CONSENSUS IN NETWORKED MULTI-AGENT SYSTEMS WITH RANDOM PACKET LOSS

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Abstract
This paper studies the consensus seeking problem for a group of agents with general discrete-time linear time-invariant dynamics over Bernoulli random communication networks. It is shown for the first time that the connection weights in a communication network should be treated as control parameters to improve the solvability of the consensus problem. For networks without packet loss, it is proved that the asymptotic consensus problem is solvable under static state feedback protocol if and only if the communication topology has spanning trees. For Bernoulli lossy networks, it is revealed that the link loss probabilities of a network have non-negligible effects on the consensus seeking ability of multi-agent systems. A packet loss probability bound is obtained to ensure the solvability of the mean square consensus problem for the case when the mean topology has spanning trees.

Key words
Consensus, multi-agent systems, random networks, weighted graphs, loss probability.

1 Introduction

Consensus seeking in multi-agent systems has received considerable attention due to its broad applications in various areas, such like load balancing of communication networks [Cybenko, 1989], sensor networks [Olfati-Saber and Shamma, 2005; Xiao, Boyd and Lall, 2005], flocking of birds [Jadbabaie, Lin and Morse, 2003], formation control[Moreau, 2005; Tanner, Jadbabaie and Pappas, 2007], cooperative control [Fax and Murray, 2004], and so on.

In load balancing [Cybenko, 1989], consensus seeking aims at assigning to each processor the same number of tasks. In flocking [Jadbabaie, Lin and Morse, 2003], consensus means each bird moving in the same direction. In both cases, each agent runs a first-order integrator model to update its state. In the formation control of mobile vehicles, agents are often treated as rigid bodies and they are often modeled as first-order or second-order integrators [Hong, Chen and Bushnell, 2008; Hu and Hong, 2007; Moreau, 2005; Ren and Beard, 2005; Olfati-Saber and Murray, 2004]. The consensus protocol, which acts as each agent's control input, drives the agents to rendezvous. Most of current works on consensus focus on first-order or second-order agents. However, there are also a number of applications in which agents are modeled by general linear dynamical systems, while the relevant results are very limited [Fax and Murray, 2004; Ma and Zhang, 2008; Tuna, 2008; Wang, Cheng and Hu, 2008].

Similar to stabilizability problem in stability theory, there is a fundamental problem in consensus seeking, i.e. whether there exist consensus protocols driving a multi-agent system to consensus. Here we briefly call it consentability problem. The problem is rarely discussed in literature. For first-order multi-agent systems, the consensus conditions proposed in [Hatano and Mesbahi, 2005; Moreau, 2005; Olfati-Saber and Murray, 2004; Porfiri and Stilwell, 2007; Ren and Beard, 2005; Tahbaz-Salehi and Jadbabaie, 2008] are actually the consentability conditions because these conditions almost have nothing to do with the parameters of consensus protocols. For the second-order case, [Zhang and Tian, 2009] reveals that under Markovian switching topologies there exist linear consensus protocols solving the mean square consensus problem, if and only if the union of the topology set has a globally reachable node. [Wang, Cheng and Hu, 2008], [Ma and Zhang, 2008] and [Tuna, 2008] discuss the consentability condition of general linear time-invariant (LTI) multi-agent systems under given connection weights. [Wang, Cheng and Hu, 2008] proves that if the adjacent topology is frequently connected then the consensus is achievable via decentralized state feedback controllers. [Ma and Zhang, 2008] and [Tuna, 2008] consider the static output feedback consensus protocol and point out that the consentability requires the fixed topology to have spanning trees.

In almost all of the existing references on consensus, connection weights are treated as a part of communication topology rather than control parameters, and the solvability of the consensus problem is independent of the connection weights. However, it will be shown by an example in this paper that such a conclusion does not hold for general discrete-time multi-agent systems, and the connection weights must be seen as control parameters in solving the consensus seeking problem.

