* Adam Hilger, Bristol.

Milner, J. (1968). *Singular Points of Complex Hypersurfaces.* Princeton University Press, Princeton.

Shiota, M. (1997). *Geometry of Subanalytic and Semi-Algebraic Sets.* Birkhäuser, P.M. 150.

Whitney, H. (1934). Analytic extensions of differentiable functions defined in closed sets. *Trans. Am. Math. Soc.*, **36**, pp. 63–89.