ON EFFICIENCY OF THE GRID OPTIMAL SYNTHESIS TO CONTROL PROBLEMS OF PRESCRIBED DURATION

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The subject of the paper is estimation of new method for constructing feedbacks. The researches follow to N.N. Krasovskii [4,5] formalization of feedbacks.

We consider optimal control problems of prescribed duration on the plane. Dynamics of controlled systems are nonlinear. Values of controls are restricted by geometrical constrains. Running cost functionals of the Bolza type are minimized along trajectories of the systems on time intervals of prescribed duration.

A new numerical method for solving optimal control problems of prescribed duration is suggested. It based on a generalization of the method of characteristics for the Hamilton – Jacobi – Bellman equation. The data of problems are assumed to be Lipschitz continuous. Constructions of optimal grid synthesis are provided and numerical algorithms are created. Efficiency of the algorithms is discussed. Estimations for difference between the optimal result and the result of control via suggested grid synthesis are obtained. Examples of solving model problems on the plane are exposed to illustrate the work of algorithms and to compare results of the new method with other known methods.

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Statement of the problem

We consider a controlled system:

$$\dot{x}(t) = f(t, x, u), \quad u \in P, \quad t \in [0, T],\ x \in \mathbb{R}^n, \qquad (1)$$
$$x(t_0) = x_0, \quad (t_0, x_0) \in [0, T] \times \mathbb{R}^n = \mathrm{cl}\,\Pi_T,$$

where control u belongs to the given compact set $P \subset \mathbb{R}^m$, a time interval $[0, T]$ is fixed. The set $U_{t_0,T}$ of admissible controls is

$$U_{t_0,T} = \{u(\cdot)\colon [t_0, T] \mapsto P \text{ — is measurable}\}.$$

Consider the cost functional of the form:

$$I_{t_0,\,x_0}(x(\cdot), u(\cdot)) = \min_{\theta \in [t_0,T]} \{\sigma(\theta,\ x(\theta; t_0, x_0, u(\cdot))) + \int_{\theta}^{t_0} g(t, x(t), u(t))dt\}, \qquad (2)$$

where $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))\colon [t_0, T] \mapsto \mathbb{R}^n$ is a trajectory of the system started at the state $x(t_0) = x_0$ under an admissible control $u(\cdot)$. We define the optimal result as follows:

$$V(t_0, x_0) = \inf_{u(\cdot) \in U_{t_0,T}} I_{t_0,\,x_0}(x(\cdot; t_0, x_0, u(\cdot)), u(\cdot)), \qquad (3)$$

The function $\mathrm{cl}\,\Pi_T \ni (t_0, x_0) \mapsto V(t_0, x_0) \in \mathbb{R}$ is called the value function of the problem (1)–(3).

Assumptions

We consider the problem (1)–(3) under the following assumptions on the data.
A1. Functions $f(t, x, u)$ and $g(t, x, u)$ in (1), (2) are defined and continuous on the set $\mathrm{cl}\,\Pi_T \times P$,

$$||f(t', x', u) - f(t'', x'', u)|| \le L_1(|t' - t''| + ||x' - x''||),$$
$$|g(t', x', u) - g(t'', x'', u)| \le L_1(|t' - t''| + ||x' - x''||),$$

where $L_1 > 0$, (t', x'), $(t'', x'') \in \mathrm{cl}\,\Pi_T$, $u \in P$.
A2. The extendibility conditions hold

$$||f(t, x, u)|| \le K_1(1 + ||x||), \quad |g(t, x, u)| \le K_1(1 + ||x||),$$

where $K_1 > 0$, $(t, x, u) \in \mathrm{cl}\,\Pi_T \times P$.
A3. The function $\sigma(t, x)$ in (2) is defined and continuous on $R \times \mathbb{R}^n$, for any $(t, x) \in \mathrm{cl}\,\Pi_T$ there exists the superdifferential $\partial\sigma(t, x)$:

$$\partial\sigma(t, x) = \{(a, p) \in \mathbb{R} \times \mathbb{R}^n\colon \quad \forall (t + \Delta t, x + \Delta x) \in B_\varepsilon(t, x)$$
$$\sigma(t + \Delta t, x + \Delta x) - \sigma(t, x) \le a\Delta t + < p, \Delta x > + o(\Delta t + \Delta x)\},$$

where $o(\Delta t + \Delta x)/(\Delta t + \Delta x) \to 0$, as $\Delta t + \Delta x \to 0$, and

$$\exists L_2 > 0 : \quad \alpha + p \le L_2 \quad \forall (\alpha, p) \in \partial\sigma(t, x).$$