This paper focuses on the agents with general discrete-time linear dynamics. We assume the agents exchange information through communication networks. There is a priori fixed communication topology, establishing which are the feasible communications among agents. For the network without packet loss, it is proved that the system is asymptotically consentable under state feedback consensus protocol if and only if the topology has spanning trees. For Bernoulli random lossy networks, it is proved that the mean square consensus problem is solvable only if the mean topology has spanning trees. What is more, we show that the packet loss probabilities of the network are vital to the solvability of mean square consensus problem. When the probabilities are larger than some critical value, the consensus seeking problem may be unsolvable. A sufficient probability bound is also obtained. In addition, for a given linear state feedback consensus protocol, a packet loss probability condition under which the system can achieve mean square consensus is also proposed.

2 Problem Formulation

This section formulates the consensus problem that we address in this paper. Here we mainly focus on a linear consensus protocol based on static state feedback.

2.1 Communication Description

First of all, some notions on graph theory are introduced. $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ often denotes a graph, where $\mathcal{V}$ is the node set, $\mathcal{E}$ denotes the edge set and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix. If the edge $\varepsilon_{ij} = (i, j) \in \mathcal{E}$, i.e. node j can obtain information from node i, then $a_{ji} = 1$, otherwise $a_{ji} = 0$. A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3) \cdots$. A *directed tree* is a directed graph in which every node has exactly one parent except for one node, called the root, which has no parent and has a directed path to every other node. A *directed spanning tree* $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{A}_0)$ of the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is a subgraph of $\mathcal{G}$ such that $\mathcal{G}_0$ is a directed tree and $\mathcal{V}_0 = \mathcal{V}$.

In this paper, the agents exchange information through a lossy communication network. We assume a Bernoulli network: 1) at each iteration, a network link is lost with some probability; 2) network links may have different but constant link probabilities; 3) links fail independently of each other.

Suppose there is a priori fixed communication topology $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, establishing which are the feasible communications among agents. So if $\varepsilon_{ji} \in \mathcal{E}$, $a_{ij}(t)$ is varying between 0 and 1, otherwise $a_{ij}(t) \equiv 0$. Define r_{ij} the packet loss probability from agent j to i, then if $\varepsilon_{ji} \in \mathcal{E}$, $\{a_{ij}(t), t \geq 0\}$ is driven by a Bernoulli process with probability $\Pr(a_{ij}(t) = 0) = r_{ij} < 1$, where $\Pr(x)$ denotes the probability of event x. The mean topology of the network is defined as a topology corresponding to adjacency matrix $E(\mathcal{A}(t))$, where $E(\cdot)$ denotes the expected value. Obviously, the mean topology has the same edge set with the given communication topology $\mathcal{G}$.

2.2 State Feedback Consensus Protocol

Consider a group of n agents, with linear dynamics

$$x_i(t+1) = Ax_i(t) + Bu_i(t) \tag{1}$$

where $x_i \in \mathbf{R}^p$, $u_i \in \mathbf{R}^q$ denote the state and control input of agent i respectively; $A \in \mathbf{R}^{p\times p}$, $B \in \mathbf{R}^{p\times q}$ are constant matrixes and (A, B) is stabilizable. Suppose the open-loop system is not asymptotically stable, i.e. $\rho(A) \geq 1$, where $\rho(\cdot)$ denotes the spectral radius of a matrix.

We say that the protocol asymptotically solves the consensus problem, if and only if for any initial state, the agents agree upon a common state, i.e. $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for any $i \neq j$. Moreover, the protocol solves the consensus problem in mean square sense, if and only if for any $i \neq j$ there holds $\lim_{t\to\infty} E(\|x_i(t) - x_j(t)\|^2) = 0$.

Here state information is exchanged via a weighted and directed random communication network, thus we apply a static state feedback consensus protocol:

$$u_i(t) = K \sum_{j=1}^{n} a_{ij}(t) w_{ij}(x_j(t) - x_i(t)) \tag{2}$$

where $K \in \mathbf{R}^{q\times p}$ is the state feedback gain; $a_{ij}(t) \in \{0, 1\}$ denotes the connection from agent j to i at time t; $w_{ij} \in \mathbf{R}$ is the connection weight of edge ε_{ji}. Here K and w_{ij} are control parameters to be designed.

