ENERGY LOSSES OF IMPACTS WITH FRICTION

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ABSTRACT
Impacts with friction are typical for very many machine and mechanism problems. They arise by short-time contacts between two or more bodies, and they generate energy losses mainly due to friction in tangential contact direction but also by small dissipative effects in normal contact direction. During the last two decades a couple of impact models were established connected with the names of Moreau, Fremond and Glocker, which all work quite satisfactorily with respect to practical applications. We shall focus on Glocker's model, for which some experimental verifications are available. A missing link are energy considerations, which are available, but nevertheless do not provide us with a complete information for all possible cases. The paper tries to fill a bit this gap, though the investigations are based on a combined phenomenological and theoretical basis. A theoretical consideration alone is not known to the author.

1 INTRODUCTION

We consider rigid bodies as part of a multibody system. which come into contact including normal and tangential features, and we focus especially on short-time contacts being interested for the energy losses accompanying such processes. The principal situation is illustrated in Figure 1. Starting with the models as developed in [3], [1] and [8] we use the following classical assumption for impacts and also for impacts with friction:

- The duration of the impact is so short, that the mathematical description may assume a zero impact time.
- As a consequence we neglect wave processes, which would take place in a finite time interval.
- Following these assumptions the mass distribution of the body does not change during the impact, the bodies remain rigid.
- All positions and orientations of the impacting bodies remain constant. The translational and rotational velocities of the bodies are finite and may change jerkyly during the impact.
- Accordingly the position of the impact point and that of the normal and tangential vectors remain constant.
- All forces and torques, which are not impulsive forces and torques, remain also constant during the impact.
- All during the impact evolving impulses act during the impact in a constant direction. Their lines of action do not change and correspond to the normal and tangential vectors in the impact point.
- The impact can be divided into two phases: the compression phase and the expansion phase.

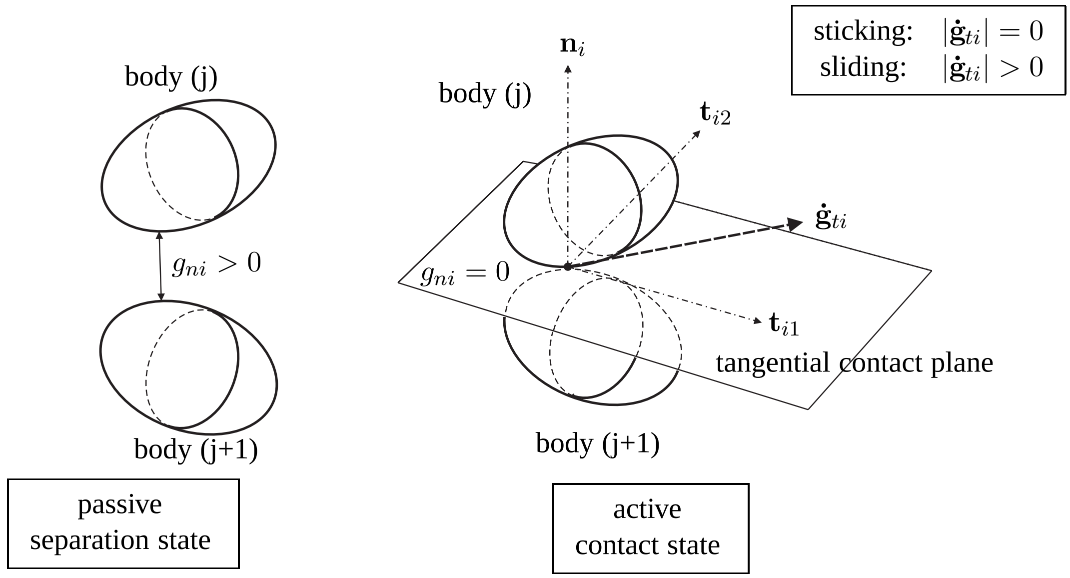

Figure 1: Principal Situation in a Multibody Contact (i)

- The compression phase starts at time t_A and ends at time t_C. The end of the compression equals the start of the expansion phase. Expansion is finished at time t_E, which is also the end of the impact.

During compression impulses in normal and tangential directions of the contact are stored, and during expansion these stored impulses are released, both processes accompanied by losses. The losses result from an application of Poisson's friction law. A detailed description of these processes may be found in the literature [8], [3], [4] and [5] with increasing depth of mathematical representation.

