ADAPTIVE OUTPUT-FEEDBACK SERVOCOMPENSATOR DESIGN USING HIGH-GAIN SCALING

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Abstract: The error-feedback servomechanism problem is addressed for a general class of strict-feedback-like systems. The design is based on our recent results on adaptive output-feedback based on dynamic dual high-gain scaling. The design technique provides strong robustness properties and allows the system to contain both unknown functions dependent on all states and uncertain parameters coupled with all states (with no *a priori* magnitude bounds required on uncertain parameters).

Keywords: Nonlinear Control, High Gain, Servomechanism.

1. INTRODUCTION

The class of systems considered are of the form:

$$\begin{aligned}
\dot{x}_1 &= \phi_1(\varpi, x_1) + \phi_{(1,2)}(x_1)x_2 \\
\dot{x}_2 &= \phi_2(\varpi, x_1, x_2) + \phi_{(2,3)}(x_1)x_3 \\
&\vdots \\
\dot{x}_i &= \phi_i(\varpi, x_1, x_2, \ldots, x_i) + \phi_{(i,i+1)}(x_1)x_{i+1} \\
&\vdots \\
\dot{x}_n &= \phi_n(\varpi, x_1, x_2, \ldots, x_n) + u \\
\tilde{x}_1 &= x_1 - \psi(\varpi) \qquad (1)
\end{aligned}$$

where [2] $\tilde{x}_1 \in \mathcal{R}$ is the tracking error which constitutes the only measurable signal available for feedback, $\varpi \in \mathcal{R}^{n_\varpi}$ is the reference (or disturbance) input, $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ is the state, and $u \in \mathcal{R}$ the input. $\phi_{(i,i+1)} : \mathcal{R} \to \mathcal{R}, i = 1, \ldots, n-1$ are known continuous functions of their arguments. $\phi_i : \mathcal{R}^i \to \mathcal{R}, i = 1, \ldots, n$, are uncertain continuous functions which can contain both functional and parametric uncertainties. ϖ is assumed to be produced by a neutrally stable (i.e., all eigenvalues of S simple and lying on the imaginary axis) exosystem of form

$$\dot{\varpi} = S\varpi. \qquad (2)$$

The error-feedback control design in this paper is based on the adaptive dual high-gain based output-feedback control design technique proposed in our recent paper (Krishnamurthy and Khorrami, 2006) for systems of the form

$$\begin{aligned}
\dot{x}_i &= \phi_i(x_1, \ldots, x_i) + \phi_{(i,i+1)}(x_1)x_{i+1}\,,\ i = 1, \ldots, n-1 \\
\dot{x}_n &= \phi_n(x_1, \ldots, x_n) + u \\
y &= x_1 \qquad (3)
\end{aligned}$$

where $y \in \mathcal{R}$ is the measured output. The required bounds on ϕ_i, $i = 1, \ldots, n$, (Assumption A2 in (Krishnamurthy and Khorrami, 2006)) allow cross-products of unknown parameters and unmeasured states with no magnitude bound or sign information on the unknown parameters being required. While earlier control design techniques such as the classical high-gain designs (Khalil and Saberi, 1987; Teel and Praly, 1994; Khalil, 1996; Ilchmann, 1996) and backstepping (Krstić *et al.*, 1995) cannot handle cross-products of unknown

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[2] The set of real numbers $(-\infty, \infty)$, the set of nonnegative real numbers $[0, \infty)$, and the set of real k-dimensional column vectors are denoted by $\mathcal{R}$, $\mathcal{R}^+$, and $\mathcal{R}^k$, respectively.

parameters and unmeasured states, the dynamic scaling-based dual high-gain observer/controller design approach developed in our recent papers (Krishnamurthy and Khorrami, 2002; Krishnamurthy and Khorrami, 2004; Krishnamurthy and Khorrami, 2006) provides a flexible framework which can accomodate such cross-products.

High gain as a technique for controller and observer designs has been investigated extensively in the literature. The well-known adaptive high-gain controller given in its basic form by $u = -ry, \dot{r} = y^2$ is applicable to minimum-phase systems with relative-degree one (Khalil and Saberi, 1987; Ilchmann, 1996). Static high-gain scaling based observers (Teel and Praly, 1994; Khalil, 1996) which introduce observer gains $r, \ldots, r^n$ with a constant r provide semiglobal solutions. In (Praly, 2003), a high-gain observer and a backstepping controller were designed for systems of form (3) with $\phi_{(i,i+1)} = 1, i = 1, \ldots, n-1$, and with $\phi_i, i = 1, \ldots, n$, being known functions of $x_1, \ldots, x_i$ incrementally linear in unmeasured states in the sense that $|\phi_i(x_1, \ldots, x_i) - \phi_i(x_1, \hat{x}_2, \ldots, \hat{x}_i)| \leq \Gamma(x_1) \sum_{j=2}^{i} |\hat{x}_j - x_j|$ with $\Gamma(x_1)$ being a known function.

