STABILITY ANALYSIS OF QUANTUM NETWORKS OF CAVITY QED AND GENERATION OF W, W-CLASS AND GHZ STATES

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Abstract

This paper is concerned with the stability analysis of the optical QED feedback control systems in interacting Fock space and a new method for generating multistate entangled states by designing quantum networks of the closed optical QED cavities connected in parallel. The Nyquist stability of the quantum feedback control system in the interacting Fock space using beam splitter device has been analyzed. The generation of W, W-class and GHZ states are discussed by constructing respective composite networks of the optical feedback QED cavities.

Key words

Stability analysis, quantum networks, cavity QED

1 Introduction

Quantum feedback analysis is increasingly being used in designing quantum gates that includes quantum components and devices [Wiseman and Milburn, 1993], [Roy and Das, 2007a], [Gardiner and Zoller, 2000]. In fact, a wide range of quantum systems can be considered as networks of quantum and classical devices that include feedback interactions of two or more optical QED cavities connected in parallel. In this paper we utilize the concept of interaction of single mode of quantized field in a cavity with a noisy external field for finding the state space model in interacting Fock space. The dynamics of the cavity in the interacting Fock space are obtained by utilizing the basic concepts of quantum stochastic process [Gardiner and Zoller, 2000]. The state space model of quantum feedback control system in interacting Fock space is shown to be a generalization of the design of quantum feedback control system of the optical cavity in the usual boson Fock space [Yanagisawa and Kimura, 2003a], [Yanagisawa and Kimura, 2003b], [Wiseman and Milburn, 1993], [Gardiner and Zoller, 2000]. The stability analysis of the feedback systems using beam splitter has been discussed by applying Nyquist stability criterion of the open-loop system of single cavity and the modelling of quantum networks of any number of optical feedback baths connected in parallel have been outlined. The physical characteristics of the feedback control system, such as, the gain margin(GM) and phase margin(PM) of the single and composite cavity QED systems are expressed in terms of the parameters of the beam splitter. The experimental problems of generating W, W-class and GHZ states [Gorbachev and Trubilko, 2006] are outlined via quantum networks of the QED cavities connected in parallel.

2 Preliminary Concepts

In this section we discuss some basic preliminaries on interacting Fock space [Accardi and Bozejko, 1998], [Das and Roy, 2006] and interaction of optical QED cavity with the external field [Wiseman and Milburn, 1993], [Gardiner and Zoller, 2000] which will be needed throughout the paper.

2.1 Interacting Fock Space

As a vector space one mode interacting Fock space $\Gamma(\mathbb{C})$ [3] is defined by

$$\Gamma(\mathbb{C}) = \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle \tag{1}$$

where $\mathbb{C}|n\rangle$ is called the n-particle subspace. The different n- particle subspaces are orthogonal, that is, the sum in (1) is also orthogonal. The norm of the vector $|n\rangle$ is given by

$$\langle n|n\rangle = \lambda_n \tag{2}$$

where $\{\lambda_n\} > 0$. The norm introduced in (2) makes $\Gamma(\mathbb{C})$ a Hilbert space.

An arbitrary vector f in $\Gamma(\mathbb{C})$ is given by

$$f \equiv c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + \ldots + c_n|n\rangle + \ldots \quad (3)$$

with $\|f\| = (\sum_{n=0}^{\infty} |c_n|^2\lambda_n)^{1/2} < \infty$. We assume also that the sequence $\{\lambda_n\}$ satisfies the condition $\inf_{n\geq 0} \lambda_n^{1/n} > 0$.

We now define following actions on $\Gamma(\mathbb{C})$

$$\begin{array}{r} a^{\dagger}|n\rangle = |n+1\rangle \\ a|n+1\rangle = \frac{\lambda_{n+1}}{\lambda_n}|n\rangle \end{array} \quad (4)$$

$a^{\dagger}$ is called the *creation operator* and its adjoint a is called the *annihilation operator*. To define the annihilation operator we have taken the convention $0/0 = 0$.