2.3 System Transformation

Define the augmented state $x(t) = [x_1^T, \cdots, x_n^T]^T$ and the Laplacian matrix $L(t) = [l_{ij}(t)]_{n\times n}$, where for $i \neq j$, $l_{ij}(t) = -a_{ij}(t)w_{ij}$, and $l_{ii}(t) = \sum_{j=1}^{n} a_{ij}(t)w_{ij}$. Then system (1) with consensus control (2) becomes

$$x(t+1) = (I_n \otimes A - L(t) \otimes BK)x(t) \tag{3}$$

Here $\otimes$ denotes Kronecker product. Obviously, for $\mathbf{1}_n = [1, \cdots, 1]^T$, $L(t)\mathbf{1}_n = 0$ holds for all graphs.

Let $z_i(t) = x_i(t) - x_1(t)$, $i = 2, \cdots, n$, then system (1)-(2) asymptotically reaches consensus is equivalent to $\lim_{t\to\infty} \|z_i(t)\| = 0$, and it achieves consensus in mean square sense if and only if $\lim_{t\to\infty} E(\|z_i(t)\|^2) = 0$. Defining $z(t) = [z_2^T, \cdots, z_n^T]^T$ we conclude that

$$z(t+1) = F(t)z(t) \tag{4}$$

where $F(t) = I_{n-1} \otimes A - \tilde{L}(t) \otimes BK$, $\tilde{L}(t) = [\tilde{l}_{ij}(t)]_{(n-1)\times(n-1)}$, $\tilde{l}_{ij}(t) = l_{(i+1)(j+1)} - l_{1(j+1)}$.

Therefore, the following lemma is obvious.

Lemma 1: Multi-agent system (1)-(2) reaches consensus asymptotically (or in mean square sense), if and only if system (4) is asymptotically stable (or mean square stable).

3 Main Results

In this section, we discuss the consentability problem defined as follows.

Definition 1: If there exists a state feedback gain $K \in \mathbf{R}^{q\times p}$ and connection weight $w_{ij} \in \mathbf{R}$ such that the protocol (2) asymptotically solves the consensus problem of system (1), we say system (1) is *asymptotically consentable* under static state feedback consensus protocol. Furthermore, if there exist K and w_{ij} such that the protocol (2) solves the consensus problem of system (1) in mean square sense, then we say system (1) is *mean square consentable* under static state feedback consensus protocol.

3.1 Communication without Packet Loss

First of all, we explain why the connection weights are considered as control parameters in Definition 1. Indeed, most of previous work on the consensus or consentability problem treated w_{ij} as fixed positive constant. Coincidentally, for the single integrator, double integrator, even high-order continuous-time LTI multi-agent systems, the consentability of the system is independent of the connection weights as long as they are positive [Ma and Zhang, 2008; Ren and Beard, 2005; Wang, Cheng and Hu, 2008; Zhang and Tian, 2009]. This gives us an impression that they can be simply absorbed into the elements a_{ij} of the adjacency matrix. However, our Example 1 will show that the connection weights play a non-negligible role for the consentability of discrete-time LTI multi-agent systems. Even if the topology has a spanning tree and each agent is stabilizable, the consensus problem may be unsolvable.

Example 1: Consider a network of 3 agents, the edge set is given as $\{\varepsilon_{12}, \varepsilon_{21}, \varepsilon_{23}\}$. Obviously, the graph has a spanning tree. Assume the connection weights are given as $w_{ij} = 1$ for all i, j, then eigenvalues of $\tilde{L}$ are 1 and 2. Given the system matrix $A = 5, B = 1$. By Lemma 1, consensus is achieved if and only if $\rho(A-BK) < 1$ and $\rho(A-2BK) < 1$, i.e. $4 < K < 6$ and $2 < K < 3$. Obviously, such a gain K doesn't exist. So even if the graph has a spanning tree and the connection weights are positive, the consensus problem may be unsolvable. □

Now let us consider the consentability condition of system (1) in the communication networks without packet loss, i.e. the connection graph is fixed.

Denote $\det(\cdot)$ as the determinant value of a matrix, $\lambda_1, \cdots, \lambda_{n-1}$ as eigenvalues of $\tilde{L}$. Then $\det(sI_{(n-1)p} - F) = \prod_{i=1}^{n-1} \det(sI_{n-1} - (A - \lambda_i BK))$, and thus the eigenvalues of F are composed of all eigenvalues of $A - \lambda_i BK$, $i = 1, \cdots, n-1$.

