PARAMETRIC DOUBLE PENDULUM

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Abstract
We have studied a parametric double pendulum which is driven by a pulsating support motion $y_p = a\cos(2\pi ft)$ in the vertical direction (see Fig. 1). The arm 1 is attached to the oscillating pivot and is heavier than the arm 2 ($M_1 \sim 3.5\,M_2$). Both arms have approximately the same length. The frequencies of the two fundamental modes are $f_{in} \approx 1.4$Hz (in phase) and $f_{out} = 2.4$Hz (out of phase). By letting θ_1 and θ_2 denote the angles that arm 1 and 2 make with the vertical, respectively, and by direct integration of the equations of motion, we have determined the borders of the stability region of the four fixed points $(\theta_1^*, \theta_2^*) = (0,0), (0,\pi), (\pi,0)$ and (π,π) in the parameter space (f,a). We have observed a small region where all the fixed points are stable and a large one where all of them are unstable.

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1 Introduction

The phenomenon of parametric instability is frequently encountered in mechanics (e.g., dynamic buckling of columns, rings and shells, water waves in vertically forced containers, stability of general motions) as well as in various areas of physics (propagation of electromagnetic waves in media with a periodic structure, motions of electrons in a crystal lattice,...). Faraday [Faraday, 1831] was one of the first to observe the phenomenon of parametric resonance noting that surface waves in a fluid-filled cylinder under vertical excitation exhibited twice the period of the excitation. Two decades later, Lord Rayleigh [Strutt, 1883] provided a theoretical basis for interpreting the parametric resonance of strings and conducted some experiments.

The studies on parametric resonance form a rich body of literature, including several books and monographs. Some of them are mostly devoted to the theory of parametrically excited linear discrete systems [Yakubovich, 1975]. A good deal of efforts has been directed towards methods for constructing the instability regions of parametrically excited systems mathematically treated as linear differential equations with periodic coefficients. In particular, a large number of works has addressed the principal parametric resonance of rods and its cancellation [Lacarbonara, 2007].

Numerous works have dealt with the parametric resonance of the simple pendulum due to its paradigmatic nature and simplicity for experimental investigations. As well known, for small oscillations, the equation of motion reduces to the Hill's equation with $g(t) = g + p(t)$, or to Mathieu's equation if $p(t)$ is a sinusoidal forcing [TerHaar, 1965; Berge, 1984; Butikov, 1984]. Some features of the chaotic dynamics of the single pendulum were discussed in [Butikov, 1984; Smith, 1992; Sanjuán, 1998] and in [Jäckel, 1998] for the double pendulum, and the forced triple pendulum with impacts has been studied theoretical and experimentally by Awrejcewicz et. al. [Awrejcewicz, 2004; Awrejcewicz 2005; Awrejcewicz 2006].

A planar double pendulum is a two degree-of-freedom system whose dynamics are described by two coupled second-order equations of motion. Our pendulum has the two arms with approximately the same length and with arm 1 heavier than arm 2 ($M_1 \sim 3.5\,M_2$). The frequencies of the two fundamental modes are $f_{in} \approx 1.44$Hz (in phase) and $f_{out} = 2.42$Hz (out phase). There are four fixed points $(\theta_1^*, \theta_2^*) = (0,0), (0,\pi), (\pi,0)$ and (π,π) with $(\omega_1^*, \omega_2^*) = (0,0)$ for all of them, where the first one is stable.

When the axis of arm 1 is attached to a pivot made to oscillate sinusoidally, $y_p = a\cos(2\pi ft)$, in the vertical direction, it exhibits some interesting features such as the stability of the fixed points depending on the values of the parameters f and a. We present a numerical study of the regions of stability in the parameter space (f, a) in the ranges $a \in [0, 5]$ cm and $f \in [0, 18]$ Hz, together with some experimental data.

