REDUCED ORDER MODEL FOR THE NONLINEAR VIBRATION ANALYSIS OF A PRESSURE LOADED CYLINDRICAL SHELL

Paulo B. Gonçalves, Frederico. M. A. da Silva
Department of Civil Engineering
Catholic University, PUC-Rio
Brazil
paulo@puc-rio.br, silvafma@gmail.com

Zenón J. G. N. Del Prado
Department of Civil Engineering
Federal University of Goiás, UFG
Brazil
zenon@eec.ufg.br

Abstract
A reduced order model for the nonlinear vibration analysis of a thin-walled, simply-supported circular cylindrical shell based on the use of a standard perturbation procedure and on the proper orthogonal decomposition is derived. First, using Donnell shallow shell nonlinear equations of motion, a modal solution is obtained by a perturbation technique leading to a multi-mode solution that captures the inherent modal coupling, thus describing correctly the nonlinear modes of vibration, and satisfies all boundary and continuity conditions. Based on this solution, the proper orthogonal modes and values are obtained, identifying the most important modes in the modal expansion. Then, different reduced models are derived and used to analyze the vibrations and resonance of the shell under a harmonic lateral pressure.

Key words
Cylindrical shells, nonlinear dynamics, reduced order models, Karhunen-Loève method, proper orthogonal decomposition.

1 Introduction

The quest for reliable reduced order models in the nonlinear dynamic analysis of continuous systems has been an active research area in recent years. Traditional perturbation techniques, the method of normal forms and the proper orthogonal decomposition based on the Karhunen-Loeve method are among the most efficient tools for the dimensional reduction of dynamical systems. Traditionally the analysis of continuous nonlinear systems has been performed by the use of nonlinear finite element formulations. However an efficient analysis usually requires a refined mesh, leading to a large number of degrees of freedom, and also a small time step is necessary to obtain a reliable time response. For example, to obtain a precise critical load for a cylindrical shell under static load hundreds of degrees of freedom are necessary and even more to obtain a dynamic nonlinear response. So it is prohibitive to perform a detailed parametric analysis of nonlinear continuous systems using a FE software. An alternative is to use a low-dimensional model. However, several examples found in literature shows that the usual technique of approximating the non-linear displacement field by a series of linear vibration modes may lead in some cases to incorrect responses for their incapacity to describe the nonlinear vibration modes or may require a large number of modes (although less than the FE) to obtain a quantitatively precise response. Recently, a lot of attention has been paid to reducing the cost of the nonlinear state solution by using reduced-order models for the state. Particularly in solid and fluid mechanics this has become a very attractive research field, enabling a deeper understanding of complex nonlinear systems. The most common approaches are the use of nonlinear normal modes [Shaw and Pierre, 1993], proper orthogonal decomposition based on Karhunen-Loève method [Steindl and Troger, 2001] and centroidal Voronoi tessellations [Burkardt et al., 2006]. Recently, Rega and Troger (2005), in an article that introduces a special issue of the journal Nonlinear Dynamics on reduced-order models have analyzed the most common methods of dimension reduction in nonlinear dynamics with emphasis on applications in mechanics. The aim of these methods is to choose a reduced basis $\mathbf{u}_i$, $i=1,\dots,n$, where n is small compared to the usual number of functions used, for example, in a finite element approximation or in a traditional Galerkin model. It is clear that the reduced basis should be chosen so that it contains all the features, e. g., the dynamics of the states encountered during the simulation. It requires some intuition about the states to be simulated. If the reduced-order model is properly selected, it should work in an interpolatory setting, but it is not clear what happens in an extrapolatory setting. One cannot hope to determine one reduced-order model capable of describing the response of a complex system for all sets of parameters. So, depending on the complexity of the systems, various reduced-order models optimized for different sets of parameters should be derived. However, one hopes that a single reduced basis can be used for several state simulations or in several design settings. In the present work a standard perturbation technique is used together with the POD to obtain an efficient reduced order model, which is then employed to analyze the non-linear vibrations and instabilities of a thin-walled circular cylindrical shell under the action of a harmonic lateral pressure.