2 IMPACT CHARACTERISTICS

According to Moreau ([6], [5]) we may express the dynamics with and without impacts by one measure differential equation in the form

$$\mathbf{M}d\mathbf{u} + \mathbf{h}dt - \mathbf{W}d\mathbf{\Lambda} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} \mathbf{M}\dot{\mathbf{u}} + \mathbf{h} - \mathbf{W}\lambda = \mathbf{0} & (t \neq t_i) \\ \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) - \mathbf{W}\mathbf{\Lambda} = \mathbf{0} & (t = t_i) \end{cases} \tag{1}$$

The part $\mathbf{W}\lambda$ contains all contact reactions due to non-impulsive contacts and the part $\mathbf{W}\Lambda$ all impulsive contact reactions. The time $t_i \in I_{kl}$ represents one of the instants (i), where an impact takes place. The vector $\mathbf{h}$ includes all non-impulsive and applied forces, whatsoever, and for multibody systems without closed loops we also include in the generalized coordinates $(\mathbf{q}, \dot{\mathbf{q}})$ all bilateral constraints.

We start with the compression phase and the normal impact direction. At the end of compression the relative normal velocity is either zero or non-negative, $\dot{g}_{Ni} \geq 0$. The tangential compression phase is characterized mainly by friction. At the end of compression we may have three states: Firstly, sliding in a positive tangential direction ($\dot{g}_{NC} > 0$), where the tangential impulse acts during this phase in opposite direction with $\Lambda_{TC} = -\mu\Lambda_{NC}$, secondly, sticking at the end of compression ($\dot{g}_{NC} = 0$), where the tangential impulse is small enough to generate sticking during the whole compression phase, and thirdly, sliding in a negative tangential direction ($\dot{g}_{NC} < 0$), where the tangential impulse acts

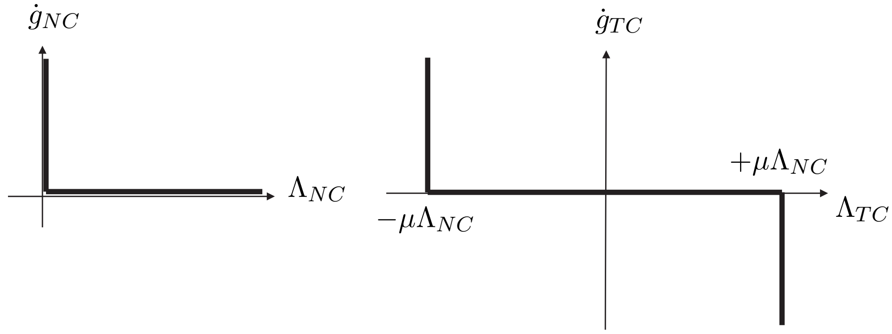

Figure 2: Contact Laws for Impacts

during this phase in opposite direction with $\Lambda_{TC} = +\mu\Lambda_{NC}$. The processes for these two directions are depicted by the well-known graphs of Figure 2.

The impulse stored during compression is released with a loss governed by Poisson's law. Restoring the tangential impulse affords some additional considerations. According to Poisson's law we get back the stored tangential impulse Λ_{TCi} of the (i)th contact with a certain loss, that is $(\varepsilon_{Ti}\Lambda_{TCi})$, where Poisson's losses are quantified by $(0 \leq \varepsilon_{Ti} \leq 1)$. The tangential friction coefficient ε_{Ti} must be measured. But this contains not all losses during expansion. The restoration of the tangential impulse possesses another quality compared with the restoration of the normal impulse, because it cannot take place independently from the normal impulse, which as a matter of fact represents the driving constraint impulse for the generation of tangential friction forces. Therefore we shall assume, that the restoration of the tangential impulse is additionally accompanied by losses in "normal direction" expressed by ε_{Ni}. Figure 3 illustrates these processes, see also [3] and [8].

