ENOC 2008, Saint Petersburg, Russia, June, 30–July, 4 2008

NECESSARY CONDITIONS FOR THE IMPULSIVE-OPTIMAL CONTROL OF MECHANICAL SYSTEMS WITH BLOCKABLE DEGREES OF FREEDOM

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Abstract

In this work, the necessary conditions for the impulsive optimal control of rigid-body mechanical systems is studied, where the dynamics is represented in the first-order form. By the application of subdifferential calculus techniques to extended-valued lower semicontinuous functionals, Pontryagin's Maximum Principle (PMP) kind conditions are obtained. This representation enables to relate the necessary conditions to the classical nonimpulsive Pontryagin's principle. The necessary conditions are derived by making use of the concepts of internal boundary variations and discontinuous transversality conditions. These concepts are derived by the author and are presented in [Yunt, 2007b] and [Yunt, 2008] in first-order and second-order representations, respectively. In this work, it is assumed that the instant of discontinuity is reduced to an instant with Lebesgue measure zero, instead of taking an interval opening approach, which is the approach considered in literature so far.

Key words

Discontinuous Transversality Conditions, Impulsive Pontryagin's Principle

1 Introduction

The optimal control of mechanical systems with impulsively blockable DOF is investigated and necessary conditions are stated. The conditions are obtained by the application subdifferential calculus techniques to extended-valued lower semi-continuous generalized Bolza functional that is evaluated on multiple intervals. Contrary to the approach in literature so far, the instant of possibly impulsive transition is considered as a Lebesgue negligible instant.

The main issue in the optimal control of impactive mechanical systems has been the blending of impact mechanics with impulsive optimal control. The crux in the derivation of these necessary conditions is to handle joint discontinuity of the state and the dual state on a Lebesgue negligible interval. In the framework of integration theory, this has long been recognized as a problem if state and costate should become concurrently discontinuous as has been addressed in [Moreau,1988a] and [Rockafellar, 1976]. Rockafellar studied in [Rockafellar, 1976] the discontinuity of the dual state in constrained convex optimal control problems but dispensed of attacking the problem concurrent discontinuity of state and costate. Moreau gave in [Moreau,1988a] partial integration formulas for differential measures in general bilinear forms. In [Bressan, 2008] several classes of impulsive Lagrangian systems are studied. The main focus is impulses generated by sudden parameter changes such as inertial parameters that affect the momentum balance, or impulses arising due to structure of constraints of a mechanical system. A certain class of impulsive systems that resemble discontinuous diffusion processes are treated in [Bensoussan, 1979]. In the approaches provided in references such as [Arutyunov, Karamzin and Pereira, 2005], [Karamzin, 2006] the impulsive control problem is transformed into a problem of an ordinary differential inclusion problem, which requires to determine trajectories for the "discontinuous" states during the "impulsive" control action. The approach of this work is in comparison to other impulsive necessary conditions consistent with mainstream hybrid system modeling methods in which transitions happen instantaneously. The necessary conditions provide necessary criteria for the determination of optimal transition times and locations. The consideration of certain type of variations at the boundaries give birth to the concepts of internal boundary variations and discontinuous transversality conditions. A transition with a discontinuity in the state can be regarded as an internal boundary in the domain of interest.

In what follows next, the new concepts required to deal with this specific problem are introduced. In order to overcome the difficulties arising from joint discontinuity of state and costate, the instant of impulsive control action where discontinuity in the generalized velocities occur is considered as an internal boundary in the time domain. In [Yunt, 2007b], the concept of internal boundary variations are introduced literally, and as an application a theorem that states the necessary conditions for the impulsive time-optimal control of finite-dimensional Lagrangian systems is stated. In the framework of these concepts, philosophically, the instant of state discontinuity constitutes an internal boundary in the optimal control problem. The essential idea is thus to consider every point of the domain where continuity and differentiability ceases to exist, as a boundary of the problem. By introducing a boundary at an instant of a discontinuity, one has to notice that it has bilateral character, in the sense that the boundary constitutes an upper boundary for one segment of the interval whereas for the other segment a lower boundary in the time domain. The necessity that at a location of transition several conditions have to be fulfilled, gives rise to the idea of some sort of transversality conditions if one begins to consider an instant of discontinuity as a two-sided boundary where to arcs are "connected" discontinuously. This dependence is embedded in the concept of internal boundary variations. In order to obtain criteria for the optimality of the transition position, transition pre-, and post-transition generalised velocities, transition time and impulsive control, variations in these entities need to be considered, which represent in the setting of this work the internal boundary variations. At the boundaries of the time domain, the pre-transition state variations are considered separately from the post-transition variations. The absolute continuity of the generalized positions means that the total variation of the generalized positions at the pre-transition and post-transition instants are equal. The pre-transition and post-transition variations are interrelated by the transition conditions which can be seen as the bases of transversality conditions that join two trajectories discontinuously. The transition conditions are introduced symmetrically with respect to pre-, and post-transition states. The transition conditions are of two types, namely, the impact equation and the constitutive impact laws.

Definition 1 Transition Time A time instant of Lebesgue measure zero is considered as a transition time $t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}$ if one of the two events occur together or for itself:

> **Event 1** Some directions of motion of the system are opened or closed by the control strategy, which entails a change in the degrees of freedom (DOF) of the system.
> **Event 2** An impulsive control action is exerted on the system, which may be accompanied by a discontinuity of the generalized velocities of the Lagrangian system.

The concurrence of both events where some directions of motion are closed is called "blocking". In the time-optimal control of dynamical systems one has to consider the variations in the end time. In the classical calculus of variations where the final state and final time are free, the variations of the final state are composed of two parts, namely, the part that arises of the variations at a given time and the part arising from variations due to final time. Since the transitions times are assumed to be free, the two-part character of the variations at pre- and post-transition states is considered. The assumptions during a possibly impactive transition are given as follows:

Assumptions 1

(1) The transitions may be impactively.
(2) The generalized position remain unchanged during transition.
(3) The support of the transition set is a set of countably many time instants t_{i} which are Lebesgue-negligible.
(4) At a possibly impactive transition, the pre-transition controller configuration is assumed to be effective.
(5) There are no transitions at t_0 and t_{f}.

The above stated assumptions are converted into requirements to the variations at the internal boundaries. At the boundaries of the time domain, the pre-transition state variations are considered separately from the post-transition variations. The impact equations relate the discontinuity in the impulse of the Lagrangian system to the impulsive forces/controls. The impact law (.i.e. the moreau-newton impact law), however, is a constitutive law which is chosen depending on the modeling approach preferred. Blocking is modeled as a fully inelastic impact in the direction of interest as discussed in [Yunt, 2007a].

2 Generalised Problem of Bolza

Let us consider a problem in Bolza form (GPB), in which the objective is to choose an absolutely continuous arc $\mathbf{x} \in \mathcal{AC}$ in order to minimize

$$P: \quad J(x) = l(\mathbf{x}(a), \mathbf{x}(b)) + \int_a^b L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \tag{1}$$

where the function $L : [a, b] \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \to \mathbb{R} \cup \{+\infty\}$ is $\mathcal{L} \times \mathcal{B}$ measurable. Here $\mathcal{L} \times \mathcal{B}$ denotes the σ-algebra of subsets of $[a, b] \times \mathbb{R}^{\mathrm{n}}$ generated by product sets $\mathcal{M} \times \mathcal{N}$, where $\mathcal{M}$ is a Lebesgue measurable subset of $[a, b]$ and $\mathcal{N}$ is a Borel subset of $\mathbb{R}^{2n}$. For each $t \in [a, b]$, the function l and L are lower semi-continuous (l.s.c) on $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$, with values in $\mathbb{R} \cup \{+\infty\}$. For each $(t, \mathbf{x})$ in $[a, b] \times \mathbb{R}^{\mathrm{n}}$, the function $L(t, \mathbf{x}, \cdot)$ is convex and l represents the endpoint cost. GPB concerns the minimization of a functional whose form is identical to that in the classical calculus of variations. The GPB is distinguished from its classical version, by the extremely mild hypotheses imposed on the endpoint cost l and the integrand L. Both are allowed to take the value $+\infty$. An important class of optimal control problems constrain the derivative of an admissible arc and they can be stated as the following Mayer problem:

$$\min\{l(\mathbf{x}(a),\mathbf{x}(b)) : \dot{\mathbf{x}}(t) \in \mathcal{F}(t,\mathbf{x}(t)),\ a.e.\, t \in [a,b]\}. \tag{2}$$