Here $\langle\cdot,\cdot\rangle$ is inner product.
A4. For all $(t,x) \in \operatorname{cl}\Pi_T,\ p \in \mathbb{R}^n$, the set

$$\operatorname{Arg}\min_{(f,g)\in E(t,x)}[\langle p,f\rangle + g] = \{(f^0(t,x,p), g^0(t,x,p))\}$$

is assumed to be a singleton. Here the set

$$E(t,x) = \{(f(t,x,u), g(t,x,u))\colon u \in P\}.$$

Preliminaries

It is known that assumptions A1–A4 in the problem (1)—(3) provide for the value function $V(t,x)$ (3) local Lipschitz continuity with constants $L_V = L_V(D) > 0,\ D \subset \operatorname{cl}\Pi_T$. At any point $(t,x) \in \Pi_T$ there exists the superdifferential $\partial V(t,x)$ [3, 9]. The value function $V(t,x)$ coincides with the unique minimax/viscosity solution [1, 8] of the Cauchy problem for the Bellman equation

$$\partial V(t,x)/\partial t + \min_{u\in P}[\langle D_x V(t,x), f(t,x,u)\rangle + g(t,x,u)] = 0, \qquad (t,x) \in \Pi_T, \quad (4)$$

$$V(T,x) = \sigma(T,x), \quad \forall x \in \mathbb{R}^n, \quad (5)$$

with the additional restriction

$$V(t,x) \le \sigma(t,x), \quad \forall (t,x) \in \operatorname{cl}\Pi_T. \quad (6)$$

Here

$$D_x V(t,x) = \left(\frac{\partial V(t,x)}{\partial x_1}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right).$$

Generalized method of characteristics

As follows from assumptions A1–A4, the Hamiltonian in problem (1)–(3) has the form

$$H(t,x,p) = \min_{u\in P}\{\langle p,\ f(t,x,u)\rangle + g(t,x,u)\} = \langle p,\ f^0(t,x,p)\rangle + g^0(t,x,p),$$

and the relations hold

$$D_p H(t,x,p) = f^0(t,x,p),$$
$$\langle p, D_p H(t,x,p)\rangle - H(t,x,p) = -g^0(t,x,p),$$

where $(f^0(t,x,p), g^0(t,x,p)) \in E(t,x)$. The Hamiltonian is Lipschitz continuous relative to (t,x) and continuous in the whole space. It is well known that a Lipschitz continuous function is differentiable almost everywhere (see, for example, [3]).
To generalize classical characteristic system of ODEs we involve the Clarke's superdifferential with respect to variable x [3]:

$$\partial_x^{\mathrm{cl}} H(t,x,p) = \operatorname{co}\left\{\lim_{x_i\to x} D_x H(t,x_i,p)\right\},$$

where (t, x_i, p) are the points of differentiability. We introduce generalized characteristics for the Bellman equation as solutions of the characteristic differential inclusions

$$\begin{cases} \frac{d\hat{x}}{dt} = & D_{\hat{p}}H(t,\hat{x},\hat{p}) = f^0(t,\hat{x},\hat{p}), \\ \frac{d\hat{p}}{dt} \in & -\partial^{\mathrm{cl}}_{\hat{x}}H(t,\hat{x},\hat{p}), \\ \frac{d\hat{z}}{dt} = & \langle \hat{p}, D_{\hat{p}}H(t,\hat{x},\hat{p})\rangle - H(t,\hat{x},\hat{p}) = -g^0(t,\hat{x},\hat{p}). \end{cases} \tag{7}$$