A dual high-gain observer/controller design approach was introduced in (Krishnamurthy and Khorrami, 2002; Krishnamurthy and Khorrami, 2004) based on the solution of a pair of coupled Lyapunov inequalities which were shown to be always solvable under a cascading dominance assumption on the upper diagonal terms $\phi_{(i,i+1)}$ (Krishnamurthy *et al.*, 2003; Krishnamurthy and Khorrami, 2004) which is closely linked to the Cascading Upper Diagonal Dominance (CUDD) condition introduced in (Krishnamurthy *et al.*, 2002). In (Krishnamurthy and Khorrami, 2004), the functions $\phi_i, i = 1, \ldots, n$, were allowed to contain functional and parametric uncertainties coupled with all the states. It was seen that a complexity of bounds on the uncertain terms ϕ_i does not result in complexity of the controller, observer, or Lyapunov function, but is instead handled through the dynamics of the high-gain scaling. However, (Krishnamurthy and Khorrami, 2004) required a magnitude bound on the uncertain parameters in the system. The requirement of a magnitude bound on unknown parameters was removed in (Krishnamurthy and Khorrami, 2005) using a time-varying dynamics of the high-gain scaling parameter with the basic idea being to asymptotically (as $t \to \infty$) guarantee sufficient gain to dominate the unknown parameters while retaining closed-loop stability. This provided the first output-feedback globally asymptotically stabilizing solution to the following benchmark open problem proposed in our earlier papers (Krishnamurthy and Khorrami, 2003; Krishnamurthy and Khorrami, 2004)

$$\dot{x}_1 = x_2 \;\; ; \;\; \dot{x}_2 = x_3 \;\; ; \;\; \dot{x}_3 = u + \theta_0 x_1^2 x_3 \tag{4}$$

with u being the input, $y = x_1$ the output, and θ_0 an uncertain parameter of unknown sign and with no available magnitude bounds. System (4) is of a very simple form with a single nonlinearity and a single unknown parameter. If any of the components of $\theta_0 x_1^2 x_3$ are dropped, the solution can be obtained using available techniques. If θ_0 is known, (Praly and Kanellakopoulos, 2000) and (Krishnamurthy *et al.*, 2003) provide controllers of dynamic orders 9 and 3, respectively. If x_1^2 is removed, the system is linear. If x_3 is removed, the system is in standard output-feedback canonical form (Marino and Toméi, 1993; Krstić *et al.*, 1995). If a magnitude bound on θ_0 is available, a solution is provided by (Krishnamurthy and Khorrami, 2004). However, with θ_0 completely unknown, no output-feedback control design technique prior to (Krishnamurthy and Khorrami, 2005) can globally asymptotically stabilize the system.

A time-invariant dynamic controller based on a factorization of the scaling parameter r into two dynamic scaling parameters as $r = LM$ was recently introduced in (Lei and Lin, 2005) for a subclass of systems of form (3) with all the upper diagonal terms $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, required to be identically equal to 1 and with the output dependence of the unknown functions $\phi_i, i = 1, \ldots, n$, required to be polynomially bounded. In (Krishnamurthy and Khorrami, 2006), it was shown that the time varying component of the scaling parameter dynamics in (Krishnamurthy and Khorrami, 2005) can be eliminated *without* requiring the restrictions on $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, and $\phi_i, i = 1, \ldots, n$, introduced in (Lei and Lin, 2005), thus providing an autonomous dynamic controller for systems of form (3) with the full generality of (Krishnamurthy and Khorrami, 2005) in terms of assumptions on system terms. In this paper, the design technique from (Krishnamurthy and Khorrami, 2006) is applied to the servomechanism problem yielding servocompensator design results for a wider class of systems than in prior results available in the literature (Davison, 1976; Francis and Wonham, 1976; Isidori and Byrnes, 1990; Priscoli, 1993; Huang and Chen, 2004).

The rest of the paper is organized as follows. The coordinate transformation of the system (1) so that the system in the error coordinates is of the form (3) is presented in Section 2 and satisfies assumptions analogous to those in (Krishnamurthy and Khorrami, 2006). The error-feedback control design is developed in Section 3. The extension of the design to systems with Input-to-State Stable (ISS) appended dynamics and inverse dynamics is briefly outlined in Section 4.

2. TRANSFORMATION INTO ERROR COORDINATES

The main assumption to ensure that the system (1) can be transformed in error coordinates into a system of form (3) is given by Assumption AT1 below:

Assumption AT1: Sufficiently smooth (possibly uncertain) functions $\overline{x}_1(\varpi),\ldots,\overline{x}_n(\varpi)$, and $\overline{u}(\varpi)$ exist such that for all $\varpi \in \mathcal{R}^{n_\varpi}$,

$$\overline{x}_1(\varpi) = \psi(\varpi) \tag{5}$$

$$\overline{x}_2(\varpi) = \frac{1}{\phi_{(1,2)}(\psi(\varpi))}\left[\frac{\partial \overline{x}_1(\varpi)}{\partial \varpi} S\varpi - \phi_1(\varpi,\psi(\varpi))\right] \tag{6}$$

$$\overline{x}_i(\varpi) = \frac{1}{\phi_{(i-1,i)}(\psi(\varpi))}\left[\frac{\partial \overline{x}_{i-1}(\varpi)}{\partial \varpi} S\varpi - \phi_{i-1}(\varpi,\psi(\varpi),\overline{x}_2(\varpi),\ldots,\overline{x}_{i-1}(\varpi))\right] \quad , \ i=3,\ldots,n \tag{7}$$

$$\overline{u}(\varpi) = \frac{\partial \overline{x}_n(\varpi)}{\partial \varpi} S\varpi - \phi_n(\varpi,\psi(\varpi),\overline{x}_2(\varpi),\ldots,\overline{x}_n(\varpi)) \tag{8}$$

To construct an internal model, the following technical assumption is also needed.