First of all, we study how the connectivity of a graph affect the eigenvalues of $\tilde{L}$.

Lemma 2: If the graph has no directed spanning trees, under any connection weight, $\tilde{L}$ has at least one zero eigenvalue.

Proof: Since for all graphs there exists a nonsingular matrix T, $T = \begin{bmatrix} 1 & 0 \\ \mathbf{1}_{n-1} & I_{n-1} \end{bmatrix}$ such that $T^{-1}LT = \begin{bmatrix} 0 & * \\ 0 & \tilde{L} \end{bmatrix}$. Thus by the results in [Ren and Beard, 2005], Lemma 2 can be obtained. □

Theorem 1: When there's no packet loss in transmissions, there exist connection weights and feedback gain K such that system (1) asymptotically achieves consensus under linear consensus protocol (2), if and only if the fixed communication topology has a directed spanning tree.

Proof: (Necessity) If the topology has no spanning trees, by Lemma 2 for any connection weight w_{ij}, $\tilde{L}$ has at least one zero eigenvalue. So for any $K \in \mathbf{R}^{q\times p}$, there holds $\rho(F) \geq \rho(A) \geq 1$, which is contrary to consensus.

(Sufficiency) The proof of sufficiency is constructive. If the topology has directed spanning trees, there must exist proper connection weights such that $\lambda_1 = \cdots = \lambda_{n-1} \neq 0$. Since (A, B) is stabilizable, there exists K stabilizing $A - \lambda_i BK$, and thus $\rho(F) < 1$. In the following, we'll give an approach to the choice of w_{ij} such that all eigenvalues of $\tilde{L}$ are nonzero and equal.

Denote $\mathcal{G}_0 = (\mathcal{V}, \mathcal{E}_0, \mathcal{A}_0)$ as a directed spanning tree, obviously $\mathcal{E}_0 \subset \mathcal{E}$. Choose

$$w_{ij} = \begin{cases} 1 & if\ \varepsilon_{ij} \in \mathcal{E}_0; \\ 0 & if\ \varepsilon_{ij} \in \mathcal{E} \setminus \mathcal{E}_0; \\ arbitrary\ else. \end{cases} \tag{5}$$

Then it can be easily obtained that all eigenvalues of $\tilde{L}$ are 1, thus the sufficiency is proved. □

Back to Example 1, if the connection weights are selected as $w_{1j} = w_{2j} = 0.5$, $w_{3j} = 1$ for $j = 1, 2, 3$, then as long as $4 < K < 6$, protocol (2) solves the consensus problem of system (1). Thus the system is asymptotically consentable.

3.2 Communication with Packet Loss

In this subsection we will discuss the consentability condition when the network has packet loss.

Theorem 2 : If there exist linear protocol (2) solving the mean square consensus problem of system (1), then the mean topology has a spanning tree.

Proof: Since $E(\|z(t)\|) \geq \|E(z(t))\|$, system (4) is mean square stable only if $\lim_{t\to\infty} E(z(t)) = 0$. Let $e(t) = E(z(t))$, then from system (4) we have

$$e(t+1) = E(F(t))e(t) \tag{6}$$

By Lemma 1 the multi-agent system (1)-(2) reaches mean square consensus only if (6) is asymptotically stable. Due to the fact that $E(L(t))$ is the Laplacian matrix of the topology $\mathcal{G}$ with the connection weight $w_{ij}(1-r_{ij})$ and $E(\tilde{L}(t))$ is its corresponding reduced Laplacian matrix, if the topology has no spanning trees, by Lemma 2 for any connection weight, $E(\tilde{L}(t))$ has at least one zero eigenvalue. Since $E(F(t)) = I_{n-1} \otimes A - E(\tilde{L}(t)) \otimes BK$, there doesn't exist K stabilizing $E(F(t))$, thus system (6) cannot be asymptotically stable, which is contrary to consentability. □

Theorem 2 provides a necessary condition of mean square consentability, which is a topology condition. For the first-order and second-order integrator agents, [Hatano and Mesbahi, 2005],[Tahbaz-Salehi and Jadbabaie, 2008] and [Zhang and Tian, 2009] have shown that the mean square consentability just depends on the topology condition while is independent of packet loss probability. For the general discrete-time LTI multi-agent systems, when the topology condition is satisfied, is the mean square consensus problem solvable for any Bernoulli networks? A negative answer is given through the following example.