2 Mathematical Formulation

2.1 Shell Equations

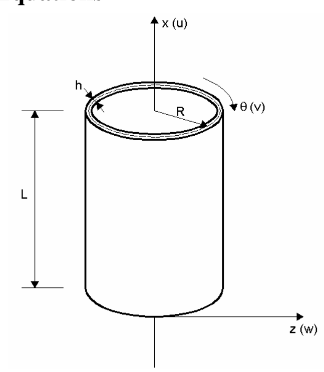


Figure 1 – Shell geometry and coordinate system

Consider a cylindrical of radius *R*, thickness *h* and length *L*, made of a linear elastic material with Young's modulus *E*, Poisson coefficient υ and mass density ρ. The three displacement components *u*, *v* and *w* are related to the cylindrical co-ordinate system *x*, θ and *z*, as shown in Figure 1.

For an isotropic shell the constitutive law is given by:

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_\theta \\ \bar{\tau}_{x\theta} \end{Bmatrix} = \frac{E}{(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{(1-\upsilon)}{2} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_\theta \\ \bar{\gamma}_{x\theta} \end{Bmatrix} \tag{1}$$

The shell deformations at an arbitrary point are given in terms of the middle surface strain and change of curvature components by:

$$\begin{aligned} \bar{\varepsilon}_x &= \varepsilon_x + \chi_x \\ \bar{\varepsilon}_\theta &= \varepsilon_\theta + \chi_\theta \\ \bar{\gamma}_{x\theta} &= \gamma_{x\theta} + 2\chi_{x\theta} \end{aligned} \tag{2}$$

These middle surface quantities are given in terms of the displacement components, according to Donnell shallow shell theory, by:

$$\begin{aligned} \varepsilon_x &= u_{,x} + \frac{1}{2} w_{,x}^2 \\ \varepsilon_\theta &= \frac{1}{R}\left(v_{,\theta} + w\right) + \frac{1}{2R^2} w_{,\theta}^2 \\ \gamma_{x\theta} &= v_{,x} + \frac{1}{R}\left(u_{,\theta} + w_{,x}\, w_{,\theta}\right) \\ \chi_x &= -w_{,xx} \\ \chi_y &= -\frac{1}{R^2} w_{,\theta\theta} \\ \chi_{xy} &= -\frac{1}{R} w_{,x\theta} \end{aligned} \tag{3}$$

The shell is subjected to a harmonic lateral pressure of the form:

$$p = P\,h \cos\left(n\,\theta\right) \sin\left(\frac{m\,\pi\,x}{L}\right) \cos\left(\omega t\right) \tag{4}$$

where *P* is the pressure magnitude in N/m²; *n* and *m* are, respectively, the number of waves in the circumferential direction and the number of half-waves in the axial direction, ω is the excitation frequency and *t* is time.

The nonlinear equations of motion, considering only the transversal inertia and damping forces, are given in terms of the force and moments resultants by:

$$\begin{aligned} & N_{x,x} + \frac{N_{x\theta,\theta}}{R} = 0 \\ & N_{x\theta,x} + \frac{N_{\theta,\theta}}{R} = 0 \\ & \rho\, h\, \ddot{w} + \beta_1\, \dot{w} - p - \frac{1}{R^2}\Big[R\, M_{x,xx} + 2 M_{x\theta,x\theta} \\ & + \frac{M_{\theta,\theta\theta}}{R} + R\, N_x\, w_{,xx} + N_\theta \left(\frac{w_{,\theta\theta}}{R} - 1\right) \\ & + 2 N_{x\theta}\, w_{,x\theta}\Big] = 0 \end{aligned} \tag{5}$$

where the force and moments resultants are obtained by the integration of the stress components along the shell thickness as follows:

$$N_x = \int_{-h/2}^{h/2} \bar{\sigma}_x \, dz \qquad M_x = \int_{-h/2}^{h/2} z\,\bar{\sigma}_x \, dz$$
$$N_\theta = \int_{-h/2}^{h/2} \bar{\sigma}_\theta \, dz \qquad M_\theta = \int_{-h/2}^{h/2} z\,\bar{\sigma}_\theta \, dz \tag{6}$$
$$N_{x\theta} = \int_{-h/2}^{h/2} \bar{\tau}_{x\theta} \, dz \qquad M_{x\theta} = \int_{-h/2}^{h/2} z\,\bar{\tau}_{x\theta} \, dz$$