Assumption AT2: A positive integer n_u and constants $a_0,\ldots,a_{n_u-1}$ exist such that

$$L^{n_u}_{S\varpi}\overline{u}(\varpi) = a_0\overline{u}(\varpi) + a_1 L_{S\varpi}\overline{u}(\varpi) + \ldots + a_{n_u-1}L^{n_u-1}_{S\varpi}\overline{u}(\varpi).$$

Under Assumption AT2, it can be seen that the exosystem with output $\overline{u}(\varpi)$ can be immersed into the system

$$\dot{\chi}_u(\varpi) = \frac{\partial \chi_u(\varpi)}{\partial \varpi} S\varpi \ = \ A_u\chi_u(\varpi)$$
$$\overline{u}(\varpi) = C_u\chi_u(\varpi) \tag{9}$$

where A_u is the $n_u \times n_u$ matrix with 1's on the upper diagonal and $[a_0,\ldots,a_{n_u-1}]$ as the last row and $C_u = [1,0,\ldots,0]$. Since (A_u,C_u) form an observable pair, an internal model can be constructed as an observer for the linear system (9) to be

$$\dot{\overline{x}}_u = G\overline{x}_u + H\overline{u}(\varpi) \tag{10}$$

where G is a $\mathcal{R}^{n_u\times n_u}$ Hurwitz matrix, H is an n_u-dimensional column vector, and (G,H) is a controllable pair. From the unique nonsingular solution $T \in \mathcal{R}^{n_u\times n_u}$ of the Sylvester equation $TA_u - GT = HC_u$, the estimate for χ_u is constructed as $T^{-1}x_u$. Since G is a Hurwitz matrix, a symmetric positive-definite matrix P_G can be found such that

$$P_G G + G^T P_G \leq -I_{n_u} \tag{11}$$

where I_k denotes an identity matrix of dimension $k\times k$. Replacing $\overline{u}(\varpi)$ with u in (10), the implementable observer (or internal model)

$$\dot{x}_u = Gx_u + Hu \tag{12}$$

is obtained where x_u is an estimate for $(\overline{x}_u + H\tilde{x}_n)$. Also, $C_uT^{-1}x_u$ serves as an estimate of $\overline{u}(\varpi)$. The observer error of the internal model is defined as

$$\tilde{x}_u = x_u - (\overline{x}_u + H\tilde{x}_n). \tag{13}$$

The dynamics of $\tilde{x}_u$ is given by

$$\dot{\tilde{x}}_u = G\tilde{x}_u + GH\tilde{x}_n - H\tilde{\phi}_n. \tag{14}$$

Defining the coordinate transformation,

$$\tilde{x}_i = x_i - \overline{x}_i(\varpi)\,,\ i=2,\ldots,n \tag{15}$$

$$\tilde{x} = [\tilde{x}_1,\ldots,\tilde{x}_n]^T \tag{16}$$

$$\tilde{u} = u - \overline{u}(\varpi) \tag{17}$$

it is seen that in the error coordinates $(\tilde{z},\tilde{x})$, the system (1) can be written in the form

$$\dot{\tilde{x}}_i = \tilde{\phi}_i(\varpi,\tilde{x}_1,\ldots,\tilde{x}_i) + \phi_{(i,i+1)}(\tilde{x}_1)\tilde{x}_{i+1}, i=1,\ldots,n-1$$
$$\dot{\tilde{x}}_n = \tilde{\phi}_n(\varpi,\tilde{x}_1,\ldots,\tilde{x}_n) + \tilde{u}$$
$$y = \tilde{x}_1 \tag{18}$$

The output-feedback control design (using output y) for system (18) is addressed in Section 3.

3. OUTPUT-FEEDBACK CONTROL DESIGN FOR THE ERROR SYSTEM

The assumptions required on the system (18) are given by Assumptions A1-A3 below which are analogous to the assumptions in (Krishnamurthy and Khorrami, 2006).

Assumption A1: System (18) is observable and controllable, i.e., a constant $\sigma > 0$ exists such that for all $\tilde{x}_1 \in \mathcal{R}$, $|\phi_{(i,i+1)}(\tilde{x}_1)| \geq \sigma$, $1 \leq i \leq n-1$.

Assumption A2: A continuous function $\Gamma : \mathcal{R} \to \mathcal{R}^+$ is known such that

$$|\phi_i(\tilde{x}_1,\ldots,\tilde{x}_i)| \leq \theta\Gamma(\tilde{x}_1)\sum_{j=1}^{i}|\tilde{x}_j|\ ,\ \ 1\leq i\leq n \tag{19}$$

for all $\tilde{x} = [\tilde{x}_1,\ldots,\tilde{x}_n] \in \mathcal{R}^n$ with $\theta \geq 0$ being an unknown parameter (with no knowledge of magnitude bounds required).