Example 2: Consider there are 2 agents in the network, and the packet loss probability of the link is $r > 0$. Agent 1 is the leader and agent 2 is the follower, then $a_{12} \equiv 0$, a_{21} is varying between 0 and 1 with probability r and $1-r$. Given the system matrix as $A = 2, B = 1$, then by Lemma 1 system achieves mean square consensus if and only if $r * A \otimes A + (1-r) * (A - w_{21}BK) \otimes (A - w_{21}BK) < 1$([Costa and Fragoso, 1993]), i.e. $4r + (1-r)(2 - w_{21}K)^2 < 1$. It is obvious that when $r \geq 0.25$, for all w_{21} and K the above inequation does not hold. So the consentability condition must depend on the packet loss probability. □

Our concerning problem is that how to search for a packet loss probability bound r^* such that for any $r_{ij} < r^*$ the system is mean square consentable under the satisfied topology condition. This problem seems to be complicated. We firstly discuss the mean square stability condition of a Bernoulli switching system. Define a system

$$y(t+1) = (A - a(t)BK)y(t) \tag{7}$$

where $A \in \mathbf{R}^{p\times p}, B \in \mathbf{R}^{p\times q}$, (A,B) is stabilizable, $\rho(A) \geq 1$; $\{a(t), t \geq 0\} \subset \{0,1\}$ is driven by a Bernoulli switching process with probability $\Pr(a(t)=0) = r$.

Lemma 3: If there exist symmetric positive definite matrixes $Q_0, Q_1 \in \mathbf{R}^{p\times p} > 0$ and matrix $Y \in \mathbf{R}^{q\times p}$ such that the following LMIs hold

$$\begin{bmatrix} Q_0 & \sqrt{r}Q_0A^T & \sqrt{1-r}(Q_0A^T) \\ * & Q_0 & 0 \\ * & 0 & Q_1 \end{bmatrix} > 0, \tag{8}$$

$$\begin{bmatrix} Q_1 & * & * \\ \sqrt{r}(AQ_1 - BY) & Q_0 & 0 \\ \sqrt{1-r}(AQ_1 - BY) & 0 & Q_1 \end{bmatrix} > 0, \tag{9}$$

then system (7) is mean square stable under state feedback gain $K = YQ_1^{-1}$. Furthermore, if the packet loss probability is $\Pr(a(t)=0) = \alpha$ and the LMI

$$Q_1 > Q_0, \tag{10}$$

holds as well, then as long as $\alpha \leq r$, system (7) is always mean square stable under $K = YQ_1^{-1}$.

Proof: Let $V(y(t), a(t)) = y^T(t)P_{a(t)}y(t)$, $P_{a(t)} = Q_{a(t)}^{-1}$, then $E(V(y(t+1), a(t+1))|y(t), a(t) = s) = y^T(t)A_s^T(rP_0 + (1-r)P_1)A_s y(t)$, where $s = 0,1$, $A_0 = A, A_1 = A - BK$.

If $A_s^T(rP_0 + (1-r)P_1)A_s < P_s$, then $E(V(y(t+1), a(t+1))|y(t), a(t)) < V(y(t), a(t))$ and thus system (7) is mean square stable. Pre- and post-multiplying the both sides of the above inequality by Q_s and denoting $Y = KQ_1$ yields $rQ_0A^TQ_0^{-1}AQ_0 + (1-r)Q_0A^TQ_1^{-1}AQ_0 < Q_0$ and $r(Q_1A^T - Y^TB^T)Q_0^{-1}(AQ_1 - BY) + (1-r)(Q_1A^T - Y^TB^T)Q_1^{-1}(AQ_1 - BY) < Q_1$, which combined with Schur complement lemma leads to (8) and (9).