For a simply-supported shell, the following boundary conditions must be satisfied:

$$v(0,\theta)=v(L,\theta)=0 \tag{7}$$
$$w(0,\theta)=w(L,\theta)=0 \tag{8}$$
$$M_x(0,\theta)=M_x(L,\theta)=0 \tag{9}$$
$$N_x(0,\theta)=N_x(L,\theta)=0 \tag{10}$$

The boundary condition (10) is a nonlinear boundary condition when written in terms of the displacements, that is:

$$N_x = \frac{E h}{1-v^2}\left[u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{\upsilon}{R}\left(v_{,\theta} + w + \frac{1}{2R} w_{,\theta}^2 \right)\right] \tag{11}$$

The displacement field, in this work, is also required to satisfy the following conditions:

$$u(L/2,\theta)=0 \tag{12}$$
$$v(x,0)=v(x,2\pi) \tag{13}$$

In the foregoing, the following non-dimensional parameters have been introduced:

$$W = \frac{w}{h} \quad \xi = \frac{x}{L}$$
$$\tau = \omega_o t \quad \Omega = \frac{\omega}{\omega_o}$$
$$\Gamma = \frac{P}{P_{cr}} \tag{14}$$
$$P_{cr} = \frac{E h}{R}\left[\frac{\left[(\pi R/L)^2 + n^2\right]^2}{n^2} \frac{(h/R)^2}{12\left(1-\upsilon^2\right)} + \frac{(\pi R/L)^4}{n^2 \left[(\pi R/L)^2 + n^2\right]^2} \right]$$

Here ω_o is the lowest natural frequency of the empty shell and P_{cr} is the classical static critical lateral pressure of the shell.

2.2 General solution of the shell displacement field by a perturbation procedure and determination of the in-plane displacements u and v

The numerical model is developed by expanding the transversal displacement component w in series in the circumferential and axial variables. From previous investigations on modal solutions for the non-linear analysis of cylindrical shells under axial loads [Hunt et al. 1986; Gonçalves and Batista, 1988; Gonçalves and Del Prado, 2002; Awrejcewicz and Krys'ko, 2003] it is observed that, in order to obtain a consistent modeling with a limited number of modes, the sum of shape functions for the displacements must express the non-linear coupling between the modes and describe consistently the unstable post-buckling response of the shell as well as the correct frequency-amplitude relation.

Based on a perturbation procedure [Gonçalves and Del Prado, 2005], the lateral deflection w can be described as:

$$W = \sum_{i=1,3,5} \sum_{j=1,3,5} \zeta_{ij}(t) \cos(i n \theta) \sin(j m \pi \xi) + \sum_{k=0,2,4} \sum_{l=0,2,4} \zeta_{ij}(t) \cos(k n \theta) \cos(l m \pi \xi) \tag{15}$$

By imposing the boundary conditions (8) and (9), and by retaining in (15) the number of modes necessary to achieve converge up to very large deflections, one obtains for the transversal displacement [Gonçalves et al., 2007]:

$$\begin{aligned} W = {} & \zeta_{11}(t)\cos(n\theta)\sin(m\pi\xi) \\ & + \zeta_{13}(t)\cos(n\theta)\sin(3m\pi\xi) \\ & + \zeta_{31}(t)\cos(3n\theta)\sin(m\pi\xi) \\ & + \zeta_{33}(t)\cos(3n\theta)\sin(3m\pi\xi) \\ & + \left(\frac{3}{4} - \cos(2m\pi\xi) + \frac{1}{4}\cos(4m\pi\xi)\right) \\ & \times\left[\zeta_{02}(t) + \zeta_{22}(t)\cos(2n\theta)\right] \end{aligned} \tag{16}$$