We also reproduced the parametric resonances when the forcing frequencies f are equal to fundamental ones, (the f resonance $f = f_0$), and when $f = 2f_0$, (the $2f$ resonance), where f_0 corresponding to the fundamental modes above, occurring when the fixed point $(0, 0)$ loses the stability for minimum values of the forcing amplitude. For the $2f$ resonances, we also applied the Floquet Theory assuming that the solutions $\theta_i(t)$ are given by $\theta_i = e^{\lambda t}\cos(2\pi ft + \phi_i)$.

2 Results and discussion

Figure 1 shows a sketch of the double pendulum. The arms are 3mm-diameter aluminum tubes and the joints are made of the phenolic resin (celeron). The axis 1 is a 4mm-diameter steel cylinder with two 8mm-diameter ball bearings and the axis 2 is a 3mm-diameter steel cylinder with two 6mm ball bearings. We have determined the two normal frequencies $f_{in} = 1.44$Hz and $f_{out} = 2.42$Hz of the in-phase and out-of-phase modes with the parameters values shown in figure 1.

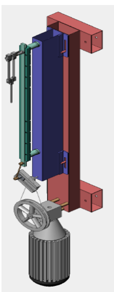

Figure 1. The double pendulum and parameters values: $L_1 = 10.86$cm, $m_1 = 18.45$g, $L_2 = 9.5$cm, $m_2 = 5.32$g, $I_2 = 79.4$g·cm^2, $I = 1342.3$g·cm^2, $C = 1325.3$g·cm, $R = 12.81$ g·cm, $\gamma = 139.3$ g·cm^2, $M_{Total} = 23.77$g

When the pendulum is harmonically forced via a vertical support motion of the form

$$y_p(t) = a\cos(2\pi ft), \qquad (1)$$

the pendulum motion is described by the following four non-autonomous ordinary-differential equations:

$$\begin{aligned}
&\dot\theta_1 = \omega_1 \\
&\dot\theta_2 = \omega_2 \\
&I\dot\omega_1 + \gamma\dot\omega_2\cos(\theta_1-\theta_2) - \gamma\sin(\theta_1-\theta_2)\dot\omega_2^2 \\
&-C(\ddot y_p - g)\sin(\theta_1) + b\omega_1 = 0 \\
&I_2\dot\omega_2 + \gamma\dot\omega_1\cos(\theta_1-\theta_2) - \gamma\sin(\theta_1-\theta_2)\dot\omega_1^2 \\
&-R(\ddot y_p - g)\sin(\theta_2) + b\omega_2 = 0.
\end{aligned} \qquad (2)$$

where M_i, I_i, L_i, G_i,and b_i are respectively, for each arm, the mass, the length and center of the mass position, the viscous friction coefficient, and $I = I_1 + M_2L_1^2$, $\gamma = L_1M_2G_2$, $C = M_2L_1 + M_1G_1$, and $R = M_2G_2$ (see Fig. 1 for the parameter values).

Without the forcing term, there are four fixed points $(\theta_1^*, \theta_2^*) = (0, 0)$, $(0, \pi)$, $(\pi, 0)$, (π, π), and all of them with $(\omega_1^*, \omega_2^*) = (0, 0)$, but just the first one is stable. However, due to an effective acceleration $g^* = g - \ddot y$ given by the modulation of the gravitational acceleration around $g = 9.78$m/s^2, the all four fixed points can be stabilized in well defined regions of the parameter space (f, a) even without friction.

By integrating Eqs. 2, for a time span of 200s, we obtained these regions of stabilization of each fixed point with the initial conditions $(\theta_1(0), \theta_2(0)) = (\theta_1^* + 10^{-280}, \theta_2^* + 10^{-280})$ and $(\omega_1(0), \omega_2(0)) = (0, 0)$, and observing the decay of the $\theta_1(t)$ and $\theta_2(t)$. To this end, we divided the parameter space, $f \in [0, 18]$ Hz and $a \in [0, 5]$ cm, in a lattice of 61×61 points. For each point the integration was done using the variable step Dormand-Prince algorithm. Thereafter, we refined the border looking for the $stable \longleftrightarrow unstable$ transitions dividing the a and f steps by 32.