On the other hand, if LMI (10) holds, i.e. $P_0 > P_1$, then for all $\alpha \leq r$, there holds $A_s^T(\alpha P_0 + (1-\alpha)P_1)A_s < P_s$, and thus when the packet loss probability is α, system (7) is still mean square stable under $K = YQ_1^{-1}$. □

We are in a position to give a sufficient mean square consentability condition for the multi-agent systems in lossy networks. By applying the results in stochastic control theory, the Bernoulli switching system (4) is mean square stable if and only if $\rho(E(F(t) \otimes F(t))) < 1$ ([Costa and Fragoso, 1993]). Therefore, we could seek for consentability condition by discussing under what condition there exist K and w_{ij} such that $\rho(E(F(t) \otimes F(t))) < 1$ holds.

Theorem 3: If the mean topology has spanning trees and the packet loss probability $r_{ij} < r^*$ holds for all $\varepsilon_{ji} \in \mathcal{E}$, then in the lossy networks the mean square consensus of system (1) is achievable under static state feedback protocol (2), where r^* is given by

$$r^* = \sup\{\beta > 0;\ LMIs\ (8) - (10)\ have\ feasible\ solutions\ for\ \forall r \in (0,\beta)\} \tag{11}$$

Proof: Here, we provide a brief constructive proof. Similar to the proof of Theorem 1, if the mean topology, i.e. the given communication topology, has directed spanning trees, denote $\mathcal{G}_0$ as one directed spanning tree, and define the connection weight w_{ij} as that in (5).

Renumber the agents such that each agent's parent node is lower numbered than itself, and obtain a new graph $\bar{\mathcal{G}_0}$. Then the corresponding reduced order Laplacian matrix is a lower triangular matrix with diagonal elements $a_2(t), a_3(t), \ldots, a_n(t) \in \{0, 1\}$. $a_i(t) = 0$ represents that at time t there's no information received by agent i from its parent node, and $\Pr(a_i(t) = 0) = \alpha_i \in \{r_{ij} | \varepsilon_{ji} \in \mathcal{E}(\bar{\mathcal{G}_0})\}$. Therefore, $F(t)$ in (4) is of the form

$$F(t) = \begin{bmatrix} A - a_2(t)BK & & 0 \\ & \ddots & \\ * & & A - a_n(t)BK \end{bmatrix},$$

and thus $F(t) \otimes F(t)$ is a lower triangular matrix with the diagonal block $(A - a_i(t)BK) \otimes (A - a_j(t)BK)$, $i, j = 2, \ldots, n$. Since for $i \neq j$, $\{a_i(t), t \geq 0\}$ and $\{a_j(t) \geq 0\}$ are independent of each other, thus the diagonal blocks of $E(F(t) \otimes F(t))$ are in form of $(A - (1 - \alpha_i)BK) \otimes (A - (1 - \alpha_j)BK)$ and $E((A - a_i(t)BK) \otimes (A - a_i(t)BK))$.

Consider the Bernoulli switching system (7). From the proof of Theorem 2, $\rho(A - (1 - r)BK) < 1$ is just its necessary mean square stability condition while $\rho(E((A - a(t)BK) \otimes (A - a(t)BK))) < 1$ is the necessary and sufficient one, thus we just study when there exists a common state feedback gain K simultaneously mean square stabilizing $E((A - a_i(t)BK) \otimes (A - a_i(t)BK))$ for all i to seek for a mean square consentability condition.

By Lemma 3, if LMIs (8)-(10) are feasible, the obtained state feedback gain $K = YQ_1^{-1}$ must simultaneously mean square stabilize all systems $y(t + 1) = (A - a_i(t)BK)y(t)$ when $\alpha_i \leq r$. In the other words, these systems are simultaneously mean square stabilizable. Obviously, when $r = 0$, LMIs (8)-(10) are always solvable. As r is increasing, LMIs (8)-(10) may be infeasible. When $r = 1$, system (7) is unstabilizable. Define r^* as that in (11), then as long as $r_{ij} < r^*$, the mean square consensus problem is solvable. □

The packet loss probability bound proposed in Theorem 3 can be easily obtained by applying the LMI toolbox in MatLab$^{\rm TM}$.

In the following, we will discuss the mean square consentability condition of some special agent dynamics.We begin with an example about double-integrator multi-agent system.

Example 3: Consider a discrete-time multi-agent system with second-order integrator dynamics. Computing r^* by using (11) we obtain that $r^* = 1$. Thus under any Bernoulli network, as long as the communication topology has spanning trees, the mean square consensus problem is solvable. This result is in accord with that proposed in [Zhang and Tian, 2009], and thus the probability bound given by (11) is not conservative. □

Theorem 4: For the networked multi-agent systems with integrator dynamics, the upper bound of packet loss probabilities is $r^* = 1$.