Assumption A3: Positive constants $\overline{\rho}_i$ and $\underline{\rho}_i$ exist such that for all $\tilde{x}_1 \in \mathcal{R}$

$$|\phi_{(i,i+1)}(\tilde{x}_1)| \geq \overline{\rho}_i|\phi_{(i-1,i)}(\tilde{x}_1)|, i=2,\ldots,n-1 \tag{20}$$

$$|\phi_{(i,i+1)}(\tilde{x}_1)| \leq \underline{\rho}_i|\phi_{(i-1,i)}(\tilde{x}_1)|, i=2,\ldots,n-1. \tag{21}$$

Remark 1: Assumption A3 requires ratios of the "upper-diagonal" terms $\phi_{(i,i+1)}$ to be bounded. The condition (20) requires the upper-diagonal terms closer to the input to be larger (in a nonlinear function sense) while condition (21) requires the upper-diagonal terms closer to the output to be larger. The conditions (20) and (21) constitute the cascading dominance assumptions (Krishnamurthy *et al.*, 2002) in the controller context and observer context, respectively, and are related to uniform solvability of coupled Lyapunov inequalities (Krishnamurthy *et al.*, 2003; Krishnamurthy and Khorrami, 2004) which are instrumental in the design of controller and observer gains in a dual dynamic high-gain design. Using Theorems A1 and A2 in (Krishnamurthy and Khorrami, 2004), the conditions on $\phi_{(i,i+1)}$ in Assumptions A1 and A3 are necessary and sufficient for the existence of functions $g_1(\tilde{x}_1),\ldots,g_n(\tilde{x}_1),k_2(\tilde{x}_1),\ldots,k_n(\tilde{x}_1)$, symmetric positive-definite matrices P_o and P_c, and positive constants ν_o, $\overline{\nu}_o$, $\underline{\nu}_o$, ν_c, $\overline{\nu}_c$, and $\underline{\nu}_c$ to satisfy for all $\tilde{x}_1 \in \mathcal{R}$

$$\left.\begin{array}{c} P_oA_o(\tilde{x}_1) + A_o^T(\tilde{x}_1)P_o \leq -\nu_o|\phi_{(1,2)}(\tilde{x}_1)|I_n \\ \underline{\nu}_oI_n \leq P_o(D_o - \frac{1}{2}I_n) + (D_o - \frac{1}{2}I_n)P_o \leq \overline{\nu}_oI_n \end{array}\right\} \tag{22}$$

$$\left.\begin{array}{c} P_cA_c(\tilde{x}_1) + A_c^T(\tilde{x}_1)P_c \leq -\nu_c|\phi_{(2,3)}(\tilde{x}_1)|I_{n-1} \\ \underline{\nu}_cI_{n-1} \leq P_c(D_c - \frac{1}{2}I_{n-1}) + (D_c - \frac{1}{2}I_{n-1})P_c \leq \overline{\nu}_cI_{n-1} \end{array}\right\} \tag{23}$$

where

$$A_o = \begin{bmatrix} -g_1 & \phi_{(1,2)} & 0 & 0 & \dots \\ -g_2 & 0 & \phi_{(2,3)} & 0 & \dots \\ \vdots & & & \ddots & \\ -g_{n-1} & & & & \phi_{(n-1,n)} \\ -g_n & 0 & & \dots & 0 \end{bmatrix} \quad (24)$$

$$A_c = \begin{bmatrix} 0 & \phi_{(2,3)} & 0 & 0 & \dots \\ 0 & 0 & \phi_{(3,4)} & 0 & \dots \\ \vdots & & & \ddots & \\ 0 & & & & \phi_{(n-1,n)} \\ -k_2 & -k_3 & & \dots & -k_n \end{bmatrix} \quad (25)$$

$$C = [1, 0, \dots, 0] \quad (26)$$

$$D_o = \mathrm{diag}(1, 1, 2, 3, \dots, n-1) \quad (27)$$

$$D_c = \mathrm{diag}(1, 2, 3, \dots, n-1), \quad (28)$$

and $\mathrm{diag}(a_1, \dots, a_k)$ denotes the $k \times k$ diagonal matrix with the i^{th} diagonal element being a_i.

Furthermore, by Theorem A1 in (Krishnamurthy and Khorrami, 2004), $g_1(\tilde{x}_1), \dots, g_n(\tilde{x}_1)$ can be picked to be linear constant-coefficient combinations of $\phi_{(1,2)}(\tilde{x}_1), \dots, \phi_{(n-1,n)}(\tilde{x}_1)$. Hence, using Assumption A3, a positive constant $\overline{G}$ exists such that

$$\sqrt{\sum_{i=1}^{n} g_i^2(\tilde{x}_1)} \leq \overline{G}|\phi_{(1,2)}(\tilde{x}_1)|. \quad (29)$$

The main result of the paper is summarized in Theorem 1.