The Mayer problem can be seen as minimizing the Bolza functional J over all arc $\mathbf{x}$. To cover the Mayer problem, it suffices to choose:

$$L(t,\mathbf{x},\mathbf{v}) = \Psi_{\mathcal{F}(t,\mathbf{x})}(\mathbf{v}) = \begin{cases} 0, \ \mathbf{v} \in \mathcal{F}(t,\mathbf{x}) \\ +\infty, \text{ otherwise.} \end{cases} \tag{3}$$

The function $\Psi_{\mathcal{C}}$ is called the indicator function of the set $\mathcal{C}$. It is evident that for any arc $\mathbf{x}$, one has

$$\int_a^b L(t,\mathbf{x},\dot{\mathbf{x}})\,dt = \begin{cases} 0, \ \dot{\mathbf{x}}(t) \in \mathcal{F}(t,\mathbf{x})\ a.e. \\ +\infty, \text{ otherwise.} \end{cases} \tag{4}$$

The Mayer type variational problem can arise from a typical dynamic constraint in controls such as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t),\boldsymbol{\tau}(t)),\ \boldsymbol{\tau}(t) \in \mathcal{C}_\tau,\ a.e. \quad t \in [a,b]. \tag{5}$$

If a control state-pair $(\boldsymbol{\tau},\mathbf{x})$ satisfies equation (5), then

$$\begin{aligned} \dot{\mathbf{x}}(t) &\in \mathcal{F}(t,\mathbf{x}(t)) \\ &:= \{\mathbf{f}(t,\mathbf{x}(t),\boldsymbol{\tau}(t)) : \boldsymbol{\tau}(t) \in \mathcal{C}_\tau\ a.e.\ t \in [a,b]\} \end{aligned} \tag{6}$$

certainly does. The well-known Fillipov's theorem is the statement that the reversal of the above statement is true as well.

In order to guarantee the well-behaving of $\mathcal{F}$ and l let following hypotheses hold:

Assumptions 2. An arc $\bar{\mathbf{x}} : [a,b] \to \mathbb{R}^{\mathrm{n}}$ is given. On some relatively open subset $\Omega \subseteq [a,b] \times \mathbb{R}^{\mathrm{n}}$ containing the graph of $\bar{\mathbf{x}}$, the following statements hold:

(1) The multifunction $\mathcal{F}$ is $\mathcal{L}\times\mathcal{B}$ measurable on Ω. For each $(t,\mathbf{x})$ in Ω, the set $\mathcal{F}(t,\mathbf{x})$ is nonempty, compact and convex.

(2) There are nonnegative integrable functions k and Φ on $[a,b]$ such that

(a) $\mathcal{F}(t,\mathbf{x}) \subseteq \Phi(t)\,\mathbb{B}$ for all $\mathbf{x}$ in Ω_t, almost everywhere, and

(b) $\mathcal{F}(t,\mathbf{x}) \subseteq \mathcal{F}(t,\mathbf{x}) + k(t)|\mathbf{y}-\mathbf{x}|\mathrm{cl}\,\mathbb{B}$ for all $\mathbf{x},\ \mathbf{y} \in \Omega_t$, almost everywhere.

The endpoint cost function l is l.s.c on $\Omega_a \times \Omega_b$.

where $\Omega_t = \{\mathbf{x} \in \mathbb{R}^{\mathrm{n}} : (t,\mathbf{x}) \in \Omega\}$ for each t in $[a,b]$ and $\mathbb{B}$ is the unit ball.

The generalized problem of many practical problems place constraints not only on the derivative of the state trajectory, but also on its endpoints. The differential inclusion problem is augmented with the additional constraint $(\mathbf{x}(a),\mathbf{x}(b)) \in \mathcal{S}$, where $\mathcal{S}$ is a given target set in $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$ and is assumed to be closed. Suppose that there is a function $\varphi(t,\mathbf{x})$ with the following properties:

1. $\varphi(t,\mathbf{x}) \in \mathcal{F}(t,\mathbf{x})$ for all $\mathbf{x} \in \Omega_t$, almost everywhere;
2. $\varphi(t,\mathbf{x})$ is a Carathéodory function, i.e., φ is $\mathcal{L}\times\mathcal{B}$ measurable on Ω, and for almost every t the function $x \mapsto \varphi(t,\mathbf{x})$ is Lipschitz on Ω_t with Lipschitz rank $k(t)$;
3. $\dot{\bar{\mathbf{x}}}(t) = \varphi(t,\bar{\mathbf{x}}(t))$ almost everywhere on $[a,b]$.

Theorem -Pontryagin's Maximum Principle Consider the optimal control problem of minimizing the endpoint function

$$l(\mathbf{x}(a),\mathbf{x}(b)) + \Psi_{\mathcal{S}}(\mathbf{x}(a),\mathbf{x}(b)) \tag{7}$$

over all arcs $\mathbf{x}$ satisfying the differential constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t),\boldsymbol{\tau}(t)),\ \boldsymbol{\tau}(t) \in \mathcal{C}_\tau, \quad a.e. \quad t \in [a,b]. \tag{8}$$

In addition, suppose that f is a Carathéodory function for which the velocity sets $\mathcal{F}(t,\mathbf{x}(t)) = \{\mathbf{v}|\mathbf{v} = \mathbf{f}(t,\mathbf{x}(t),\boldsymbol{\tau}(t)),\ \boldsymbol{\tau}(t) \in \mathcal{C}_\tau\}$ satisfy assumptions 2. If an arc $\bar{\mathbf{x}}$, together with a corresponding control function $\bar{\boldsymbol{\tau}}$, solves this problem, then there exist an arc $\mathbf{p} \in \mathcal{AC}$ on $[a,b]$ and a scalar λ equal to either 0 or 1 for which one has, for almost every $t \in [a,b]$,

the adjoint equation,

$$-\dot{\mathbf{p}}(t) \in \bar{\partial}_x\langle \mathbf{p}(t),\mathbf{f}(t,\bar{\mathbf{x}}(t),\boldsymbol{\tau}(t))\rangle \tag{9}$$

the maximum condition

$$\langle \mathbf{p}(t),\mathbf{f}(t,\bar{\mathbf{x}},\bar{\boldsymbol{\tau}})\rangle = \sup\{\langle \mathbf{p},\mathbf{f}(t,\bar{\mathbf{x}},\boldsymbol{\tau})\rangle : \boldsymbol{\tau} \in \mathcal{C}_\tau\}, \tag{10}$$

the transversality conditon

$$(\mathbf{p}(a),-\mathbf{p}(b)) \in \lambda\,\partial l(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b)) + \mathcal{N}_{\mathcal{S}}(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b)). \tag{11}$$

Here $\mathcal{N}_{\mathcal{S}}(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b))$ denotes the limiting normal cone to the set $\mathcal{S}$ at $(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b))$. The operator $\bar{\partial}$ denotes generalized subdifferential in the sense of Clarke. The above stated form of the Pontryagin's maximum principle (PMP) defines the necessary conditions for an arc $\bar{\mathbf{x}} \in \mathcal{AC}$ to extremize the Mayer problem subject to the constraints $\mathcal{S}$.