3 ENERGY LOSSES

The loss of energy is the difference of the total system energy after an impact and before an impact. In terms of the generalized velocities $\dot{\mathbf{q}}$ we write

$$\begin{aligned} \Delta \mathrm{T} =& \mathrm{T}_E - \mathrm{T}_A \quad \leq \quad 0 \\ \Delta \mathrm{T} =& \frac{1}{2}\dot{\mathbf{q}}_E^T \mathbf{M}\dot{\mathbf{q}}_E - \frac{1}{2}\dot{\mathbf{q}}_A^T \mathbf{M}\dot{\mathbf{q}}_A = \frac{1}{2}(\dot{\mathbf{q}}_E + \dot{\mathbf{q}}_A)^T \mathbf{M}(\dot{\mathbf{q}}_E - \dot{\mathbf{q}}_A). \end{aligned} \tag{2}$$

These are expressions considering scleronomic systems without an excitation by external kinematical sources. Applying the relations as presented in [8], we get for the energy expression the form

$$\begin{aligned} 2\Delta \mathrm{T} =& 2\Delta \mathrm{T}_1 + \Delta \mathrm{T}_2 = +2 \begin{pmatrix} \dot{\mathbf{g}}_{NE} \\ \dot{\mathbf{g}}_{TE} \end{pmatrix}^T \left[\begin{pmatrix} \mathbf{\Lambda}_{NC} \\ \mathbf{\Lambda}_{TC} \end{pmatrix} + \begin{pmatrix} \mathbf{\Lambda}_{NE} \\ \mathbf{\Lambda}_{TE} \end{pmatrix} \right] - \\ & - \left[\begin{pmatrix} \mathbf{\Lambda}_{NC} \\ \mathbf{\Lambda}_{TC} \end{pmatrix} + \begin{pmatrix} \mathbf{\Lambda}_{NE} \\ \mathbf{\Lambda}_{TE} \end{pmatrix} \right]^T \mathbf{G} \left[\begin{pmatrix} \mathbf{\Lambda}_{NC} \\ \mathbf{\Lambda}_{TC} \end{pmatrix} + \begin{pmatrix} \mathbf{\Lambda}_{NE} \\ \mathbf{\Lambda}_{TE} \end{pmatrix} \right] \end{aligned}$$

$$\text{with} \qquad \boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_{NN} & \boldsymbol{G}_{NT} \\ \boldsymbol{G}_{TN} & \boldsymbol{G}_{TT} \end{pmatrix} \quad \text{where} \quad \boldsymbol{G}_{ij} = \boldsymbol{W}_i^T \boldsymbol{M}^{-1} \boldsymbol{W}_j, \quad i,j \in \{N,T\} \tag{3}$$

$\boldsymbol{G}$ is the mass projection matrix, which is quadratic and positive definite with the exception of dependent constraints, where it is semidefinite. The $\dot{\mathbf{g}}$ are relative contact velocities and the $\mathbf{\Lambda}$ impulses. The

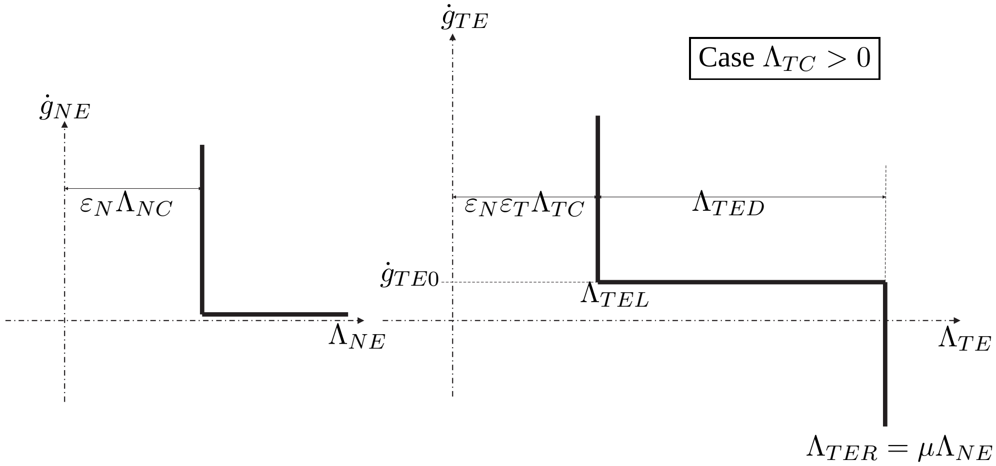

Figure 3: Shifted Normal and Tangential Characteristics for Impact Expansion

indices N,T stand for normal and tangential, respectively, the indices C,E for the end of compression and the end of expansion, respectively. The second term of the energy equation is a quadratic form and for itself always positive or zero, and from this we have $\Delta \mathrm{T}_2 \leq 0$, always. The energy loss has to be negative, which will be decided by the first term of the above relations. If this term is negative or at least zero, the condition $\Delta \mathrm{T} \leq 0$ will hold. Therefore we shall concentrate on the first term which writes in more detail