Theorem 1: Under Assumptions AT1, AT2, A1, A2, and A3, positive constants a and b and continuous functions $\zeta : \mathcal{R}^2 \to \mathcal{R}$, $\Theta_1 : \mathcal{R} \to \mathcal{R}^+$, $\Theta_2 : \mathcal{R}^2 \to \mathcal{R}^+$, $\gamma : \mathcal{R}^3 \to \mathcal{R}^+$, $g_i : \mathcal{R} \to \mathcal{R}, i = 1, \dots, n$, and $k_i : \mathcal{R} \to \mathcal{R}, i = 2, \dots, n$ can be chosen such that all solution trajectories of the closed-loop system formed by the dynamic controller given by

$$\left.\begin{aligned} \dot{\hat{x}}_1 &= \phi_{(1,2)}(\tilde{x}_1)\hat{x}_2 - \frac{\dot{r}}{r}(\hat{x}_1 - \tilde{x}_1) - rg_1(\tilde{x}_1)[\hat{x}_1 - \tilde{x}_1] \\ \dot{\hat{x}}_i &= \phi_{(i,i+1)}(\tilde{x}_1)\hat{x}_{i+1} \\ &\quad - r^i g_i(\tilde{x}_1)[\hat{x}_1 - \tilde{x}_1] \ , \ i = 2, \dots, n-1 \\ \dot{\hat{x}}_n &= \mu_0(\tilde{x}_1)[u - C_u T^{-1} x_u] - r^n g_n(\tilde{x}_1)[\hat{x}_1 - \tilde{x}_1] \end{aligned}\right\} \quad (30)$$

$$u = -r^n \sum_{i=2}^{n} k_i(\tilde{x}_1)\eta_i + C_u T^{-1} x_u \quad (31)$$

$$\eta_2 = \frac{\hat{x}_2 + \zeta(\tilde{x}_1, \hat{\theta})}{r} \ ; \ \eta_i = \frac{\hat{x}_i}{r^{i-1}} \ , \ i = 3, \dots, n \quad (32)$$

$$\dot{r} = r[-a(r-1) + b\gamma(\tilde{x}_1, \hat{x}_1, \hat{\theta})] \ ; \ r(0) \geq 1 \quad (33)$$

$$\dot{\hat{\theta}} = \Theta_1(\tilde{x}_1)x_1^2 + \Theta_2(\tilde{x}_1, r)(\hat{x}_1 - \tilde{x}_1)^2 \ ; \ \hat{\theta}(0) > 0, \quad (34)$$

in closed loop with the internal model (12) and the system (1) starting from any initial condition $(x(0), \hat{x}(0), r(0), \hat{\theta}(0), x_u(0)) \in \mathcal{R}^n \times \mathcal{R}^n \times [1, \infty) \times (0, \infty) \times \mathcal{R}^{n_u}$ where $\hat{x} = [\hat{x}_1, \dots, \hat{x}_n]^T$ have the following properties:

- Solution trajectories exist on the time interval $[0, \infty)$ and all closed-loop signals are bounded on $[0, \infty)$.
- The tracking error $\tilde{x}_1 = x_1 - \psi(\varpi(t))$ asymptotically converges to zero as $t \to \infty$.

Observer Design: A full-order observer for system (18) is given by (30) where r is the dynamic high-gain scaling parameter and $g_1, \dots, g_n$ are functions chosen as in Remark 1. The observer errors e_i and scaled observer errors ϵ_i are defined as

$$e_i = \hat{x}_i - \tilde{x}_i \ ; \ \epsilon_i = \frac{e_i}{r^{i-1}} \ , \ 1 \leq i \leq n. \quad (35)$$

From (35), $\epsilon_1 = e_1$. The dynamics of the scaled observer error vector $\epsilon = [\epsilon_1, \dots, \epsilon_n]^T$ are given by

$$\dot{\epsilon} = rA_o\epsilon - \frac{\dot{r}}{r}D_o\epsilon - \overline{\Phi} + B\frac{1}{r^{n-1}}[\overline{u}(\varpi) - C_u T^{-1} x_u] \quad (36)$$

$$\overline{\Phi} = [\overline{\Phi}_1, \dots, \overline{\Phi}_n]^T \ , \ \overline{\Phi}_i = \frac{\phi_i}{r^{i-1}} \quad (37)$$

where A_o and D_o are defined in (24) and (27), respectively. The term $-\frac{\dot{r}}{r}e_1$ introduced in the dynamics of $\hat{x}_1$ in (30) contributes a value of 1 to the $(1, 1)$ element of D_o thus ensuring the positive-definiteness of the matrix $(D_o - \frac{1}{2}I_n)$. This is crucial to the solvability of the second Lyapunov inequality in (22).

Controller Design: The control law is given by (31) where the term $C_u T^{-1} x_u$ is used as an estimate of $\overline{u}(\varpi)$. The controller gain functions $k_2, \dots, k_n$ are chosen as in Remark 1 and $\eta_2, \dots, \eta_n$ are given by (32). The design function ζ is picked to be of the form $\zeta(\tilde{x}_1, \hat{\theta}) = (1 + \hat{\theta})\tilde{x}_1\zeta_1(\tilde{x}_1)$ with ζ_1 being a continuously differentiable function and $\hat{\theta}$ a parameter estimator. The signals η_i, $i = 2, \dots, n$, are scaled observer estimates of the states $\hat{x}_i$ with an additional design freedom ζ incorporated into η_2. The dynamics of $\eta = [\eta_2, \dots, \eta_n]^T$ are

$$\dot{\eta} = rA_c\eta - \frac{\dot{r}}{r}D_c\eta - rG_2\epsilon_1 + H(\eta_2 - \epsilon_2) + \Xi \quad (38)$$

where

$$G_2 = [g_2, \dots, g_n]^T \quad (39)$$

$$H = \left[(1 + \hat{\theta})\left\{\zeta_1'\tilde{x}_1 + \zeta_1\right\}\phi_{(1,2)}, 0, \dots, 0\right]^T \quad (40)$$

$$\Xi = \frac{1}{r}\left[\dot{\hat{\theta}}\zeta_1\tilde{x}_1 + (1+\hat{\theta})\left\{\zeta_1'\tilde{x}_1 + \zeta_1\right\} \times\left\{\phi_1 - (1 + \hat{\theta})\zeta_1\tilde{x}_1\phi_{(1,2)}\right\}, 0, \dots, 0\right]^T, \quad (41)$$