In the impulsive optimal control, it is assumed that the control horizon is composed of N different phases, which are separated from each other by $N-1$ possibly discontinuous transitions. The importance of the transition process becomes clear if one considers the fact that at pre-transition and post-transition states the values of several functions may differ due to discontinuities. The right-continuous and left-continuous regularizations of a function f, which is a mapping of $\mathcal{I}$ to a Hausdorff topological space $\mathcal{E}$; becomes important if one considers that for every $t_{\rm i} \in \mathcal{I}_{\rm T}$ the right-side limit given by:

$$f^{+}(t_{\rm i}) = \lim_{s \to t_{\rm i},\, s > t} f(s) \tag{12}$$

may differ from $f(t_{\rm i})$, if it exists. Symmetrically, the left-side limit, if it exists, is denoted by $f^{-}(t_{\rm i})$. Following proposition is used often in this work:

Proposition 1 [Moreau,1988a] Let $\mathcal{E}$ be regular and let $f : \mathcal{I} \to \mathcal{E}$ be such that for every $t \in \mathcal{I}$ different from the possible right end of $\mathcal{I}$, there exists $f^{+}(t)$; then

$$f^{+}(t) = \lim_{s \to t,\, s > t} f^{+}(s). \tag{13}$$

If, in addition, for every t different from the possible left end of $\mathcal{I}$, there exists ; then

$$f^{-}(t) = \lim_{s \to t,\, s > t} f^{+}(s). \tag{14}$$

As short hand notation one has:

$$(f^{+})^{+} = f^{+},\, (f^{-})^{+} = f^{+},$$
$$(f^{-})^{-} = f^{-},\, (f^{+})^{-} = f^{-}.$$

A good overview on the topic of treatment of functions of bounded variation in time is provided in [Moreau,1988a]. A transition process is common to the pre-transition configuration and post-transition configuration. Each problem $P_{\rm i}$ with a unique mechanical configuration is defined on a closed time domain $\mathrm{dom}(P_{\rm i})$ with variable boundary which is partitioned as follows:

$$\mathrm{dom}(P_{\rm i}) = \left\{t_{\rm i}^{-},\, t_{\rm i}^{+}\right\} \bigcup \left(,t_{\rm i}^{+}, t_{\rm i+1}^{-}\right) \bigcup \left\{t_{\rm i+1}^{-},\, t_{\rm i+1}^{+}\right\}. \tag{15}$$

The boundary of the domain $\mathrm{dom}(P_{\rm i})$ is given by:

$$\mathrm{bdy\ dom}(P_{\rm i}) = \left\{t_{\rm i}^{-},\, t_{\rm i}^{+}\right\} \bigcup \left\{t_{\rm i+1}^{-},\, t_{\rm i+1}^{+}\right\}. \tag{16}$$

The interior of the domain is given by:

$$\mathrm{int\ dom}(P_{\rm i}) = \left(t_{\rm i}^{+}, t_{\rm i+1}^{-}\right). \tag{17}$$

The domain of the overall problem P is given by the union:

$$\mathrm{dom}(P_{\rm Tot}) = \bigcup_{\forall\,\rm i} \mathrm{dom}(P_{\rm i}). \tag{18}$$

However, the domains of successive problems $P_{\rm i}$ and $P_{\rm i+1}$ are not disjoint:

$$\mathrm{dom}(P_{\rm i}) \bigcap \mathrm{dom}(P_{\rm i+1}) = \tag{19}$$
$$\mathrm{bdy\ dom}(P_{\rm i}) \bigcap \mathrm{bdy\ dom}(P_{\rm i+1}) = \left\{t_{\rm i+1}^{-},\, t_{\rm i+1}^{+}\right\},$$

The set $\mathrm{bdy\ dom}(P_{\rm i}) \bigcap \mathrm{bdy\ dom}(P_{\rm i+1}) = \left\{t_{\rm i+1}^{-},\, t_{\rm i+1}^{+}\right\}$ is the support of the transition process and is Lebesgue-negligible.

Having set the stage, the necessary conditions for the impulsive optimal control of structure-variant rigidbody mechanical systems is formally derived by considering a problem in Bolza form (GPB), in which the objective is to choose an arc $\mathbf{x} \in \mathcal{BV}$ in order to minimize

$$J(\mathbf{x}, \{\mathbf{x}(t_{\rm i}^{-}),\, \mathbf{x}(t_{\rm i}^{+}),\, t_{\rm i}\}) = \tag{20}$$
$$\sum_{i=1}^{N} l_{\rm i}(\mathbf{x}(t_{\rm i}^{-}), \mathbf{x}(t_{\rm i}^{+})) + \int_{t_{\rm i}^{+}}^{t_{i+1}^{-}} L_{\rm i}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt.$$

The overall problem as stated in (20) is seen as the union of several problems in the generalized Bolza form. The theory at hand treats optimal solutions as solutions of multi-point boundary value problems (MBVP) with discontinuous transitions in the state. In this setting, the prespecification of the mode sequence and number of intervals must be given in advance. Here it is assumed that the control horizon is composed of N different phases, which are separated from each other by $N-1$ possibly discontinuous transitions in the generalized velocities. The extended-valued integrand may differ on each interval based on the structure of the equations of motion. The difference in structure may arise due to change in parameters (i.e. mass, inertia) or degrees of freedom. In [Yunt, 2008] a projection approach is presented in case, the mechanical configurations in successive intervals differ based on change in the number of degrees of freedom.

3 Analysis of Rigidbody Lagrangian Systems with Impactively Blockable Degrees of Freedom

Let $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ represent the position, velocity and acceleration in the generalised coordinates of a scleronomic rigidbody mechanical system with n degrees of freedom (DOF), respectively. The equations of motion are given by:

$$\mathbf{M}(\mathbf{q})\, \ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{B}(\mathbf{q})\, \boldsymbol{\tau} = \mathbf{0}, \tag{21}$$

where $\mathbf{q}$ and $\dot{\mathbf{q}}$ denote the absolutely continuous generalised positions and bounded variation generalised velocities, respectively. Here $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{\rm n \times n}$ is the mass matrix, $\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{\rm n \times 1}$ denote the vector of gyroscopical and coriolis, smooth potential (gravity, spring etc.)

forces and $\boldsymbol{\tau} \in \mathbb{R}^{\mathrm{s}\times 1}$ is the vector of Lebesgue measurable controls. The linear operator $\mathbf{B}(\mathbf{q}) \in \mathbb{R}^{\mathrm{n}\times\mathrm{s}}$ includes the generalised control directions. This representation is the maximal representation, meaning that all directions are unblocked. In order to describe the transition conditions properly following index sets are defined:

$$\mathcal{C}_{\mathrm{B}} = \{\, i \,|\gamma_{\mathrm{i}}^{+} = 0,\, \dot{\gamma}_{\mathrm{i}}^{+} = 0, \forall t \in \left(t_{\mathrm{i}}^{+}, t_{i+1}^{-}\right)\},$$
$$\mathcal{C}_{\mathrm{P}} = \{\, i \,|\gamma_{\mathrm{i}}^{+} = 0,\, \dot{\gamma}_{\mathrm{i}}^{+} \in \mathbb{R}\}.$$

Here $\mathcal{C}_{\mathrm{B}}$ denotes the index set of directions that remain blocked after a transition time until another possible transition time. The set $\mathcal{C}_{\mathrm{P}}$ is the index set of directions at which a blocking action takes place at a transition time, such that the relative post-transition directional velocity is nullified. The relative velocity at any direction is given by (22):

$$\gamma_{\mathrm{i}} = \mathbf{d}_{\mathrm{i}}^{\mathrm{T}}(\mathbf{q})\,\dot{\mathbf{q}}. \tag{22}$$

The difference between the pre-transition and post-transition relative velocities is related to the post-, and pre-transition generalised velocities of the mechanical system by expression (23):

$$\boldsymbol{\gamma}^{+} - \boldsymbol{\gamma}^{-} = \mathbf{D}^{\mathrm{T}}(\mathbf{q})\,(\dot{\mathbf{q}}^{+} - \dot{\mathbf{q}}^{-}). \tag{23}$$