The commutation relation of the operators then takes the form

$$[a, a^{\dagger}] = \frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}} \quad (5)$$

where N is the number operator defined by $N|n\rangle = n|n\rangle$.

2.2 Interaction of Cavity and the External Field

We consider the interaction of an interacting single-mode of quantized field confined in an optical cavity with a noisy external field. Let $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ be Hilbert spaces of the cavity and the external field respectively. The composite system is expressed by the tensor product space $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$. The total Hamiltonian is given by

$$H_{total} = H_A + H_B + H_{int} \quad (6)$$

H_A being described the Hamiltonian of the cavity mode and may be further subdivided into two parts H_{cav} and H. Here H is referred to as a *free Hamiltonian* determined by the optical medium in the cavity. H_B is the Hamiltonian of the external field.

After dropping the energy non-conserving terms in H_{int} corresponding to the rotating-wave approximation we obtain the simplified Hamiltonian

$$H_{int}(t) = i\sqrt{\gamma}[a(t)b^{+}(t) - a^{+}(t)b(t)] \quad (7)$$

with $[b(t), b^{+}(t')] = \delta(t - t')$, γ being a coupling constant. The operators a and b are respectively the annihilation operators of the cavity and the external field.

3 States Space Model of Cavity QED via Stochastic Process

In deriving the dynamics of the QED cavity in interacting Fock space, let us define quantum stochastic process by the operator

$$B_{in}(t, t_0) = \int_{t_0}^{t} b_{in}(s)ds \quad (8)$$

where the field $b_{in}(t)$ being the input to the cavity satisfies the commutation relation of section 2.2 and represents also the field immediately before it interacts with the system.

It is easy to show that [Das and Roy, 2006], [Gardiner and Zoller, 2000]

$$[dB_{in}(t), dB_{in}^{+}(t)] = dt \quad (9)$$

The relation (9) leads to the natural definition of quantum stochastic process referred in [Yanagisawa and Kimura, 2003a], [Yanagisawa and Kimura, 2003b].

The evolution of an arbitrary operator X is written as $X(t) = U^{+}(t)XU(t)$ in which the unitary operator $U(t)$ is generated by the Hamiltonian in (6). The Hamiltonians H_{cav} and H_B drive the cavity and the external field respectively. We shall assume here H to be zero. The unitary operator of the system is then given by

$$U(dt) = e^{\sqrt{\gamma}(adB_{in}^{+} - a^{+}dB_{in})} \quad (10)$$

The increment of an arbitrary operator r of the system driven by the stochastic input b_{in} is given by

$$\begin{array}{l} dr(t) = \sqrt{\gamma}[a^{+}dB_{in} - adB_{in}^{+}, r(t)] + \\ \quad +\frac{\gamma}{2}\{(N'+1)(2a^{+}ra - a^{+}ar - ra^{+}a) \\ \quad +N'(2ara^{+} - aa^{+}r - raa^{+}) \\ \quad +M[a^{+}, [a^{+}, r]] + M^{*}[a, [a, r]]\}dt \end{array} \quad (11)$$

The dynamical behaviour of the optical cavity in interacting Fock space is now described on replacing the general operator $r(t)$ in equation (11) by the operator $a(t)$ of the QED bath. Then using the commutation relations (9) and the stochastic process given in [Yanagisawa and Kimura, 2003a], [Yanagisawa and Kimura, 2003b] we get

$$\begin{array}{rl} da &= a(t + dt) - a(t) \\ &= \{-\frac{\gamma}{2}(\frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}})a \\ &\quad -\sqrt{\gamma}(\frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}})b_{in}(t)\}dt \end{array} \quad (12)$$

This implies

$$\begin{array}{rl} \dot{a}(t) = &-\frac{\gamma}{2}(\frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}})a(t) \\ &-\sqrt{\gamma}(\frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}})b_{in}(t) \end{array} \quad (13)$$

The state equation represented by (13) of the dynamics of the cavity A in the interacting Fock space is a generalization of the well known Langevin equation of the cavity in boson Fock space [Yanagisawa and Kimura, 2003a], [Yanagisawa and Kimura, 2003b], [Gardiner and Zoller, 2000]. In case of boson Fock space the commutator described in equation (5) becomes unity, that is,

$$[a, a^*] = \frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}} = \Lambda_N^{'} = 1$$