A_c and D_c are as defined in (25) and (28), respectively, and $\zeta_1'(\tilde{x}_1)$ denotes the partial derivative evaluated at $\tilde{x}_1$ of ζ_1 with respect to its argument.

Stability Analysis: To analyze closed-loop stability, the observer and controller Lyapunov functions are defined as

$$V_o = r\epsilon^T P_o\epsilon \ ; \ V_c = r\eta^T P_c\eta + \frac{\beta_1}{2}\tilde{x}_1^2 \quad (42)$$

where β_1 is a positive constant free to be picked by the designer. Differentiating V_o and V_c and using (36) and (38),

$$\begin{aligned} \dot{V}_o &= r^2\epsilon^T[P_oA_o + A_o^T P_o]\epsilon - 2r\epsilon^T P_o\overline{\Phi} \\ &\quad - \dot{r}\epsilon^T\left[P_o(D_o - \frac{1}{2}I_n) + (D_o - \frac{1}{2}I_n)P_o\right]\epsilon \\ &\quad + 2r\epsilon^T P_o B\frac{1}{r^{n-1}}[\overline{u}(\varpi) - C_u T^{-1} x_u] \end{aligned} \quad (43)$$

$$\begin{aligned} \dot{V}_c &= r^2\eta^T[P_cA_c + A_c^T P_c]\eta \\ &\quad - \dot{r}\eta^T\left[P_c(D_c - \frac{1}{2}I_{n-1}) + (D_c - \frac{1}{2}I_{n-1})P_c\right]\eta \\ &\quad - 2r^2\eta^T P_c G_2\epsilon_1 + 2r\eta^T P_c H(\eta_2 - \epsilon_2) + 2r\eta^T P_c\Xi \\ &\quad + \beta_1\tilde{x}_1[\phi_1 + (r\eta_2 - r\epsilon_2 - \zeta)\phi_{(1,2)}]. \end{aligned} \quad (44)$$

A Lyapunov function for the observer error of the internal model (12) is introduced as

$$V_{\tilde{x}_u} = \frac{1}{r^{2n-3}} \tilde{x}_u^T P_G \tilde{x}_u. \quad (45)$$

Using (11) and (14), the derivative of $V_{\tilde{x}_u}$ is given by

$$\dot{V}_{\tilde{x}_u} \leq -(2n-3)\frac{\dot{r}}{r^{2n-2}} \tilde{x}_u^T P_G \tilde{x}_u - \frac{1}{r^{2n-3}} |\tilde{x}_u|^2 + \frac{2}{r^{2n-3}} \tilde{x}_u^T P_G [GH\tilde{x}_n - H\tilde{\phi}_n]. \quad (46)$$

Closed-loop stability is analyzed using the Lyapunov function

$$V = cV_o + V_c + V_{\tilde{x}_u} \quad (47)$$

with c being a design parameter picked such that $c \geq \frac{8}{\nu_o \nu_c \overline{\rho}_2} \lambda_{max}^2(P_c) \overline{G}^2$ where $\lambda_{max}(P)$ denotes the maximum eigenvalue of a symmetric positive-definite matrix P. Using algebraic manipulations along the same lines as in (Krishnamurthy and Khorrami, 2006), the details of which are omitted here for brevity, the Lyapunov inequality

$$\begin{aligned}\dot{V} \leq &-\frac{c\nu_o}{4} r^2 |\phi_{(1,2)}||\epsilon|^2 - \frac{\nu_c}{2} r^2 |\phi_{(2,3)}||\eta|^2 - \beta_1 \tilde{x}_1 \phi_{(1,2)} \zeta \\ &-(2n-3)\frac{\dot{r}}{r^{2n-2}} \tilde{x}_u^T P_G \tilde{x}_u - \frac{1}{2r^{2n-3}} |\tilde{x}_u|^2 \\ &+[q_1(\tilde{x}_1) + \theta^* q_2(\tilde{x}_1)]\tilde{x}_1^2 \\ &-c\dot{r}\epsilon^T \left[P_o(D_o - \frac{1}{2}I_n) + (D_o - \frac{1}{2}I_n)P_o\right]\epsilon \\ &-\dot{r}\eta^T \left[P_c(D_c - \frac{1}{2}I_{n-1}) + (D_c - \frac{1}{2}I_{n-1})P_c\right]\eta \\ &+r[w_1(\tilde{x}_1, \hat{\theta}, \dot{\hat{\theta}}) + \theta^* w_2(\tilde{x}_1, \hat{\theta})]\{|\epsilon|^2 + |\eta|^2\} \end{aligned} \quad (48)$$

can be obtained where $\theta^* \stackrel{\triangle}{=} \max\{1, \theta + \theta^2\}$, and q_1, q_2, w_1, and w_2 are continuous functions of the indicated arguments.