Here $\boldsymbol{\gamma} \in \mathbb{R}^{\mathrm{n}}$ is such that $\mathbf{d}_{\mathrm{i}}(\mathbf{q}) \in \mathrm{col}\{\mathbf{D}\}$ and $\mathbf{D} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Here $\mathrm{col}\{\cdot\}$ denotes the set of column vectors of the relevant linear operator. Let at a transition, which is accompanied by an impact, which is induced by the sudden blocking of directions of motion, p of the directions, characterised by their force directions, be active. Then, the vector $\boldsymbol{\gamma}$ is decomposed in the following manner:

$$\begin{bmatrix} \boldsymbol{\gamma}_{\mathrm{b}}^{+} \\ \boldsymbol{\gamma}_{\mathrm{f}}^{+} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\gamma}_{\mathrm{b}}^{-} \\ \boldsymbol{\gamma}_{\mathrm{f}}^{-} \end{bmatrix} = \mathbf{D}(\mathbf{q})^{\mathrm{T}}\,(\dot{\mathbf{q}}^{+} - \dot{\mathbf{q}}^{-}), \tag{24}$$

where $\boldsymbol{\gamma}_{\mathrm{b}}^{+}$ and $\boldsymbol{\gamma}_{\mathrm{b}}^{-}$ denote the relative post-, and pre-transition directional velocities at the blocked/active directions, and $\boldsymbol{\gamma}_{\mathrm{f}}^{+}$ and $\boldsymbol{\gamma}_{\mathrm{f}}^{-}$ denote the relative post-, and pretransition velocities at the free/passive directions. Here the linear operator $\mathbf{D}(\mathbf{q})$ is then partitioned as:

$$\mathbf{D}(\mathbf{q}) = \left[\mathbf{D}_{\mathrm{b}}(\mathbf{q}) \;\; \vdots \;\; \mathbf{D}_{\mathrm{f}}(\mathbf{q})\right]. \tag{25}$$

Here $\mathbf{D}_{\mathrm{b}} \in \mathbb{R}^{\mathrm{n}\times\mathrm{p}}, \mathbf{D}_{\mathrm{f}} \in \mathbb{R}^{\mathrm{n}\times(\mathrm{n}-\mathrm{p})}$ denote the linear operators, consisting columnwise of blocked and unblocked generalised directions such that $\mathrm{col}\{\mathbf{D}_{\mathrm{f}}\}\bigcup\mathrm{col}\{\mathbf{D}_{\mathrm{b}}\} = \mathrm{col}\{\mathbf{D}\}$ and $\mathrm{col}\{\mathbf{D}_{\mathrm{f}}\}\bigcap\mathrm{col}\{\mathbf{D}_{\mathrm{b}}\} = \emptyset$. Let $p < s$ the number DOF which are being blocked impactively. The impact equation is given by the following expression:

$$\mathbf{M}(\mathbf{q})\,(\dot{\mathbf{q}}^{+} - \dot{\mathbf{q}}^{-}) - \mathbf{D}_{\mathrm{b}}(\mathbf{q})\,\boldsymbol{\Gamma} = \mathbf{0}, \tag{26}$$

where $\boldsymbol{\Gamma} \in \mathbb{R}^{\mathrm{p}}$ are blocking impulses that can be generated at the directions, which participate in blocking and are elements of the index set $\mathcal{C}_{\mathrm{P}}$ and the linear operator $\mathbf{D}_{\mathrm{b}} \in \mathbb{R}^{\mathrm{n}\times\mathrm{p}}$ denotes the generalised force direction of the blocking forces, such that $\mathrm{col}\{\mathbf{D}_{\mathrm{b}}\} \subset \mathrm{col}\{\mathbf{B}\}$. Further, it is assumed that $\mathrm{col}\{\mathbf{B}\} \subset \mathrm{col}\{\mathbf{D}\}$ for convenience and without loss of generality. The equation (26) can be solved for the jump in the generalised velocities of the system:

$$\dot{\mathbf{q}}^{+} - \dot{\mathbf{q}}^{-} = \mathbf{M}^{-1}(\mathbf{q})\,\mathbf{D}_{\mathrm{b}}(\mathbf{q})\,\boldsymbol{\Gamma}. \tag{27}$$

Inserting this expression in (23) reveals the jump in the vector of relative velocity vector:

$$\boldsymbol{\gamma}^{+} - \boldsymbol{\gamma}^{-} = \mathbf{D}^{\mathrm{T}}(\mathbf{q})\,(\dot{\mathbf{q}}^{+} - \dot{\mathbf{q}}^{-}) = \mathbf{D}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{D}\,\begin{bmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{\Delta} \end{bmatrix}. \tag{28}$$

By making use of the decomposition of the relative velocities into blocked and free directions as introduced in equation (24) following is obtained:

$$\begin{bmatrix} \boldsymbol{\gamma}_{\mathrm{b}}^{+} - \boldsymbol{\gamma}_{\mathrm{b}}^{-} \\ \boldsymbol{\gamma}_{\mathrm{f}}^{+} - \boldsymbol{\gamma}_{\mathrm{f}}^{-} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{D}_{\mathrm{b}} & \mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{D}_{\mathrm{f}} \\ \mathbf{D}_{\mathrm{f}}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{D}_{\mathrm{b}} & \mathbf{D}_{\mathrm{f}}^{\mathrm{T}}\,\mathbf{M}^{-1}\,\mathbf{D}_{\mathrm{f}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{\Delta} \end{bmatrix}. \tag{29}$$

In order to simplify the notation matrices $\mathbf{G}_{\mathrm{bb}}$, $\mathbf{G}_{\mathrm{b\,f}}$, $\mathbf{G}_{\mathrm{fb}}$ and $\mathbf{G}_{\mathrm{f\,f}}$ are introduced as follows:

$$\begin{bmatrix} \boldsymbol{\gamma}_{\mathrm{b}}^{+} - \boldsymbol{\gamma}_{\mathrm{b}}^{-} \\ \boldsymbol{\gamma}_{\mathrm{f}}^{+} - \boldsymbol{\gamma}_{\mathrm{f}}^{-} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{\mathrm{bb}} & \mathbf{G}_{\mathrm{b\,f}} \\ \mathbf{G}_{\mathrm{fb}} & \mathbf{G}_{\mathrm{f\,f}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{\Delta} \end{bmatrix}. \tag{30}$$

Since the impulse at non-blocked directions $\boldsymbol{\Delta}$ which do not participate at the blocking and post-transition velocity at the blocked directions is zero, following is valid:

$$\boldsymbol{\Delta} = \mathbf{0}, \tag{31}$$
$$\boldsymbol{\gamma}_{\mathrm{b}}^{+} = \mathbf{0}. \tag{32}$$

The impulse vector $\boldsymbol{\Gamma}$ can be eliminated by considering of (31) and (32) together with (30), which boils down to:

$$-\boldsymbol{\gamma}_{\mathrm{b}}^{-} = \mathbf{G}_{\mathrm{bb}}\boldsymbol{\Gamma}, \tag{33}$$
$$\boldsymbol{\gamma}_{\mathrm{f}}^{+} - \boldsymbol{\gamma}_{\mathrm{f}}^{-} = \mathbf{G}_{\mathrm{fb}}\boldsymbol{\Gamma}. \tag{34}$$

Equation (33) when solved for $\boldsymbol{\Gamma}$ reveals:

$$\boldsymbol{\Gamma} = -\mathbf{G}_{\mathrm{bb}}^{-1}\,\boldsymbol{\gamma}_{\mathrm{b}}^{-}, \tag{35}$$

and insertion into equation (34) eliminates the impulse and establishes the relation between post-, and pre-transition relative velocities:

$$\gamma_{\mathrm{f}}^{+} = \gamma_{\mathrm{f}}^{-} - \mathbf{G}_{\mathrm{fb}}\,\mathbf{G}_{\mathrm{bb}}^{-1}\,\gamma_{\mathrm{b}}^{-}. \quad (36)$$

This equation can be rewritten in terms of the post-, and pre-transition generalised velocities by making use of equation (24) as given in (37):

$$\mathbf{D}_{\mathrm{f}}^{\mathrm{T}}\,\dot{\mathbf{q}}^{+} = \mathbf{D}_{\mathrm{f}}^{\mathrm{T}}\,\dot{\mathbf{q}}^{-} - \mathbf{G}_{\mathrm{fb}}\,\mathbf{G}_{\mathrm{bb}}^{-1}\,\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\,\dot{\mathbf{q}}^{-}, \quad (37)$$

and by defining $\mathbf{K}(\mathbf{q}) = \mathbf{D}_{\mathrm{f}}^{\mathrm{T}} - \mathbf{G}_{\mathrm{fb}}\,\mathbf{G}_{\mathrm{bb}}^{-1}\,\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}$ reveals following expression:

$$\mathbf{D}_{\mathrm{f}}^{\mathrm{T}}(\mathbf{q})\,\dot{\mathbf{q}}^{+} - \mathbf{K}(\mathbf{q})\,\dot{\mathbf{q}}^{-} = \mathbf{0}. \quad (38)$$