The in-plane displacements u and v are obtained by substituting (16) into the in-plane equilibrium equations (5.a) and (5.b) and solving the system of linear partial differential equations in u and v and imposing the relevant boundary, symmetry and continuity conditions. Based on this procedure one selects the necessary number of in-plane modes and write their modal amplitudes in terms of the modal amplitudes ζ_{ij} in (16) [Gonçalves et al. 2007]. It is important to notice that the harmonic terms in the modal expansion for u and v derived by this procedure are similar to those derived by the perturbation procedure. Finally, by substituting the adopted expansion for the transversal displacement w together with the obtained expressions for u and v

into the equation of motion in the transversal direction, Eq. (5.3), and by applying the standard Galerkin method, a consistent discretized system of ordinary differential equations of motion is derived.

2.3 Reduction of the problem by Karhunen-Loève decomposition

In order to construct a theoretically well founded low-dimensional model, it is important to identify the relative importance of each mode to the total energy of the system as a function of the vibration amplitude and the participation of each term of the modal expansion (16) in the nonlinear vibration modes. Also, the modal basis may contain redundant information in the sense that the dynamics of the system can be approximated with accuracy by a set of functions of much lower dimension.

One way of solving this problem is to use the Karhunen-Loève method also known as proper orthogonal decomposition (POD). Various applications of the POD method for the reduced-order modeling of complex systems can be found in literature [Steindl and Troger, 2001; Rega and Troger, 2005; Amabili et. al., 2006]. The POD method is based on the analysis of a series of snapshots of the system response obtained from a high-fidelity solution of the mathematical model. Experimental data have also been used to determine the snapshot sets. A detailed mathematical formulation of the Karhunen-Loève method can be found, for example, in Sirovich [1987a, b, c] and Bellizzi and Sampaio [2006]. In this work the so-called direct method is employed.

The continuous displacement field at a certain instant is approximated by a discrete field. To obtain a vector field representative of the shell displacements, the surface of the shell is discretized and the displacements are evaluated at N_T spatial points uniformly spaced along the x and θ axis, as follows:

$$\mathbf{w}^* = w\left(x_i, \theta_j, t\right), \begin{cases} x_i = \dfrac{iL}{n_x}, i=0,\ldots,n_x \\ \theta_j = \dfrac{j2\pi}{n_\theta}, j=0,\ldots,n_\theta \end{cases} \tag{17}$$

where $\mathbf{w}^*$ is the vector of the components of the transversal displacements measured at each point (x_i, θ_j) using (16). The modal time-dependent amplitudes $\zeta_{ij}(t)$ in (16) are obtained by the solution of the discretized equations of motion of the shell. So, for each time interval a vector with n_x x n_θ = N_T elements ordered as $w_1^*(t),\ldots,w_{N_T}^*(t)$ is obtained. Taking M snapshots at $t_m = m\tau$ ($t_1,\ldots,t_M$), where τ is the sampling period, which must be greater than the correlation time, the following ensemble matrix of dimension $M \times N_T$ is obtained:

$$\mathbf{w}^* = \begin{bmatrix} w_1(t_1) & \ldots & w_{N_T}(t_1) \\ \vdots & \ddots & \vdots \\ w_1(t_M) & \ldots & w_{N_T}(t_M) \end{bmatrix} \tag{18}$$

where each column represents the temporal variation of the displacement at a certain point in space and each row represents the displacement field at a certain instant t_m.