Picking b to be an arbitrary positive constant, choose $a > 0$ small enough to ensure that

$$\max\left(-\frac{\sigma c\nu_o}{4} + ac\overline{\nu}_o, -\frac{\sigma\nu_c}{2} + a\overline{\nu}_c, -\frac{1}{2} + a(2n-3)\lambda_{max}(P_G)\right) = -a^* < 0, \quad (49)$$

and choose $\zeta_1(\tilde{x}_1)$ such that

$$-\beta_1 \zeta_1(\tilde{x}_1)\phi_{(1,2)}(\tilde{x}_1) + q_1(\tilde{x}_1) + q_2(\tilde{x}_1) \leq -\zeta_1^*(\tilde{x}_1) \quad (50)$$

with ζ_1^* being a positive function of $\tilde{x}_1$ bounded below by a positive constant $\underline{\zeta}_1^*$. The parameter estimator dynamics are chosen as shown in (34) with $\Theta_1(\tilde{x}_1) = \frac{1}{\beta_\theta} q_2(\tilde{x}_1)$ and $\Theta_2(\tilde{x}_1, r) = \frac{1}{\beta_\theta} r^2 \phi_{(1,2)}^2$ where $\beta_\theta > 0$ is a design parameter. Note that the parameter estimate $\hat{\theta}(t)$ with dynamics (34) is a monotonically nondecreasing function of time. The design function γ is picked to be

$$\begin{aligned}\gamma(\tilde{x}_1, \hat{x}_1, \hat{\theta}) = &\frac{1}{b\min(c\underline{\nu}_o, \underline{\nu}_c)} \\ &\times \Bigg[w_1\left(\tilde{x}_1, \hat{\theta}, \Theta_1(\tilde{x}_1)\tilde{x}_1^2 + \Theta_2(\tilde{x}_1, r)(\hat{x}_1 - \tilde{x}_1)^2\right) \\ &+\frac{1}{\beta_3} w_2^2(\tilde{x}_1, \hat{\theta})(1 + \tilde{x}_1^2) + \hat{\theta} + \beta_4 \Bigg] \end{aligned} \quad (51)$$

with β_4 being a design parameter which can be picked to be any positive constant. Using (33), (49), (50) and (51), (48) reduces to

$$\begin{aligned}\dot{V} \leq &-a^* r^2 \left[|\epsilon|^2 + ||\eta|^2 + \frac{|\tilde{x}_u|^2}{r^{2n-1}}\right] - \tilde{x}_1^2 \zeta_1^*(\tilde{x}_1) \\ &+(\theta^* - \hat{\theta})q_2(\tilde{x}_1)\tilde{x}_1^2 + r\Bigg\{\theta^* w_2(\tilde{x}_1, \hat{\theta}) \\ &-\frac{1}{\beta_3} w_2^2(\tilde{x}_1, \hat{\theta})(1 + \tilde{x}_1^2) - \hat{\theta} - \beta_4\Bigg\} \times [|\epsilon|^2 + |\eta|^2]. \end{aligned} \quad (52)$$

The *parameter estimation error* is defined to be $(\hat{\theta} - \overline{\theta})$ with $\overline{\theta} \stackrel{\triangle}{=} \max\{\theta^*, \frac{\beta_3}{4}\theta^{*2}\}$. Note that $\overline{\theta} \geq 1$ since θ^* was defined as $\max\{1, \theta + \theta^2\}$. A new Lyapunov function is defined including a quadratic of the parameter estimation error $(\hat{\theta} - \overline{\theta})$ as $\overline{V} = V + \frac{\beta_\theta}{2}(\hat{\theta} - \overline{\theta})^2$. Using (52),

$$\begin{aligned}\dot{\overline{V}} \leq &-a^* r^2 \left[|\epsilon|^2 + ||\eta|^2 + \frac{|\tilde{x}_u|^2}{r^{2n-1}}\right] - \tilde{x}_1^2 \zeta_1^*(\tilde{x}_1) \\ &+(\hat{\theta} - \overline{\theta})r^2 \phi_{(1,2)}^2 [\hat{x}_1 - \tilde{x}_1]^2 \\ &+r\left\{\theta^* w_2(\tilde{x}_1, \hat{\theta}) - \frac{1}{\beta_3} w_2^2(\tilde{x}_1, \hat{\theta})(1 + \tilde{x}_1^2) - \hat{\theta} - \beta_4\right\} \\ &\times[|\epsilon|^2 + |\eta|^2]. \end{aligned} \quad (53)$$

Local existence of solutions is guaranteed by the assumptions on the functions ϕ_i and $\phi_{(i,i+1)}$. Let the maximal interval of existence of solutions be $[0, t_f)$. The proof of Theorem 1 utilizes Lemmas 1-4 given below to infer that $t_f = \infty$ (i.e., solutions exist for all time) and that in the limit as $t \to \infty$, the signals $\tilde{x}_1, \ldots, \tilde{x}_n$, $\tilde{e}_1, \ldots, \tilde{e}_n$, $\tilde{x}_u$, converge to zero. The Lemmas 1-4 can be proved along similar lines as in (Krishnamurthy and Khorrami, 2006) and the details are omitted here for brevity.