$$\begin{aligned} 2\Delta \mathrm{T}_1 &= +2 \begin{pmatrix} \dot{\mathbf{g}}_{NE} \\ \dot{\mathbf{g}}_{TE} \end{pmatrix}^T \left[\begin{pmatrix} \mathbf{\Lambda}_{NC} \\ \mathbf{\Lambda}_{TC} \end{pmatrix} + \begin{pmatrix} \mathbf{\Lambda}_{NE} \\ \mathbf{\Lambda}_{TE} \end{pmatrix} \right] = \\ &= \quad 2\dot{\mathbf{g}}_{NE}^T(\mathbf{\Lambda}_{NC} + \mathbf{\Lambda}_{NE}) + \dot{\mathbf{g}}_{TE}^T(\mathbf{\Lambda}_{TC} + \mathbf{\Lambda}_{TE}) \end{aligned} \tag{4}$$

For the evaluation of this equation we have to discuss the models. The compression/expansion model as considered here is a very powerful one and approved by many applications, but it does not give any evidence of the points where transition from friction to sliding, or vice versa, take place. But this is decisive for evaluating energies. As we are dealing with models, we are free to define that in a way, which is physically plausible. And we define, that transitions occur always at the end of the phases compression and expansion in an infinitesimal short instant of time not influencing the impact dynamics but only going from one branch of the corner laws to another branch, which means, transitions take place in the corners of the contact laws.

An example illustrates it: Having sliding or sticking during the expansion, we need to have also a zero normal velocity $\dot{\mathbf{g}}_N$ for realizing these states. On the other hand, if we have detachment at the end of expansion we need to have a normal velocity $\dot{\mathbf{g}}_N \neq \mathbf{0}$, which contradicts the first necessity. A way out of this dilemma can only consist in a transition definition taking place firstly at the very end of any of the two phases and taking place secondly extremely shortly without energy losses.

So it can be shown, that the first term $\dot{\mathbf{g}}_{NE}^T(\mathbf{\Lambda}_{NC} + \mathbf{\Lambda}_{NE})$ of the energy equation (4), last line, is not zero due to positive normal impulses $(\mathbf{\Lambda}_{NC} + \mathbf{\Lambda}_{NE})$ and due to a non-zero end velocity $\dot{\mathbf{g}}_{NE}$ after the impact, which is physically reasonable for a separation of the two contacting bodies. But on the other hand sliding or sticking during expansion requires a zero normal relative velocity $\dot{\mathbf{g}}_{NE}$ in the contact, which makes the above mentioned term to zero. The $\mathbf{\Lambda}_{NE}$-value slips into the corner of Figure (3) allowing the system to build up the necessary separation velocity. From this we assume, that during the expansion phase the term $\dot{\mathbf{g}}_{NE}^T(\mathbf{\Lambda}_{NC} + \mathbf{\Lambda}_{NE}) = 0$ is zero.

As a result of these arguments and of the last condition of continual contact during the impact we get for compression and expansion $\mathbf{\Lambda}_N > 0$ and $\dot{\mathbf{g}}_N = 0$, which is also part of the complementarity, and therefore simply

$$2\Delta \mathrm{T}_1 = 2\dot{\mathbf{g}}_{TE}^T(\mathbf{\Lambda}_{TC} + \mathbf{\Lambda}_{TE}), \tag{5}$$

the sign of which we have to investigate. For this purpose we consider this equation with respect to the following four cases, see for the arguments always the Figures 2 and 3:

- **sticking during compression, sticking during expansion**

  The tangential impulses have to be within the appropriate friction cones. The tangential velocities are zero, therefore we need not to consider the magnitudes of the impulses.

  $$-\mathrm{diag}(\mu_0)\mathbf{\Lambda}_{NC} \leq \mathbf{\Lambda}_{TC} \leq +\mathrm{diag}(\mu_0)\mathbf{\Lambda}_{NC}, \quad \mathbf{\Lambda}_{TEL} \leq \mathbf{\Lambda}_{TE} \leq \mathbf{\Lambda}_{TER}$$
  $$\Longrightarrow \quad \dot{\mathbf{g}}_{TE}^T(\mathbf{\Lambda}_{TC} + \mathbf{\Lambda}_{TE}) = 0$$