Using the ergodicity hypothesis, the mean value of the field is obtained by summing all components of **w*** and dividing the result by the number of rows M. The variation of the field with respect to the mean value of each row is obtained by:

$$\mathbf{v}^* = \mathbf{w}^* - \frac{1}{M}\begin{bmatrix} \sum_{i=1}^{M} w_1(t_i) & \ldots & \sum_{i=1}^{M} w_{N_T}(t_i) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{M} w_1(t_i) & \ldots & \sum_{i=1}^{M} w_{N_T}(t_i) \end{bmatrix} \tag{19}$$

Finally, using again an ergodicity assumption, the spatial correlation matrix can be written as follows:

$$\mathbf{R} = \frac{1}{M}\left(\mathbf{v}^*\right)^T\left(\mathbf{v}^*\right) \tag{20}$$

where **R** is a symmetric, positive-definite matrix.

The eigenvectors of (20), which are orthogonal due to its symmetry, are the POMs and the associated eigenvalues, the POVs. An eigenvalue (POV) has the interpretation of giving the mean energy of the system projected on the associated eigenvector-axis in function space [Sirovich, 1987a, b, c]. The mean energy of a flow should therefore be equal to the sum of the eigenvalues.

Using the eigenvalues and eigenvectors of the spatial correlation matrix, the dynamics of the original system can be reconstructed as:

$$\mathbf{w} \approx \sum_{k=1}^{K} a_k(t)\varphi_k(\mathbf{x},\boldsymbol{\theta}) + E\left[\mathbf{w}^*\right] \tag{21}$$

where $\varphi_k(\mathbf{x},\theta)$ is the *k-th* eigenvector and $a_k(t)$ is the *k-th* coefficient which is a function of the temporal dependence and is defined as:

$$a_k(t) = \left\langle \mathbf{v}^*(\mathbf{x},\boldsymbol{\theta},t), \varphi_k(\mathbf{x},\boldsymbol{\theta})\right\rangle \tag{22}$$

3 Numerical Results

Consider a cylindrical shell of radius R = 0.2 m, length L = 0.4 m and thickness h = 0.002 m. The shell material has the following properties: E = 2.1 x 10^8 kN/m², υ = 0.3 and ρ = 7850 kg/m³. For this shell geometry the lowest buckling load as well as the lowest natural frequency are obtained for $m=1$ and $n=5$ [Gonçalves and Del Prado, 2005]. These values will be used throughout the present numerical analysis.

In Figure 2 the frequency-amplitude relation obtain with expansion (16) – which includes ten modes of the complete solution (15) and gives a precise solution up to very large deflections – is compared with three different approximations: one using the first two POMs, the second using the first three POMs and the third using a single DOF model. The time-dependent modal amplitudes $\zeta_{ij}(t)$ are, in each case, exactly obtained by the use of the shooting method [Seydel, 1988].

The single DOF model is obtained based on the perturbation technique which shows that the higher order modal amplitudes ζ_{ij} can be in fact considered as slave co-ordinates and written as a polynomial function of the amplitude of the seminal mode ζ_{11} [Gonçalves and Del Prado, 2005]. For the shell under consideration, these relations are given by:

$$\begin{aligned}
\zeta_{02}(t) &= 3.8\times10^{-2}\zeta_{11}^2(t)+8.8\times10^{-4}\zeta_{11}^4(t) \\
&\quad +3.4\times10^{-5}\zeta_{11}^6(t) \\
\zeta_{22}(t) &= 8.9\times10^{-3}\zeta_{11}^2(t)+5.2\times10^{-4}\zeta_{11}^4(t) \\
&\quad -7.0\times10^{-4}\zeta_{11}^6(t) \\
\zeta_{13}(t) &= -2.2\times10^{-2}\zeta_{11}^3(t)+1.1\times10^{-3}\zeta_{11}^5(t) \\
&\quad -2.6\times10^{-4}\zeta_{11}^7(t) \\
\zeta_{31}(t) &= 5.0\times10^{-4}\zeta_{11}^3(t)+2.6\times10^{-4}\zeta_{11}^5(t) \\
&\quad -4.6\times10^{-5}\zeta_{11}^7(t) \\
\zeta_{33}(t) &= -2.3\times10^{-5}\zeta_{11}^3(t)-2.5\times10^{-4}\zeta_{11}^5(t) \\
&\quad +4.3\times10^{-5}\zeta_{11}^7(t)
\end{aligned} \tag{23}$$

As shown in Figure 2, all reduced models are capable of describing the frequency-amplitude relation up vibration amplitudes of the order of the shell thickness. For vibration amplitudes of the order of two times the shell thickness, there is a small difference between the reduced models, but all approximations are still reasonably good and capture the type and degree of non-linearity of the response.