Lemma 1: If $\sup_{t\in[0,t_f)} V(t) < \infty$ and $\sup_{t\in[0,t_f)} \hat{\theta}(t) < \infty$, then $\sup_{t\in[0,t_f)} r(t) < \infty$.

Lemma 2: If $\sup_{t\in[0,t_f)} \hat{\theta}(t) > \overline{\theta}$, then $t_f = \infty$, $\lim_{t\to\infty} V(t) = 0$, $\int_0^\infty V(t)dt < \infty$, and $\sup_{t\in[0,\infty)} \hat{\theta}(t) < \infty$.

Lemma 3: If $\sup_{t\in[0,t_f)} \hat{\theta}(t) \leq \overline{\theta}$, then $\sup_{t\in[0,t_f)} V(t) < \infty$, and $\int_0^{t_f} V(t) < \infty$.

Lemma 4: If $\sup_{t\in[0,t_f)} \hat{\theta}(t) \leq \overline{\theta}$, then $t_f = \infty$ and $\lim_{t\to\infty} V(t) = 0$.

Proof of Theorem 1: With the maximal interval of existence of solutions denoted by $[0, t_f)$, one of the following possibilities should hold: Case $\mathcal{A}_1$: $\sup_{t\in[0,t_f)} \hat{\theta}(t) \leq \overline{\theta}$; Case $\mathcal{A}_2$: $\sup_{t\in[0,t_f)} \hat{\theta}(t) > \overline{\theta}$. If Case $\mathcal{A}_2$ holds, then Lemma 2 guarantees that $t_f = \infty$. On the other hand, under Case $\mathcal{A}_1$, Lemma 4 implies that $t_f = \infty$. Hence, the possibility of finite escape time is ruled out, i.e., $t_f = \infty$. Furthermore, from Lemmas 2-4, it is seen that $\sup_{t\in[0,\infty)} V(t) < \infty$, $\int_0^\infty V(t)dt < \infty$, $\sup_{t\in[0,\infty)} \hat{\theta}(t) < \infty$, and $\lim_{t\to\infty} V(t) = 0$. Also, it can be shown that the boundedness of all closed-loop signals on the time interval $[0, \infty)$ follows from the boundedness of $\hat{\theta}(t)$, $V(t)$, and $\int_0^t V(\tau)d\tau$ on $t \in [0, \infty)$. The asymptotic convergence of $V(t)$ to zero as $t \to \infty$ implies the asymptotic convergence of the signals $\tilde{x}_1, \ldots, \tilde{x}_n$, $\tilde{e}_1, \ldots, \tilde{e}_n$, $\tilde{x}_u$ to zero as $t \to \infty$, thus completing the proof of Theorem 1. $\diamond$

4. EXTENSION TO SYSTEMS WITH ISS APPENDED DYNAMICS

The error-feedback control design technique presented in this paper can be extended to the more general class of systems

$$\begin{aligned}
\dot{z}_i &= q_i(\varpi, z, x)\ ,\ i = 1, \ldots, n \\
\dot{x}_i &= \phi_i(\varpi, z, x, u) + \phi_{(i,i+1)}(x_1)x_{i+1}\ ,\ i = 1, \ldots, s-1 \\
\dot{x}_i &= \phi_i(\varpi, z, x, u) + \phi_{(i,i+1)}(x_1)x_{i+1} \\
&\quad +\mu_{i-s}(x_1)u\ ,\ i = s, \ldots, n \\
\tilde{x}_1 &= x_1 - \psi(\varpi)
\end{aligned} \tag{54}$$

where $z_i \in \mathcal{R}^{n_{z_i}}$ are the (unmeasurable) states of appended ISS dynamics(Sontag, 1995) and $z = [z_1^T, \ldots, z_n^T]^T$. s is the relative degree of the system and $[x_{s+1}, \ldots, x_n]^T$ is the state of the inverse dynamics. The appended dynamics are driven by all the system states with a triangular structure of ISS interconnections. While previous techniques (Praly and Jiang, 1993) required ISS dynamics and inverse dynamics to be driven only by x_1, (Krishnamurthy and Khorrami, 2004) provided an output-feedback control method using the dynamic high-gain scaling approach to handle ISS appended dynamics and inverse dynamics driven by all the system states. The design in (Krishnamurthy and Khorrami, 2004) utilized dynamics of the high-gain scaling parameter of the form $\dot{r} = \lambda(R(x_1, \hat{\theta}, \dot{\hat{\theta}}) - r)\Omega(r, x_1, \hat{\theta}, \dot{\hat{\theta}})$ with R, λ, and Ω being suitably chosen functions. The Lyapunov function in (Krishnamurthy and Khorrami, 2004) incorporates appropriately scaled versions of the ISS Lyapunov functions of the inverse dynamics and the appended dynamics. By using the techniques in (Krishnamurthy and Khorrami, 2004; Krishnamurthy and Khorrami, 2006) and the servocompensator design technique in this paper, the proposed error-feedback controller can be extended to systems of form (54). The details are omitted here for brevity.

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