On the other hand by insertion of the impulse obtained in (35) into (27) reveals the relation between post- and pretransition generalised velocities:

$$\dot{\mathbf{q}}^{+} - \dot{\mathbf{q}}^{-} = -\mathbf{M}^{-1}\,\mathbf{D}_{\mathrm{b}}\mathbf{G}_{\mathrm{bb}}^{-1}\,\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}}^{-}. \quad (39)$$

This equation can be rewritten in the following form:

$$\dot{\mathbf{q}}^{+} = (I - \mathbf{M}^{-1}\,\mathbf{D}_{\mathrm{b}}\mathbf{G}_{\mathrm{bb}}^{-1}\,\mathbf{D}_{\mathrm{b}}^{\mathrm{T}})\,\dot{\mathbf{q}}^{-} = \mathbf{P}_{\perp}^{\mathrm{T}}(\mathbf{q})\,\dot{\mathbf{q}}^{-}. \quad (40)$$

The value of impulse established in (35) represents the minimal value to induce full blocking in every direction in the index set $\mathcal{C}_{\mathrm{P}}$, beyond which no difference in action is observed. So $\mathbf{\Gamma}$ is assumed to attain the minimal value in evaluating γ_{f}^{+}.

3.1 Lagrangian Dynamics in Different Phases of Motion

After the possibly impactive transition the equations of motion on acceleration level may differ from the pre-transition equations of motion based on the closed directions of motion. A direction of interest γ, which for example can be the relative velocity at a blockable joint, is expressed as a linear combination of generalized velocities. The generalized acceleration of the finite-dimensional Lagrangian system when some DOF are closed by $\boldsymbol{\tau}_{\mathrm{b}}$, is given by (41):

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})\,(\mathbf{h}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{D}_{\mathrm{b}}(\mathbf{q})\,\boldsymbol{\tau}_{\mathrm{b}} + \mathbf{B}(\mathbf{q})\,\boldsymbol{\tau})\ . \quad (41)$$

The controls $\boldsymbol{\tau}_{\mathrm{b}}$ represent the forces which are required to constrain the vector field from evolving in certain directions. The linear operator $\mathbf{D}_{\mathrm{b}}$ denotes the generalized force direction of the constraining forces, such that $\mathrm{col}\{\mathbf{D}_{\mathrm{b}}\} \subset \mathrm{col}\{\mathbf{B}\}$. The linear operator $\mathbf{D}_{\mathrm{b}}$ consists here columnwise of the directions in the set $\mathcal{C}_{\mathrm{B}}$. The accelerations in the closed directions must be zero, as a consequence one has:

$$\dot{\gamma}_{\mathrm{b}} = \mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\ddot{\mathbf{q}} + \dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}} = \mathbf{0}\ . \quad (42)$$

The insertion of equation (41) in equation (42) reveals:

$$\begin{aligned}&\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\ddot{\mathbf{q}} + \dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}} = \mathbf{0} = \\ &\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\mathbf{M}^{-1}\,(\mathbf{h} + \mathbf{D}_{\mathrm{b}}\,\boldsymbol{\tau}_{\mathrm{b}} + \mathbf{B}\,\boldsymbol{\tau}) + \dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}}\ .\end{aligned} \quad (43)$$

The equation (43) can be solved for the blocking forces/moments as below:

$$\begin{aligned}&\boldsymbol{\tau}_{\mathrm{b}} = \\ &-(\mathbf{G}_{\mathrm{bb}})^{-1}\left(\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{h} + \mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{B}\,\boldsymbol{\tau} + \dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}}\right)\ .\end{aligned} \quad (44)$$

Defining the projector $\mathbf{P}_{\|}$ as

$$\mathbf{P}_{\|} = \mathbf{D}_{\mathrm{b}}(\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{D}_{\mathrm{b}})^{-1}\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\mathbf{M}^{-1} \quad (45)$$

and inserting into equation (41) gives the projected dynamics:

$$\mathbf{M}\,\ddot{\mathbf{q}} - \mathbf{h} - \mathbf{P}_{\|}\,(\mathbf{h} + \mathbf{B}\,\boldsymbol{\tau}) + \mathbf{D}_{\mathrm{b}}\mathbf{G}_{\mathrm{b\,b}}^{-1}\dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}} - \mathbf{B}\,\boldsymbol{\tau} = \mathbf{0}\ . \quad (46)$$

By defining the projector $\mathbf{P}_{\perp}$ as

$$\mathbf{P}_{\perp} = \mathbf{I} - \mathbf{P}_{\|}, \quad (47)$$

where $\mathbf{I}$ is an identity linear operator of appropriate size. The equations of motion after the directions $\mathbf{D}_{\mathrm{b}}$ are closed in the generalized coordinates can be rearranged as below:

$$\mathbf{M}\,\ddot{\mathbf{q}} - \mathbf{P}_{\perp}\,\mathbf{h} - \mathbf{P}_{\perp}\,\mathbf{B}\,\boldsymbol{\tau} + \mathbf{D}_{\mathrm{b}}\mathbf{G}_{\mathrm{bb}}^{-1}\dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}} = \mathbf{0}\ . \quad (48)$$

The new vector of coriolis and gyroscopical forces as well as the linear operator of generalized control directions can be redefined as:

$$\mathbf{h}_{\mathrm{b}} = \mathbf{P}_{\perp}\,\mathbf{h} - \mathbf{D}_{\mathrm{b}}(\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{D}_{\mathrm{b}})^{-1}\dot{\mathbf{D}}_{\mathrm{b}}^{\mathrm{T}}\dot{\mathbf{q}}\ , \quad (49)$$

$$\mathbf{B}_{\mathrm{b}} = \mathbf{P}_{\perp}\,\mathbf{B}\ , \quad (50)$$

to yield

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} - \mathbf{h}_{\mathrm{b}}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{B}_{\mathrm{b}}(\mathbf{q})\,\boldsymbol{\tau} = \mathbf{0}\ . \quad (51)$$

4 Impulsive Optimal Control Problem in First-Order Form

The optimal control problem with free end-time t_{f} and free transition times t_{i} and locations

$\{\mathbf{q}(t_{\rm i}),\, \dot{\mathbf{q}}(t_{\rm i}^-),\, \dot{\mathbf{q}}(t_{\rm i}^+)\}$ is considered. The goal function is given by:

$$\min \int_{t_0}^{t_{\rm f}} g(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau})\, dt. \tag{52}$$

The goal function is subject to the mechanical system dynamics stated in the first-order measure-differential equation form:

$$d\mathbf{q} = \mathbf{y}\, dt, \tag{53}$$
$$d\mathbf{y} = \tag{54}$$
$$(\mathbf{f}_{\rm i}(\mathbf{q}(t), \mathbf{y}(t)) + \mathbf{G}_{\rm i}(\mathbf{q}(t))\boldsymbol{\tau}(t))\; dt + \mathbf{V}_{\rm i}(\mathbf{q}(t))\boldsymbol{\zeta}'\, d\sigma.$$