- **sliding during compression, sliding during expansion**

  Sliding means single-valued impulse laws according to Coulomb's law. Some difficulties will appear for the cases with reversed sliding, that means, with a tangential relative velocity the sign of which is different during compression and during expansion. Therefore we have to consider the two cases without and with tangential reversibility. For the first case we do not have a change of sign of the relative tangential velocity, which gives $\mathrm{sign}(\dot{\mathbf{g}}_{TC}) = \mathrm{sign}(\dot{\mathbf{g}}_{TE})$. This comes out with the relations:

  $$\dot{\mathbf{g}}_{TE}^T\mathbf{\Lambda}_{TC} = -\dot{\mathbf{g}}_{TE}^T[\mathrm{diag}(\mu)\mathrm{sign}(\dot{\mathbf{g}}_{TE})\mathbf{\Lambda}_{NC}] = -\mathrm{diag}(\mu)|\dot{\mathbf{g}}_{TE}|\mathbf{\Lambda}_{NC} \leq \mathbf{0},$$
  $$\Longrightarrow \quad \dot{\mathbf{g}}_{TE}^T(\mathbf{\Lambda}_{TC} + \mathbf{\Lambda}_{TE}) < 0$$

  The case with tangential reversibility is more complicated, because it includes a change of sign of the tangential relative velocity and thus at least an extremely short stiction phase, which we put exactly at the point (end of compression)/(beginning of expansion). The sliding velocity during compression decreases until it arrives at one of the corners of Figure 2, then we get an extremely short shift from this corner to the other one, which allows the contact to build up a tangential velocity with an opposite sign, then valid for the expansion phase. Only by such a short stiction phase a reversal of tangential velocity is possible. On the other hand such a transition from stick to slip, as short as it might be, follows the same process as for the next case sticking/sliding. Therefore it is dissipative:

  $$\Longrightarrow \quad \dot{\mathbf{g}}_{TE}^T(\mathbf{\Lambda}_{TC} + \mathbf{\Lambda}_{TE}) < 0$$

- **sticking during compression, sliding during expansion**

  The transition from sticking in compression and sliding in expansion follows the mechanism (Figure 2): If $\mathbf{\Lambda}_{TC} \gtrless \mathbf{0}$, then sliding is only possible for being at the very end of compression on the friction cone boundary with $\mathbf{\Lambda}_{TC} = \pm\mathrm{diag}(\mu)\mathbf{\Lambda}_{NC}$ and $\dot{\mathbf{g}}_{TC-at} \lessgtr \mathbf{0}$ (at = after transition stick-slip). This results always in a negative sign of the expression $(\dot{\mathbf{g}}_{TE}^T\mathbf{\Lambda}_{TC})$. For the rest we assume a continuation of the signs after going from stick to slip $[\mathrm{sign}(\dot{\mathbf{g}}_{TE}) = \mathrm{sign}(\dot{\mathbf{g}}_{TC-at})]$. Then we arrive at:

  $$\Longrightarrow \quad \dot{\mathbf{g}}_{TE}^T(\mathbf{\Lambda}_{TC} + \mathbf{\Lambda}_{TE}) < 0$$

- **sliding during compression, sticking during expansion**

  This case is again simpler, because we get sticking at the end with a zero relative tangential velocity. Therefore we need not to consider the impulses.

$$\Longrightarrow \quad \dot{\mathbf{g}}_{TE}^{T}(\mathbf{\Lambda}_{TC}+\mathbf{\Lambda}_{TE})=0$$

- **summarized result for all cases**

$$\Longrightarrow \quad \dot{\mathbf{g}}_{TE}^{T}(\mathbf{\Lambda}_{TC}+\mathbf{\Lambda}_{TE})\leq 0 \quad \Longrightarrow \quad \Delta \mathrm{T}_1 \leq 0 \quad \Longrightarrow \quad \Delta \mathrm{T} \leq 0$$

One may object that the above considerations assume in the case of multiple impacts the same impact structure for all simultaneously appearing impacts, which is usually not true. But even any combination of the above four cases for simultaneous impacts gives a loss of energy. Practical experience indicates in addition that the simultaneous appearance of impacts is extremely scarce, it is an event, which nearly does not happen.

As a final result we may state that the above evaluation confirms the physical argument, that any impact processes are accompanied by energy losses. This confirms also the well-known statement of Carnot, that *”in the absence of impressed impulses, the sudden introduction of stationary and persistent constraints that change some velocity reduces the kinetic energy. Hence, by the collision of inelastic bodies, some kinetic energy is always lost”*.

The above considerations and the underlying theory have been confirmed not by the experimental work of Beitelschmidt [1], but also by many industrial projects where the non-smooth methods were applied [9], [2].

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