Due to the boundary condition (5) and restrictions (6) for the value function the following boundary conditions arise:

$$\hat{x}(T,y) = y,\ \hat{p}(T,y) \in \partial_y\sigma(T,y),\ \hat{z}(T,y) = \sigma(T,y),\ y \in \mathbb{R}^n, \tag{8}$$

$$\min_{\hat{x}(t,y^*)=x} \hat{z}(t,y^*) = \sigma(t,x) \Rightarrow \quad \hat{p}(t,y^*) \in \partial_y\sigma(t,x). \tag{9}$$

Definfition. 1 *Absolutely continuous functions*

$$\hat{x}(\cdot,y),\ \hat{p}(\cdot,y),\ \hat{z}(\cdot,y): \quad [0,T] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

satisfying the Hamiltonian differential inclusions (7) and the boundary conditions (9) are called generalized characteristics for the Bellman equation (4).

It is known [3, 7], that necessary optimality conditions in the problem (1)–(3) can be expressed in the form of differential Hamiltonian inclusions. It means that extremals and coextremals for any initial point $(t_0, x_0) \in \mathrm{cl}\,\Pi_T$ can be considered as generalized characteristics (7)–(9) crossed at the initial point. The following theorems are proven [9, 10]

Theorem 1 *If conditions* A1–A4 *are satisfied in the problem* (1)–(3)*, then the value function* $(t_0, x_0) \mapsto V(t_0, x_0)$ *has the representation*

$$V(t_0,x_0) = \min\Big\{\sigma(t_0,x_0),\ \min_{y\in\mathbb{R}^n:\ \hat{x}(t_0,y)=x_0} \hat{z}(t_0,y)\Big\}, \tag{10}$$

where $(\hat{x}(t_0,y),\ \hat{p}(t_0,y),\ \hat{z}\big(t_0,y)\big)$ *are generalized characteristics (7, 9).*

Theorem 2 *If conditions* A1–A4 *are satisfied in the problem* (1)–(3)*, then the optimal synthesis* $u^0(t,x)\colon \mathrm{cl}\,\Pi_T \mapsto \mathrm{P}$ *can be defined as follows*

$$(-H(t,x,p_*),\ p_*) \in \partial V(t,x), H(t,x,p_*) = \langle p_*, f^0(t,x,p_*)\rangle + g^0(t,x,p_*),$$
$$(f^0, g^0) = \big(f^0(t,x,p_*), g^0(t,x,p_*)\big) \in E(t,x),$$
$$u^0(t,x) \in \operatorname*{Arg}_{u\in P}\{f(t,x,u) = f^0,\ g(t,x,u) = g^0\}.$$

These facts are the basis of numerical algorithms for solving the optimal control problem (1)–(3). The new numerical method consists of a backward procedure of integrating generalized characteristic system and applying theorems 1–2 to construct a numerical approximation of the value function and an optimal grid synthesis $u^0_*(\cdot)$ on adaptive current grids. Nodes of the grids are points $(\hat{x}^j_i, \hat{p}^j_i, \hat{z}^j_i)$ on the generalized characteristics at instants t_i of a time partition $\Gamma = \{t_0 < t_1 < \ldots < t_N = T\} \subset [0,T]$.

Estimations

Estimations of the method is the important problem [6]. Following to N.N. Krasovskii [5] let us define the cost $\tilde{C}_{t_0,x_0}\big(\Gamma; u^0_*(\cdot)\big)$ of the grid feedback $u^0_*(t_i, \hat{x}^j_i)$

$$\tilde{C}_{t_0,x_0}\big(\Gamma; u^0_*(\cdot)\big) = I_{t_0,x_0}\big(x_\Gamma(\cdot), u_\Gamma(\cdot)\big)$$
$$u_\Gamma(t) = u_{i-1} = u\big(t_{i-1}, x_\Gamma(t_{i-1})\big) = \text{const} \quad \forall t \in [t_{i-1}, t_i),$$
$$\dot{x}_\Gamma(t) = f\big(t, x_\Gamma(t), u_{i-1}\big) \quad \forall t \in [t_{i-1}, t_i),\ t_i \in \Gamma,\ i = 1, \ldots, N.$$