The smooth dynamics of the rigidbody mechanical system is characterized in every interval of motion $(t_{\rm i}^+, t_{i+1}^-)$ by a triplet $\{\mathbf{f}_{\rm i}(\mathbf{q}(t), \mathbf{y}(t)), \mathbf{G}_{\rm i}(\mathbf{q}(t)), \mathbf{V}_{\rm i}(\mathbf{q}(t))\}$. By the Lebesgue-Stieltjes integration of equation (54) over an atom of time instant $t_{\rm i} \in \mathcal{I}_{\rm T}$ one obtains:

$$\int_{\{t_{\rm i}\}} d\mathbf{y} - (\mathbf{f}_{\rm i} + \mathbf{G}_{\rm i}\boldsymbol{\tau}(t))\; dt - \mathbf{V}_{\rm i}\boldsymbol{\zeta}'\, d\sigma = \tag{55}$$
$$\mathbf{y}(t_{\rm i}^+) - \mathbf{y}(t_{\rm i}^-) - \mathbf{V}_{\rm i}(\mathbf{q}(t_{\rm i}))\left(\boldsymbol{\zeta}_{\rm i}^+ - \boldsymbol{\zeta}_{\rm i}^-\right)$$

which is an impact equation in first order form. In performing the integration in (55), it is assumed that the generalised impulsive control force directions $\mathbf{V}_{\rm i}(\mathbf{q}(t_{\rm i}))$ do not change their structure. The vector controls $\boldsymbol{\tau}$ is assumed to be constrained in a polytopic compact convex set denoted by $\mathcal{C}_\tau$. The set of transition conditions at each transition instant $t_{\rm i}$ are denoted by $\mathcal{C}_{\rm T_i}^+$ and $\mathcal{C}_{\rm T_i}^-$ are stated in terms of generalized positions $\mathbf{q}(t_{\rm i})$, and generalized post-, and pre-transition velocities $\dot{\mathbf{q}}(t_{\rm i}^+)$, $\dot{\mathbf{q}}(t_{\rm i}^-)$. Here the sets are defined as below:

$$\mathcal{C}_{\rm T_i}^+ = \tag{56}$$
$$\{\left(\mathbf{q}(t_{\rm i}^+), \dot{\mathbf{q}}(t_{\rm i}^-), \dot{\mathbf{q}}(t_{\rm i}^+)\right) \mid \mathbf{D}_{\rm b}(\mathbf{q}(t_{\rm i}^+))\dot{\mathbf{q}}(t_{\rm i}^+) = \mathbf{0}\},$$
$$\mathcal{C}_{\rm T_i}^- = \tag{57}$$
$$\{\left(\mathbf{q}(t_{\rm i}^+), \dot{\mathbf{q}}(t_{\rm i}^-), \dot{\mathbf{q}}(t_{\rm i}^+)\right) \mid \mathbf{D}_{\rm b}(\mathbf{q}(t_{\rm i}^-))\dot{\mathbf{q}}(t_{\rm i}^+) = \mathbf{0}\},$$
$$\mathcal{C}_{\rm f} = \{(\mathbf{q}(t_{\rm f}), \dot{\mathbf{q}}(t_{\rm f})) \mid \mathbf{q}(t_{\rm f}) = \mathbf{q}_{\rm f}, \quad \dot{\mathbf{q}}(t_{\rm f}) = \dot{\mathbf{q}}_{\rm f}\}, \tag{58}$$
$$\mathcal{C}_\tau = \{\boldsymbol{\tau} \,|\, \boldsymbol{\tau} \in \mathcal{K}, \text{convex, polytopic, compact}\}, \tag{59}$$
$$\mathcal{C}_{\rm I_i}^+ = \{\left(\mathbf{q}(t_{\rm i}^+), \mathbf{y}(t_{\rm i}^+), \mathbf{y}(t_{\rm i}^-)\right) \mid \tag{60}$$
$$\mathbf{y}(t_{\rm i}^+) - \mathbf{y}(t_{\rm i}^-) - \mathbf{V}_{\rm i}(\mathbf{q}(t_{\rm i}^+))\,\left(\boldsymbol{\zeta}_{\rm i}^+ - \boldsymbol{\zeta}_{\rm i}^-\right) = \mathbf{0}\},$$
$$\mathcal{C}_{\rm I_i}^- = \{\left(\mathbf{q}(t_{\rm i}^-), \mathbf{y}(t_{\rm i}^+), \mathbf{y}(t_{\rm i}^-)\right) \mid \tag{61}$$
$$\mathbf{y}(t_{\rm i}^+) - \mathbf{y}(t_{\rm i}^-) - \mathbf{V}_{\rm i}(\mathbf{q}(t_{\rm i}^-))\,\left(\boldsymbol{\zeta}_{\rm i}^+ - \boldsymbol{\zeta}_{\rm i}^-\right) = \mathbf{0}\}.$$

The overall value functional is given by:

$$J = \Psi_{\mathcal{C}_{\rm f}} + \sum_{\rm i \in \mathcal{I}_{\rm T}} \Psi_{\mathcal{C}_{\rm T_i}^-} + \Psi_{\mathcal{C}_{\rm T_i}^+}$$
$$+ \int_{(t_0, t_{\rm f})} \Psi_{\mathcal{C}_\tau}\, dt + dH - \boldsymbol{\eta}_1\, d\mathbf{q} - \boldsymbol{\eta}_2\, d\mathbf{y}. \tag{62}$$

The differential measure of the Hamiltonian is defined as:

$$d\,H = H_{\rm t}\, dt + H_\sigma\, d\sigma = \boldsymbol{\eta}_2(t)\mathbf{V}_{\rm i}\boldsymbol{\zeta}'\, d\sigma + \tag{63}$$
$$(\lambda(t)\, g(\mathbf{q},\, \mathbf{y}, \boldsymbol{\tau}) + \boldsymbol{\eta}_1(t)\, \mathbf{y}(t) + \boldsymbol{\eta}_2(t)\; (\mathbf{f}_{\rm i} + \mathbf{G}_{\rm i}\boldsymbol{\tau}(t)))\; dt$$

where $\boldsymbol{\eta}_1(t) \in \mathcal{LCBV}^\star(\mathbb{R}^{1\times \rm n})$ and $\boldsymbol{\eta}_2(t) \in \mathcal{LCBV}^\star(\mathbb{R}^{1\times \rm n})$ are the dual states. The unconstrained functional in (62) is equivalent to (64) under assumption (1.3):

$$J = \Psi_{\mathcal{C}_{\rm f}} + \sum_{\rm i \in \mathcal{I}_{\rm T}} \Psi_{\mathcal{C}_{\rm I_i}^-} + \Psi_{\mathcal{C}_{\rm I_i}^+} + \Psi_{\mathcal{C}_{\rm T_i}^-} + \Psi_{\mathcal{C}_{\rm T_i}^+}$$
$$+ \int_{t_{\rm i}^+}^{t_{\rm i+1}^-} \Psi_{\mathcal{C}_\tau} + H_{\rm t} - \boldsymbol{\eta}_1\, \dot{\mathbf{q}} - \boldsymbol{\eta}_2\, \dot{\mathbf{y}}\, dt. \tag{64}$$

Following structure for various differential measures is noted:

$$d\mathbf{q} = \dot{\mathbf{q}}\, dt + \boldsymbol{\rho}'\, d\sigma, \quad d\mathbf{y} = \ddot{\mathbf{q}}\, dt + \boldsymbol{\chi}'\, d\sigma\,,$$
$$d\boldsymbol{\eta}_1 = \dot{\boldsymbol{\eta}}_1\, dt + \boldsymbol{\xi}_1'\, d\sigma, \quad d\boldsymbol{\eta}_2 = \dot{\boldsymbol{\eta}}_2\, dt + \boldsymbol{\xi}_2'\, d\sigma.$$

The necessary conditions are derived by making use of following assumptions on the general problem:

Assumptions 3

1. the dual states $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are assumed left-continuous locally bounded variation functions ($\mathcal{LCLBV}$), and the generalized velocities $\dot{\mathbf{q}}$ of the Lagrangian system is assumed right-continuous locally bounded variation functions ($\mathcal{RCLBV}$), whereas the generalized positions are in class $\mathcal{AC}$.
2. The mode sequence and number of intervals for the MBVP constitute a feasible hybrid trajectory.
3. The set $\mathcal{C}_{\rm I_i}^+ \cap \mathcal{C}_{\rm T_i}^+$ is closed and nonempty.
4. The set $\mathcal{C}_{\rm I_i}^- \cap \mathcal{C}_{\rm T_i}^-$ is is closed and nonempty.
5. The goal functional $g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau})$ is continuously differentiable forall $t \in \Omega_{\rm t}$ and $t_{\rm i} \in \mathcal{I}_{\rm T}$.
6. The partial derivatives $\partial_{\mathbf{y}} g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau})$ are bounded for all $t \in \Omega_{\rm t}$ and $t_{\rm i} \in \mathcal{I}_{\rm T}$.
7. Each $L_{\rm i} : (t_{\rm i}^+, t_{\rm i+1}^-) \times \mathbb{R}^{\rm n} \times \mathbb{R}^{\rm n} \times \mathbb{R}^{\rm n} \times \mathbb{R}^{\rm n} \to \mathbb{R} \cup \{+\infty\}$ is a Lebesgue normal integrand.
8. Each $L_{\rm i}(\mathbf{q}(s), \mathbf{y}(s), \cdot)$ is convex for each $(\mathbf{q}(s), \mathbf{y}(s))$.
9. Each $l_{\rm i} : \mathbb{R}^{\rm n} \times \mathbb{R}^{\rm n} \times \mathbb{R}^{\rm n} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

The value function J has some pleasant regularity properties if assumptions (3) hold. Making use of these regularity properties, ”sharp” necessary conditions are derived.