The dynamics of the cavity given by (13) then reduces to the usual quantum Langevin form in boson Fock space

$$\dot{a}(t) = -\frac{\gamma}{2}a(t) - \sqrt{\gamma}b_{in}(t). \tag{14}$$

Due to interaction of the evolving incoming field with the optical cavity an outgoing field is produced and is given by

$$B_{out}(t, t_0) = \int_{t_0}^{t} b_{out}(s)ds, \tag{15}$$

$$b_{out}(t) = U^{+}(dt)b_{in}(t)U(dt) \tag{16}$$

The input-output relation after the interaction at time t is represented [Das and Roy, 2006], using (9), by the following equation:

$$b_{out}(t) = \sqrt{\gamma}a(t) + b_{in}(t). \tag{17}$$

We have seen that the cavity dynamics may be thought of as an operator equation in Hilbert space. The equations (13) and (17) give the state equation and the system output of a cavity in different modes. The state equation of the cavity dynamics along with the output equation can be represented with usual notation as

$$\begin{aligned} \dot{a}(t) &= A^{'}a(t) + B^{'}b_{in}(t) \\ b_{out}(t) &= C^{'}a(t) + D^{'}b_{in}(t) \end{aligned} \tag{18}$$

The dynamics of the cavity in the interacting Fock space is a first order differential equation of the system operators with variable space parameter $\Lambda_N^{'}$ as coefficient.

Applying Laplace transform in (18), assuming zero initial state of the QED bath, we get the transfer function representation of the optical QED system in interacting mode as,

$$b_{out}(s) = G(s)b_{in}(s), \tag{19}$$

$$G(s) = \frac{s - \frac{\gamma_A}{2}\Lambda_N^{'}}{s + \frac{\gamma_A}{2}\Lambda_N^{'}}. \tag{20}$$

We have seen that a cavity QED in interacting mode in some way closely analogous to the classical one in which the input and the output are described by operators in Hilbert space.

4 Stability of Feedback Control System of QED

The state space modelling of the optical QED bath in the interacting Fock Space is taken to be the basis of designing closed-loop feedback control system. However, the transfer function representations of quantum feedback control experience system gain functions that provide various valuable insight into the problems such as, stability, gain margin and phase margin of the systems. These problems of quantum mechanical systems can be solved by analyzing open loop transfer functions of the feedback systems using Nyquist stability criterion.

Utilizing the general procedure of classical feedback control theory, the mathematical model of the cavity QED system with unity feedback may be described using a quantum device, such as, beam splitter as shown on Fig. 1.

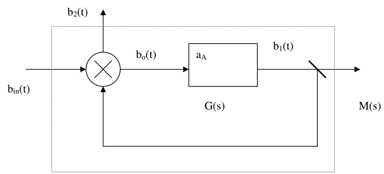

Figure 1. The design of the cavity QED system using beam splitter.

The two input signals b_{in} and b_1 to the beam splitter are related to the output signals b_0 and b_2 by

$$\begin{bmatrix} b_2 \\ b_0 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} b_{in} \\ b_1 \end{bmatrix} \tag{21}$$

where α and β are real, positive and satisfy $\alpha^2 + \beta^2 = 1$. From the input-output relation (17) of the cavity, we have

$$b_1 = \sqrt{\gamma_A}a_A + b_0 \tag{22}$$

Each signal in the cavity feedback loop can now be written as

$$\begin{aligned} b_0 &= \tfrac{\beta}{1+\alpha} b_{in} - \tfrac{\alpha}{1+\alpha}\sqrt{\gamma_A} a_A \\ b_1 &= \tfrac{\beta}{1+\alpha} b_{in} + \tfrac{1}{1+\alpha}\sqrt{\gamma_A} a_A \\ b_2 &= b_{in} + \tfrac{\beta}{1+\alpha}\sqrt{\gamma_A} a_A \end{aligned} \quad (23)$$

The closed-loop transfer function of the feedback control system with unity feedback of the optical cavity QED using beam splitter is described as

$$M_1(s) = \frac{\beta G(s)}{1 + \alpha G(s)} \quad (24)$$

The stability of the closed-loop system is characterized by the zeros of $1 + \alpha G(s) = 0$. In this problem of examining the stability of a complex feedback control system there arises some difficulty in solving the characteristic roots of the system. However, this problem of complex system may easily be solved with the help of Nyquist stability criterion.