The following estimates hold for difference between the optimal result $V(t_0,\, x_0)$ and the cost $\tilde{C}_{t_0,x_0}\big(\Gamma; u^0_*(\cdot)\big)$ of grid feedback $u^0_*(t_i, \hat{x}^j_i)$

$$|V(t_0,\, x_0) \;-\; \tilde{C}_{t_0,x_0}\big(\Gamma; u^0_*(\cdot)\big)| \;\leq\; C_1\Delta t \;+\; C_2\Delta t\,\omega(F\Delta t) \;+\; C_3\Delta x \;+\; C_4,$$

where F, C_1, C_2, C_3, C_4 are constants, $C_4 = 10^{-7}$, Δt is a diameter of Γ, Δx is a diameter of adaptive grids in the phase space and $\omega(\cdot)$ is the modulus of continuity of functions $p \mapsto f^0(t, x, p)$, $p \mapsto g^0(t, x, p)$ defined in A4.

Example

Dynamics

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 + u, \quad \|u\| \leq 1;\ t \in [0, 3.0].$$

The payoff functional

$$I_{t_0,\, x_0}\big(x(\cdot), u(\cdot)\big) = \min_{\theta \in [t_0, 3.0]} \Big\{ \frac{(x_1 - \sin\theta)^2 + (x_2 - \cos\theta)^2}{2} + \theta^2 - \int_{t_0}^{\theta} \sqrt{1 - u^2(t)}\,dt \Big\}.$$

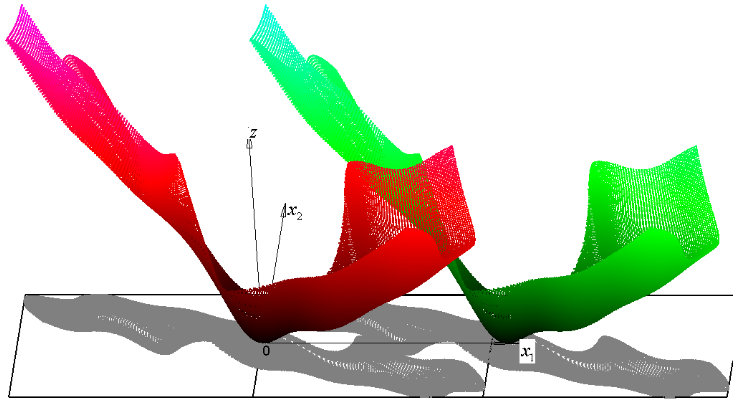
The Hamiltonian is $H(x, p) = p_1 x_2 - p_2 \sin x_1 - \sqrt{p_2^2 + 1}$. The characteristic system

$$\begin{cases} \dfrac{d\hat{x}_1}{dt} = \hat{x}_2, & \dfrac{d\hat{x}_2}{dt} = -\sin\hat{x}_1 - \dfrac{\hat{p}_2}{\sqrt{\hat{p}_2^2 + 1}}, \\ \dfrac{d\hat{p}_1}{dt} = \hat{p}_2 \cos\hat{x}_1, & \dfrac{d\hat{p}_2}{dt} = -\hat{p}_1, \quad \dfrac{d\hat{z}}{dt} = \dfrac{1}{\sqrt{\hat{p}_1^2 + 1}}. \end{cases}$$

Boundary conditions

$$\hat{x}_1(3.0, y) = y_1,\ \hat{x}_2(3.0, y) = y_2,$$
$$\hat{p}_1(3.0, y) = y_1 - \sin 3.0,\ \hat{p}_2(3.0, y) = y_2 - \cos 3.0,$$
$$\hat{z}(3.0, y) = \frac{(y_1 - \sin 3.0)^2 + (y_2 - \cos 3.0)^2}{2} + (3.0)^2,$$
$$\hat{z}(t, y) = \frac{(\hat{x}_1 - \sin t)^2 + (\hat{x}_2 - \cos t)^2}{2} + t^2 \Rightarrow \begin{cases} \hat{p}_1(t, y) = \hat{x}_1 - \sin t, \\ \hat{p}_2(t, y) = \hat{x}_2 - \cos t. \end{cases}$$

Parameters of approximation $\Delta x = 0.04$, $\Delta t = 0.01$.



The graphs of $\tilde{V}(t,x)$ and $I_{t,x}\big(\tilde{x}(\cdot),\tilde{u}(\cdot)\big)$ at instant $t=0$.

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