One cannot hope to determine one reduced-order model capable of describing the response of a complex system for all sets of parameters. However, it is expected that the present model can be successfully used to study the forced response of the shell and identify the relevant bifurcations within this range of amplitude and be as precise as the response obtained using the modal expansion (16).

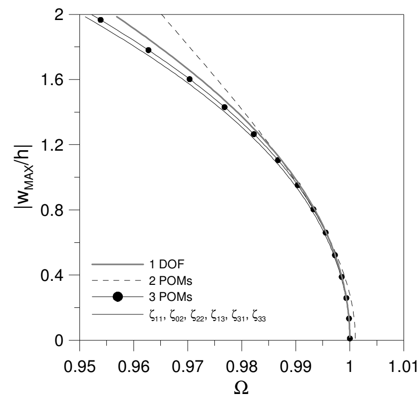
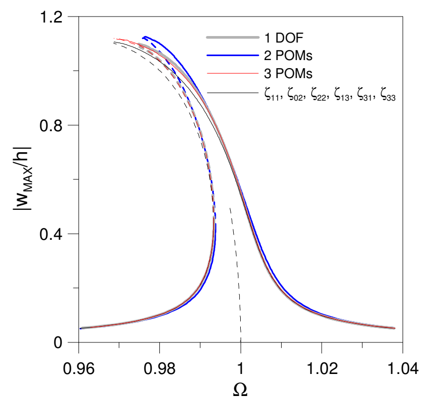
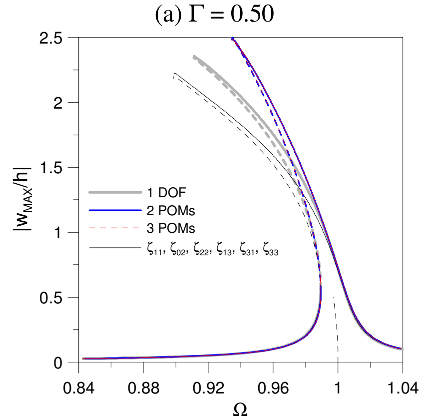


Figure 2 – Frequency-amplitude relation for the cylindrical shell. Softening behavior.



(a) Γ = 0.50



(b) Γ = 1.0

Figure 3 – Frequency-response curves (maximum lateral deflection versus excitation frequency) for increasing values of the magnitude of the external lateral pressure, Γ. ($\beta_1 = 2\varepsilon\rho h\omega_0$ with ε = 0.001). Solid line: stable. Dashed line: unstable.

Figure 3 shows the resonance curve of the shell subjected to a lateral pressure, as described by equation (4). The magnitude of the load is in Fig.

(3.a) equal to 50% of the static critical load of the shell under a uniform lateral pressure (Γ = 0.50), while in Fig. (3.b) the load magnitude is equal to the critical load Γ = 1.00. These load levels are of course well beyond the values expected in normal engineering applications, but shows the quality of the selected reduced order models. For Γ = 0.50, all models agree up to the maximum vibration amplitude, with only a small difference in the upper fold point for the model using two POMs. For Γ = 1.00, a limit case, the reduced order models show good agreement up to vibration amplitudes of the order of the shell thickness but lead to slightly less softening responses for higher vibration amplitudes.