The results are shown in Fig. 2 together with some experimental data. Below the dotted blue line the fixed point $(0, 0)$ is stable, the regions between the green lines, between the red lines and between the light blue ones the fixed points $(0, \pi)$, $(\pi, 0)$, (π, π) are stable, respectively. It should be noticed that the (π, π) fixed point stability region is embedded in the $(\pi, 0)$ region which, in turn, is also embedded in the $(0, \pi)$ region. Above 14 Hz, there exists a small region, (a is in the range $[1.3, 1.75]$ mm for $f = 18$ Hz), where all the stable regions overlap, so all fixed points are stable; on the other hand, there is a large region where no overlapping of the stability regions exists, hence, all fixed points are unstable.

In Fig. 2 we can also see two $1f$ resonances corresponding to the eigenfrequencies $f_{in} = 1.44$Hz and $f_{out} = 2.42$ Hz and two $2f$ resonances at 2.88 Hz and 4.84 Hz for the in-phase and out-of-phase modes, respectively. Our experimental data of the borders of stability regions is reasonably described by the simulations performed without friction despite the presence of viscous and Coulomb friction.

With an encoder with resolution of $2\pi/2500 rads$ we have measured the arm 1 velocity ω_1 for $f = 2.72Hz$ and amplitude equal $1.0cm$, see the red triangle point in

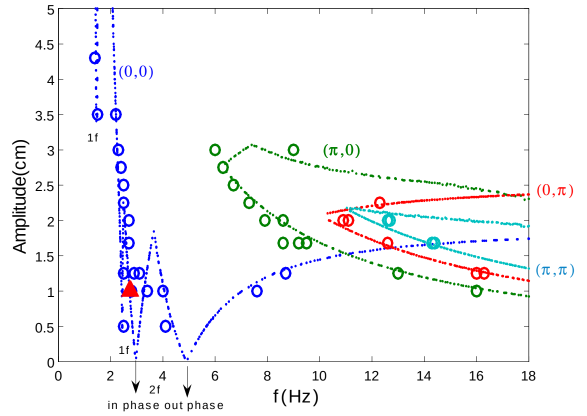

Figure 2. Parameter space (a, f). The dotted lines are the borders of the stabilization regions obtained by integration of Eqs. 2 without friction. The blue, green, red and light blue circles are experimental observations of the borders of the fixed points $(0, 0)$, $(0, \pi)$, $(\pi, 0)$, (π, π), respectively.

Fig. 2. After a transient time bigger than $20s$, the fixed point (0,0) loses its stability, as illustrated in Fig. 3, and thereafter an in phase mode oscillation takes place with a frequency which is half of the pivot oscillation, as shown by the Fourier transformer of ω_1 in Fig.4.

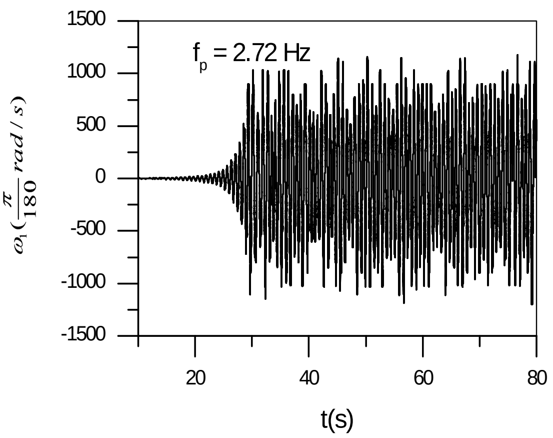

Figure 3. Illustration of the transition from the stable region of the fixed point (0,0) to the unstable region with the pivot oscillating with frequency $f_p = 2.72$ Hz and amplitude of 1.0 cm.