Let us now discuss the problem of stability of the QED system when designed as on Fig. 1 using a beam splitter. In this case, the Nyquist plot of $G(s)$ with the parameter $\Lambda_N^{'} = 1$ cuts the real axis at the points $E = -\alpha + j0$ and $F = \alpha + j0$. As $0 < \alpha < 1$, the stability of the closed-loop system is assured when a beam splitter is used.

The *phase margin*(PM) of the feedback control system is zero. The *gain margin*(GM) of the system is given by

$$GM = \frac{1}{|OE|} = \frac{1}{\alpha} > 1. \quad (25)$$

In the special case of $50/50$ beam splitter, $\alpha = \frac{1}{\sqrt{2}}$. And so, $GM = \sqrt{2}$. If it is expressed in decibel, then

$$GM = 20log_{10}\frac{1}{|OE|} = 3.01 > 3\,dB \quad (26)$$

Note that the measurements of GM and PM of a closed-loop feedback control system indicate the information about the degree of stability of the control system.

5 Quantum Networks of Cavities Connected in Parallel

In describing quantum network of any number of optical QED baths connected in parallel, we consider a network of two closed cavities of fields a_A and a_B as shown on Fig. 2.

The initial input to the network is the field $b_{in}(t)$. The outputs leaving out of the beam splitters are shown in the figure.

In deriving the resultant transfer function $H(s)$ of the composite feedback system let us assume,for simplicity, the reflectivity (α) and transmissivity (β) parameters of all beam splitters to be the same.

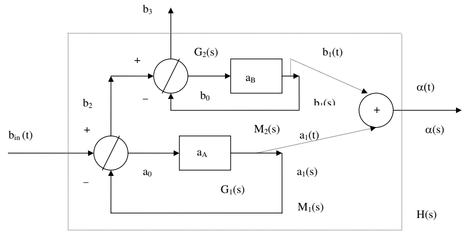

Figure 2. Network of two parallel QED.

For generalization we use the following notations for computing the transfer functions of the feedback QED system. The open-loop and closed-loop transfer functions of the optical cavity A are given by

$$a_1(s) = G_1(s)a_0(s), \quad a_1(s) = M_1(s)b_{in}(s), \quad (27)$$

$$G_1(s) = \frac{s - \frac{\gamma_A}{2}\Lambda_N^{'}}{s + \frac{\gamma_A}{2}\Lambda_N^{'}}, \quad M_1(s) = \frac{\beta G_1(s)}{1 + \alpha G_1(s)}$$

The corresponding transfer functions of other cavities are derived similarly. Using the input-output relation defined by (21) of the beam splitter we get

$$b_1(s) = G_2(s)b_0(s), \quad b_1(s) = M_2(s)b_2(s), \quad (28)$$

$$G_2(s) = \frac{s - \frac{\gamma_B}{2}\Lambda_N^{'}}{s + \frac{\gamma_B}{2}\Lambda_N^{'}}, \quad M_2(s) = \frac{\beta G_2(s)}{1 + \alpha G_2(s)}$$

Therefore the output of the second bath is given by

$$b_1(s) = \{M_2(s)[\alpha + \beta M_1(s)]\}b_{in}(s) \quad (29)$$

Then the resultant transfer function of the output $a_2(s)$ to the input $b_{in}(s)$ of the composite network of two baths connected in parallel is computed using an optocoupler (Σ) as

$$\begin{aligned} a_2(s) &= a_1(s) + b_1(s) \\ &= H(s)b_{in}(s), \end{aligned} \quad (30)$$

$$H(s) = M_1(s) + M_2(s)[\alpha + \beta M_1(s)] \quad (31)$$

In a similar way we can compute by means of optocoupler the resultant transfer function of the closed network of n parallel baths and represent as

$$H(s) \equiv \{M_1(s) + M_2(s)[\alpha + \beta M_1(s)] + \ldots + M_n(s)[\alpha + \beta M_1(s)]. \ldots [\alpha + \beta M_{n-1}(s)]\} \quad (32)$$

The stability of the quantum network of composite cavities follows from the stability of the individual baths.