5 Necessary Conditions in First-Order Form

Theorem [Yunt] Let assumptions (1), (2) and (3) be valid for the optimal control problem. If trajectories of generalized positions $\mathbf{q}^*(t^+) \in \mathcal{AC}[\mathbb{R}^{\mathrm{n}}]$, velocities $\mathbf{y}^*(t^+) \in \mathcal{RCLBV}[\mathbb{R}^{\mathrm{n}}]$ provide a minimum for the described optimal control problem, then there exist optimal transition times $t_{\mathrm{i}}^* \in \mathcal{I}_{\mathrm{T}}$, optimal controls $\boldsymbol{\tau}^*(t)$, optimal impulsive controls $\boldsymbol{\zeta}_{\mathrm{i}}^{+*} - \boldsymbol{\zeta}_{\mathrm{i}}^{-*}$, $\forall t_{\mathrm{i}}^* \in \mathcal{I}_{\mathrm{T}}$, dual multipliers ${}_{\mathrm{i}}\boldsymbol{\xi}^{+*}$, ${}_{\mathrm{i}}\boldsymbol{\xi}^{-*}$, $\forall t_{\mathrm{i}}^* \in \mathcal{I}_{\mathrm{T}}$, transition location triplets $\{\mathbf{q}^*(t_{\mathrm{i}}),\, \mathbf{y}^*(t_{\mathrm{i}}^+),\, \mathbf{y}^*(t_{\mathrm{i}}^-)\}$, dual states $\boldsymbol{\eta}_1^*(t^-) \in \mathcal{LCLBV}^\star[\mathbb{R}^{1\times \mathrm{n}}]$ and $\boldsymbol{\eta}_2^*(t^-) \in \mathcal{LCLBV}^\star[\mathbb{R}^{1\times \mathrm{n}}]$ (where $\star$ denote dual space) and a scalar $\lambda(t^+) \in \{0,1\}$, such that $\lambda^*(t^+) + |\boldsymbol{\eta}_1^*(t^-)| + |\boldsymbol{\eta}_2^*(t^-)| > 0$ for all $t \in \Omega_{\mathrm{t}} \cup \mathcal{I}_{\mathrm{T}}$, which fulfill:

(1) the dynamics of the mechanical system stated in first-order differential equation form on every interval $t \in (t_{\mathrm{i}}^+, t_{\mathrm{i+1}}^-)$:

$$\dot{\mathbf{q}}^*(t) = \mathbf{y}^*(t), \quad \text{a.e.}, \tag{65}$$

$$\dot{\mathbf{y}}^*(t) = \mathbf{f}_{\mathrm{i}}(\mathbf{q}^*(t), \mathbf{y}^*(t)) + \mathbf{G}_{\mathrm{i}}(\mathbf{q}^*(t))\,\boldsymbol{\tau}^*, \quad \text{a.e.} \tag{66}$$

(2) the costate dynamics on every interval $t \in (t_{\mathrm{i}}^+, t_{\mathrm{i+1}}^-)$:

$$\dot{\boldsymbol{\eta}}_1^*(t) = -\nabla_{\mathbf{q}} H = \tag{67}$$
$$-\nabla_{\mathbf{q}} \left(\mathbf{f}(\mathbf{q}^*(t), \mathbf{y}^*(t)) + \mathbf{g}(\mathbf{q}^*(t))\,\boldsymbol{\tau}^*(t)\right)^{\mathrm{T}}\, \boldsymbol{\eta}_2^*(t),$$
$$-\lambda(t)\,\nabla_{\mathbf{q}}\, g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau}) \quad \text{a.e.},$$

$$\dot{\boldsymbol{\eta}}_2^*(t) = -\nabla_{\mathbf{y}} H = \tag{68}$$
$$-\boldsymbol{\eta}_1^*(t) - (\nabla_{\mathbf{y}}\, \mathbf{f}(\mathbf{q}^*(t), \mathbf{y}^*(t)))^{\mathrm{T}} \boldsymbol{\eta}_2^*(t)$$
$$-\lambda(t)\,\nabla_{\mathbf{y}}\, g(\mathbf{q}, \mathbf{y}, \boldsymbol{\tau}), \text{ a.e.},$$

(3)the control constraints:

$$\mathcal{C}_\tau = \{\boldsymbol{\tau}\,|\,\boldsymbol{\tau} \in \mathcal{K}, \textit{compact}, \textit{polytopic}\}, \tag{69}$$

(4)the control law on $\forall t \in (t_{\mathrm{i}}^+, t_{\mathrm{i+1}}^-)$ given by:

$$-\nabla_\tau H_{\mathrm{t}}^{\mathrm{i}} \in \mathcal{N}_{\mathcal{C}_\tau}(\boldsymbol{\tau}^*(t)),$$

(5) the transition and impact conditions as given in sets (70), (71) $\forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}$:

$$\mathcal{C}_{\mathrm{T_i}} = \mathcal{C}_{\mathrm{T_i}}^+ \cup \mathcal{C}_{\mathrm{T_i}}^-, \forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}} \tag{70}$$

$$\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{I}}^+ \cup \mathcal{C}_{\mathrm{I}}^-, \forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}, \tag{71}$$

(6) the jump of the Lebesgue measurable part of the differential measure of the Hamiltonian:

$$H_{\mathrm{t}}^+ - H_{\mathrm{t}}^- = -\boldsymbol{\xi}_{\mathrm{i}}^{*+}\,\boldsymbol{\Omega}\,\mathbf{y}^*(t_{\mathrm{i}}^+) - \boldsymbol{\xi}_{\mathrm{i}}^{*-}\,\boldsymbol{\Omega}\,\mathbf{y}^*(t_{\mathrm{i}}^-)$$
$$-\boldsymbol{\xi}_{\mathrm{i}}^*\left(\begin{bmatrix}\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}(\mathbf{q}^*) & \mathbf{0}\\ \mathbf{D}_{\mathrm{f}}^{\mathrm{T}}(\mathbf{q}^*) & -\mathbf{K}(\mathbf{q}^*)\end{bmatrix}\begin{bmatrix}\dot{\mathbf{y}}^*(t_{\mathrm{i}}^+)\\ \dot{\mathbf{y}}^*(t_{\mathrm{i}}^-)\end{bmatrix}\right),$$

where $\boldsymbol{\Omega}$ is given by:

$$\boldsymbol{\Omega} = \nabla_{\mathbf{q}}\left(\begin{bmatrix}\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}(\mathbf{q}^*) & \mathbf{0}\\ \mathbf{D}_{\mathrm{f}}^{\mathrm{T}}(\mathbf{q}^*) & -\mathbf{K}(\mathbf{q}^*)\end{bmatrix}\begin{bmatrix}\mathbf{y}^*(t_{\mathrm{i}}^+)\\ \mathbf{y}^*(t_{\mathrm{i}}^-)\end{bmatrix}\right)$$

and $\boldsymbol{\xi}_{\mathrm{i}}^* \in \mathbb{R}^{1\times\mathrm{n}}$ by $\boldsymbol{\xi}_{\mathrm{i}}^* = \boldsymbol{\xi}_{\mathrm{i}}^{+*} + \boldsymbol{\xi}_{\mathrm{i}}^{-*}$,