The border of the stable region and the unstable one of the fixed point $(0, 0)$ was also obtained by applying the Floquet theory for the $2f$ resonance, in which we supposed that the frequency forcing is twice the oscillation frequency

$$y_p = a\cos(4\pi f t) \tag{3}$$

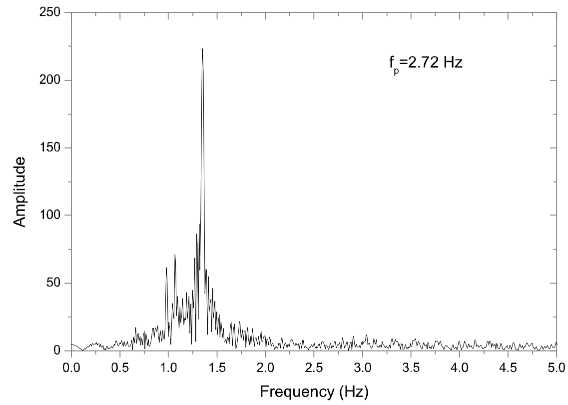

Figure 4. Amplitude of the ω_1 Fourier transformer showing the peak position at $f = f_p/2$.

for small oscillations approximation with solutions given by:

$$\theta_i = e^{\lambda t}\cos(2\pi f t + \phi_i) \tag{4}$$

where $i = 1, 2$, ϕ_i are the phase differences, λ is the Lyapunov exponent.

The border is detected by calculating numerically the eigenvalues of the characteristic matrix (eq. 5) when a sign change occurs.

The borders of the stability regions, also referred to as transition curves, could alternatively be obtained via application of an asymptotic technique as the method of multiple scales. The limitation of such an approach is that it is valid for small excitation amplitudes. However, the virtue is in that the asymptotic approximation is in closed form and provides also the post-critical responses of the double pendulum which account for the nonlinearities associated to the geometric and inertia terms.

$$\begin{bmatrix} \begin{pmatrix} I(u^2-\omega^2)+bu \\ +2Ca\omega^2+978C \end{pmatrix} & (-b-2Iu)\omega & -(\omega^2-u^2)\gamma & -2\gamma u\omega \\ (-b-2Iu)\omega & \begin{pmatrix} -I(u^2-\omega^2)-bu \\ 2Ca\omega^2-978C \end{pmatrix} & -2\gamma u\omega & (\omega^2-u^2)\gamma \\ -(\omega^2-u^2)\gamma & (-2Ju-b)\omega & \begin{pmatrix} J(u^2-\omega^2)+bu \\ 2Ra\omega^2+978R \end{pmatrix} & (-2Ju-b)\omega \\ -2\gamma u\omega & (\omega^2-u^2)\gamma & (-2Ju-b)\omega & \begin{pmatrix} -J(u^2-\omega^2)-bu \\ 2Ra\omega^2-978R \end{pmatrix} \end{bmatrix} \tag{5}$$

The results are shown in figure 5, together with the data obtained from the direct integration.

3 Conclusions

We have established the borders of the stabilization regions of the four fixed points of a parametric double

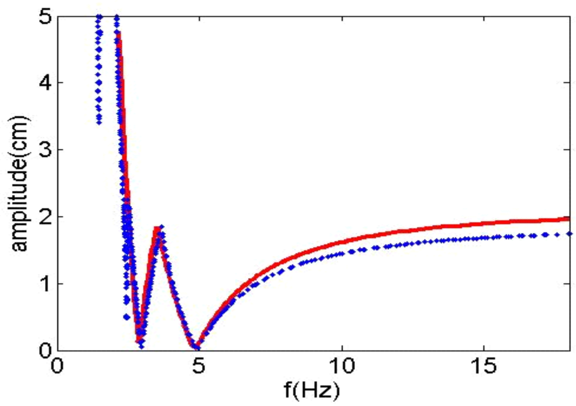

Figure 5. Parameter space (f, a).for the frictionless case. The blue line was obtained by direct integration of eq. 2. Below this line the fixed point $(0, 0)$ is stable and the other ones unstable. The red line was obtained by the approximate Floquet Theory.

pendulum by numerical integration of the equations of motion. We observed that the stable region of the fixed point (π, π) is embedded in the $(\pi, 0)-$region which, in turn, is also embedded in the $(0, \pi)$-region and all these regions overlap for $f \gtrsim 14$ Hz implying that all fixed points are stable in within this region. We also observed a big region, in which there is no overlapping, so all fixed points are unstable. The experimental data are reasonably described by these simulations without consideration of the friction.

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