Proof: This can be proved by perturbation argument and hypothetico-deductive method.

Denote $A = I_p + J, B = [0, \cdots, 0, 1]^T, K = [k_1, \cdots, k_p]$, where I_p is an identity matrix with dimension p, J is a Jordan block. If the gain K is small enough, i.e. k_i is small enough, $E((A - a_i(t)BK) \otimes (A - a_i(t)BK)) = A \otimes A - (1 - \alpha_i)(A \otimes BK + BK \otimes A) + (1 - \alpha_i)BK \otimes BK$ is a perturbation of $A \otimes A$ by two terms depending on K, and the latter perturbation term can be neglected comparing with the former one, and thus we just consider $A \otimes A - (1 - \alpha_i)(A \otimes BK + BK \otimes A)$ to verify whether its spectral radius is less than 1. Meantime, $E(A - a_i(t)BK) \otimes E(A - a_i(t)BK)$ is also a perturbation of $A \otimes A$ by two terms depending on K. For small enough K, just $A \otimes A - (1 - \alpha_i)(A \otimes BK + BK \otimes A)$ should be considered. Therefore, if there exists a small enough K stabilizing $E(A - a_i(t)BK)$, it can stabilize $E((A - a_i(t)BK) \otimes (A - a_i(t)BK))$ as well.

Denote $\Delta_i = J - (1 - \alpha_i)BK$, thus $E(A - a_i(t)BK) = I_p + \Delta_i$. Under a small enough K, as long as Δ_i is Hurwitz, $\rho(E(A - a_i(t)BK)) < 1$ holds for all $\alpha_i < 1$. Since $\det(sI_p - \Delta_i) = s^p + (1 - \alpha_i)k_p s^{p-1} + \cdots + (1 - \alpha_i)k_2 s + (1 - \alpha_i)k_1$, by applying hypothetico-deductive method, it can be proven that there exist small enough $k_1, \ldots, k_p$ such that all solutions lie in the left half-plane. The detailed process is omitted here. Therefore, $\rho(E(A - a_i(t)BK)) < 1$ and sequentially $\rho(E((A - a_i(t)BK) \otimes (A - a_i(t)BK))) < 1$. Similar to the proof of Theorem 3, Theorem 4 has been proved. □

The above discussions focus on how to choose a linear state feedback protocol to make the system achieve mean square consensus. For a given protocol, under what network conditions can the system achieve mean square consensus? By applying the results in [Hu and Yan, 2007], the following remark can be obtained to answer this question.

Remark 2: The proofs of Theorem 1 and Theorem 3 provide an approach to choosing connection weights. In fact the least neighbor information is demanded in this approach. If the gain K is given and the given communication topology has spanning trees, under these connection weights the networked multi-agent system can achieve mean square consensus if the packet loss probability $r_{ij} < \alpha$, where
$\alpha = \frac{1}{\mu(H)}$, $\mu(\cdot)$ denotes the maximum positive eigenvalue of a matrix,

$$H = \begin{bmatrix} (H_1 \otimes H_2 + H_2 \otimes H_1)H_3 & H_2 \otimes H_2 \\ H_3 & 0 \end{bmatrix},$$

$H_1 = (A - BK) \otimes (A - BK)$,
$H_2 = A \otimes A - (A - BK) \otimes (A - BK)$,
$H_3 = (I - H_1 \otimes H_1)^{-1}$.

4 Conclusion

A general discrete-time LTI multi-agent system in a Bernoulli network is discussed. The main contribution of this paper is to reveal the following two facts: 1) unlike the continuous-time LTI multi-agent systems, in discrete-time dynamics the connection weights should be treated as control parameters, otherwise consensus may be unsolvable even if the topology has spanning trees; 2) unlike the integrator case, for the general LTI multi-agent systems the packet loss probability of network is vital to consentability. A sufficient condition of the packet loss probability bound is also obtained to ensure that when the mean topology has spanning trees, the mean square consensus problem is solvable. It is not necessary for the interaction topology to be instantaneously connected.

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