An application of the quantum networks of the optical QED cavities is outlined in the following section by generating W, W-class and GHZ states.

6 Generation of $|W>$, W-class and $|GHZ>$ States

The most experimental demonstration of entanglement of quqbits require nonclassical devices and correlated measurements of simple photon detection events. Many nonclassical features of three or more photon absorptions have been described including an enhanced rate of photon absorption when the incident photons are entangled. The fact that the photons are incident on any given atom in a QED cavity at the same type while their total energy is still well defined giving rise to the rate of photon absorption.

For constructing entangled states we use a number of optical QED baths connected in parallel. The initial input to the quantum network of parallel baths is the field $b_{in}(t)$. The outputs leaving out of the beam splitter are shown in generating the respective entangled states in different figures.Finally performing concatenation first and then superposition via optocoupler of the output of the beam splitter and system outputs of the cavities A, B and C, as shown on the figure, we can generate the W state and GHZ state.

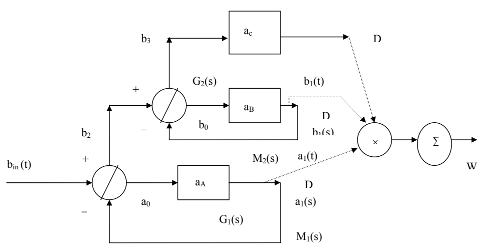

Figure 3. Generation of W state.

The W state is an entangled state of three qubits and we thus require three QED baths whose total energy $\hbar w$ as shown on Figure 2. If a photon is the input source, it is injected either in the first bath A where a photon is detected and the other two baths B and C are empty and so we get the state $|100>$ by concatenation of three baths, or in other baths we then get the states $|010>$ or $|001>$. Then finally taking superposition rules by means of optocoupler we get the W state which is

$$|W>= \frac{1}{\sqrt{3}}(|100> + |010> + |001>)$$

.

Again for generating W-class let us also put a beam splitter in the last cavity of Fig. 3. If at the outputs of the cavities we do not observe any photons then the code $|000>$ is generated. In other cases we can compute as explained in case of W state the states $|100>$, $|010>$ and $|001>$. In the extreme case, if the photon is going out through the output of the last beam splitter of bath C then to generate the nonzero weighted code we take the output of the beam splitter of bath C and the output of the other two baths. Then taking superposition by means of optocoupler of the concatenated states we get the state of W-class as

$$|W>= a_1|000> + a_2|100> + a_3|010> + a_4|001>$$

where $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$.

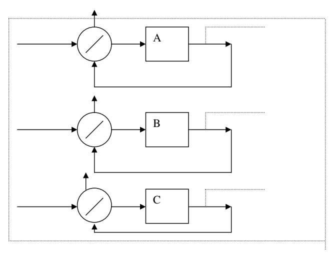

Figure 4. Generation of GHZ state.

The generation of GHZ state may be discussed as follows. Let us assume that all the outputs of the beam splitters contain three photons. If not, in case of a bath, then the output of that bath contains the photon. We then concatenate the output of the bath and the outputs of other two beam splitters yielding the state $|111>$. Then the outputs of the baths generate the state $|000>$ by concatenation. Finally taking superposition by means of optocoupler of these states as in W state we get the GHZ state as

$$|GHZ>= \frac{1}{\sqrt{2}}(|111> + |000>)$$

.

7 Conclusion

This study mainly explores the input-output relation of the cavity in noisy stochastic field and the modelling of quantum network of optical QED cavities and the stability analysis of feedback QED cavities in interacting Fock space. As an application we outlined the generation of entangled W, W-class and GHZ states using the photonic device optocoupler.

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