(7) the discontinuities in $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ must fulfill:

$$\boldsymbol{\eta}_1^*(t_{\mathrm{i}}^+) - \boldsymbol{\eta}_1^*(t_{\mathrm{i}}^-) = -\boldsymbol{\xi}_{\mathrm{i}}^*\,\boldsymbol{\Omega} \tag{72}$$

and

$$\boldsymbol{\eta}_2^*(t_{\mathrm{i}}^+) - \boldsymbol{\eta}_2^*(t_{\mathrm{i}}^-) = \tag{73}$$
$$-\boldsymbol{\xi}_{\mathrm{i}}^*\begin{bmatrix}\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}(\mathbf{q}^*(t_{\mathrm{i}}))\\ \mathbf{G}_{\mathrm{fb}}(\mathbf{q}^*(t_{\mathrm{i}}))\,\mathbf{G}_{\mathrm{bb}}^{-1}(\mathbf{q}^*(t_{\mathrm{i}}))\mathbf{D}_{\mathrm{b}}^{\mathrm{T}}(\mathbf{q}^*(t_{\mathrm{i}}))\end{bmatrix},$$

(8) the boundary conditions:

$$\mathcal{C}_{\mathrm{f}} = \left\{\begin{pmatrix}\mathbf{q}^*(t_{\mathrm{f}})\\ \dot{\mathbf{q}}^*(t_{\mathrm{f}})\end{pmatrix} |\, \mathbf{q}^*(t_{\mathrm{f}}) = \mathbf{q}_{\mathrm{f}},\, \dot{\mathbf{q}}^*(t_{\mathrm{f}}) = \dot{\mathbf{q}}_{\mathrm{f}}\right\}, \tag{74}$$

(9) the Hamiltonian condition at final time

$$H_{\mathrm{t}}(\mathbf{q}^*(t_{\mathrm{f}}), \mathbf{y}^*(t_{\mathrm{f}}), \boldsymbol{\eta}_1^*(t_{\mathrm{f}}), \boldsymbol{\eta}_2^*(t_{\mathrm{f}}), \boldsymbol{\tau}^*(t_{\mathrm{f}})) = 0, \tag{75}$$

(10) the transversality condition at final state:

$$(\boldsymbol{\eta}_1^*(t_{\mathrm{f}}), \boldsymbol{\eta}_2^*(t_{\mathrm{f}})) \in \mathcal{N}_{\mathcal{C}_{\mathrm{f}}}(\mathbf{q}^*(t_{\mathrm{f}}), \dot{\mathbf{q}}^*(t_{\mathrm{f}})). \tag{76}$$

6 Discussion and Conclusion

The well-known PMP entails the necessary conditions for optimal control problems with differential constraints and end-point constraints with sufficient regularity properties in the space of absolutely continuous arcs ($\mathcal{AC}$). However, impulsive optimal control requires to search extremizing arcs in the space of bounded variation arcs ($\mathcal{BV}$). So the obtained necessary conditions resemble the PMP conditions with conditions added by the optimality conditions of impulsive transitions. Since the class of $\mathcal{BV}$ arcs totally encompass the class of $\mathcal{AC}$ arcs, it would be expected that in the case of a control problem where set of transitions is empty, that the necessary conditions should overlap with the classical PMP. Indeed, the proposed necessary conditions differ from the nonimpulsive PMP conditions by the conditions given in (70), (71), (72), (72) and (73). In order to obtain criteria for the optimality of the transition position, transition pre-, and post-transition generalised velocities, transition time and impulsive control, variations in these entities need to be considered, which represent in the setting of this work the internal boundary variations. These additional conditions are obtained by considering the internal boundary variations at the transitions. They are obtained as a

result of discontinuous transversality conditions.
The proposed necessary conditions are for strong local minimizers and are valid in singular intervals as well. The optimal control law as stated in equation (70) is valid in singular intervals, since the zero vector at the origin belongs to the normal cone as well. The discontinuity in the controls of a bang-bang type controller are on Lebesgue negligible intervals so the control law is valid in the ”almost everywhere” sense.

References

Arutyunov, A., Karamzin, D., Pereira, F. A. (2005). Nondegenerate Maximum Principle for Impulse Control Problem with State Constraints. *SIAM J. Control Optim.*, **43**, pp. 1812–1843.

Branicky, M. S., Borkar V. S., Mitter S. M. (1998). A unified framework for hybrid control: Model and optimal theory. *IEEE Transactions on Automatic Control*, **43**, pp. 31–45.

Brogliato, B. (1996). *Non-smooth Impact Mechanics*. Lecture Notes in Control and Information Sciences. Springer Verlag.

Clarke, F. H. (1983). *Optimization and Nonsmooth Analysis*. SIAM Classics in Applied Mathematics. Wiley. New York.

Glocker, Ch, An Introduction to Impacts, In: Nonsmooth Mechanics and Solids, CISM Courses and Lectures Vol. 485 Edts: J. Haslinger and G. Stavroulakis Springer Verlag,(2006) 45–101

Miller, B. M., Bentsman, J., Optimal Control Problems in Hybrid Systems with Active Singularities. Nonlinear Analysis. **65** (2006) 999–1017

Karamzin, D. Y. , Necessary Conditions of the Minimum in an Impulse Optimal Control Problem, Journal of Mathematical Sciences, **139** (2006)

Loewen, P. D. , Rockafellar, R. T. , Bolza Problems with General Time Constraints, SIAM J. Control Optim. **35** (1997) 2050–2069

Moreau, J. J. (1988). Bounded Variations in time. In: Topics in Non-smooth Mechanics, Edts: J. J. Moreau, P. D. Panagiotopoulos, G. Strang: 1–74 , Birkhäuser, Basel

Moreau, J. J. (1988). Unilateral Contact and Dry Friction in Finite Freedom Dynamics. Non-smooth Mechanics and Applications, CISM Courses and Lectures. **302** Springer Verlag Wien (1988)

Rockafellar R. T., Convex Analysis, Princeton Landmarks in Mathematics. Princeton University Press. (1970)

Rockafellar R. T., Generalized Directional Derivatives and Subgradients of Nonconvex Functions, Can. J. Math.,**32** (1980) 257–280

Rockafellar R. T., Directionally Lipschitzian Functions and Subdifferential Calculus, Proc. London Math. Soc. **39** (1979) 331–355

Rockafellar R. T. , Dual Problems of Lagrange for Arcs of Bounded Variation, In: Calculus of Variations and Control Theory. Edts: D.L. Russell, Academic Press (1976) 155–192

Rockafellar R. T. , Integral functionals, normal integrands and measurable selections, In Nonlinear Operators and the Calculus of Variations, L. Waelbroeck, ed., Springer-Verlag, Berlin, (1976), pp. 157–207.

Rockafellar R. T. , Loewen, P. D. , The Adjoint Arc in Nonsmooth Optimization, Trans. Amer. Math. Soc. **325** (1991) 39–72

Yunt, K., Glocker, Ch., Modeling and Optimal Control of Hybrid rigidbody Mechanical Systems, Hybrid System Computation and Control (HSCC07), A. Bemporad, A. Bichi and G. Buttazzo (eds.), Springer, Lecture Notes in Computer Science (LNCS) 4416 2007 614-627

Yunt, K (2007). Impulsive Time-Optimal Control of Underactuated Manipulators with Impactively Blockable Degrees of Freedom. European Control Conference (ECC’07) 2007 3977-3984

Yunt, K (2007). Impulsive Time-Optimal Control of Structure-Variant rigidbody Mechanical Systems. 3rd International IEEE Scientific Conference on Physics and Control (Physcon 2007), IPACS Open Library \ http://lib.physcon.ru

Yunt, K., Necessary Conditions for Impulsive Time-Optimal Control of Finite-Dimensional Lagrangian Systems. Hybrid System Computation and Control (HSCC08), M. Egerstedt and B. Mishra (eds.), Springer, Lecture Notes in Computer Science (LNCS) 4981 2008 556-569

Bensoussan, A. , Optimal impulsive control theory, Lecture Notes in Control and Information Sciences, Volume 16/1979, In Book: Stochastic Control Theory and Stochastic Differential Systems, Springer (2006) 17–41

Bressan, A., Impulsive Control of Lagrangian Systems and Locomotion in Fluids, Journal of Discrete and Continuous Dynamical Systems, Vol. 20, Number 1, (2008) 1–35

Yunt, K., Impulsive Optimal Control of Hybrid Multibody Mechanical Systems, (submittted)