4 Conclusion

In the present paper, Donnell shallow shell theory has been applied to model the dynamics of a thin-walled circular cylindrical shell under lateral pressure. Based on a modal solution obtained from perturbation techniques, a general solution for the displacement field is obtained satisfying all boundary, symmetry and continuity conditions of a simply-supported shell. The discretized shell equations are solved and non-linear frequency-amplitude relation is obtained. Then, the Karhunen-Loève method is employed to obtain the proper orthogonal modes and values. This procedure allows the quantification of the influence of each mode of the perturbation solution on the convergent response of the shell. The results also corroborate the coupling between asymmetric and symmetric modes. Based on this analysis, three different reduced order models are derived and compared. The results show that a small number of properly selected modes can describe the nonlinear behavior of the shell up to very large deflections. Finally, the results show that this technique can be used to derive consistent low-dimensional models for the non-linear static and dynamic analysis of cylindrical shells. It can also be extended to other structural components such as beams and plates, leading to efficient and well-founded low-dimensional models.

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References

Amabili, M., Sarkar, A., Païdoussis, M. P. (2006) Chaotic vibrations of circular cylindrical shells: Galerkin versus reduced-order models via the proper orthogonal decomposition method. *Journal of Sound and Vibration*, (290), pp. 736-762.

Awrejcewicz, J., Krys'ko, A. V. (2003) Analysis of complex parametric vibrations of plates and shells using Bubnov-Galerkin approach. *Archive of Applied Mechanics*, (73), pp. 495-504.

Bellizzi, S., Sampaio, R. (2006) POMs analysis of randomly vibrating systems obtained from Karhunen–Loève expansion. *Journal of Sound and Vibration*, (297), pp. 774-793.

Burkardt, J., Gunzburger, M., Lee, H. (2006), Centroidal Voronoi Tessellation-Based Reduced-Order Modeling of Complex Systems, *SIAM Journal on Scientific Computing*, 28(2) pp. 459-484

Gonçalves, P. B., Batista, R. C. (1988) Non-linear vibration analysis of fluid-filled cylindrical shells. *Journal of Sound and Vibration*, (127), pp. 133-143.

Gonçalves, P. B., Del Prado, Z. J. G. N. (2002) Nonlinear oscilations and stability of parametrically excited cylindrical shells. *Meccanica,* (37), pp. 569-597.

Gonçalves, P. B., Del Prado, Z. J. G. N. (2005) Low-dimensional Galerkin model for nonlinear vibration and instability analysis of cylindrical shells. *Nonlinear Dynamics*, (41), pp. 129-145.

Gonçalves, P. B., Silva, F. M. A., Del Prado, Z. J. G. N. (2007) A low dimensional model for nonlinear vibration analysis of cylindrical shells base don Karhunen-Loève decomposition. *EUROMECH colloquium 483*, Porto, Portugal, pp. 65-68.

Hunt, G. W., Williams, K. A. J., Cowell, R. G. (1986) Hidden symmetry concepts in the elastic buckling of axially loaded cylinders. *International Journal of Solid and Structures*, (22), pp.1501-1515.

Rega, G., Troger, H. (2005) Dimension reduction of dynamical systems: Methods, models, applications. *Nonlinear Dynamics,* (41), pp. 1-15.

Seydel, R. (1988) *From Equilibrium to Chaos: Practical Bifurcation and Stability Analysis.* Elsevier, Amsterdam.

Shaw, S. W., C. Pierre, C. (1993), Normal modes for nonlinear vibratory systems, *Journal of Sound and Vibration* 164(1) pp. 85-124

Sirovich, L. (1987a) Turbulence and the dynamics of coherent structures part I: coherent structures. *Quartely of Applied Mathematics,* (45), pp. 561-571.

Sirovich, L. (1987b) Turbulence and the dynamics of coherent structures part II: symmetries and transformations. *Quartely of Applied Mathematics,* (45), pp. 573-582.

Sirovich, L. (1987c) Turbulence and the dynamics of coherent structures part III: dynamics and scaling. *Quartely of Applied Mathematics*, (45), pp. 583-590.

Steindl, A., Troger, H. (2001) Methods for dimension reduction and their applications in nonlinear dynamics. *International Journal of Solids and Structures,* (38), pp. 2131-2147.