CONDITIONS FOR TOPOLOGICAL EQUIVALENCE OF LOCAL QUALITATIVE SINGULARITIES OF DYNAMICAL SYSTEMS WITH IMPACT INTERACTIONS

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Abstract

For the introduced class of dynamical systems with impact interactions, local singularities (six types) are determined. Properties that allow us to prove the topological equivalence of these singularities are described for them. A number of unsolved problems are formulated, which are adjacent to the problems considered in the article.

Key words

dynamical system with impact interactions, vibro-impact system, local singularity, topological equivalence, infinite-impact motion

1 Introduction

The author dedicates this article to the memory of Professor Ilya Israelevich Blekhman, who showed an example of attitude to person and science with his life. He was the author of completely unexpected ideas. For example [Blekhman, 2012], where the definition of ”oscillatory strobodynamics” is given as an interdisciplinary field of knowledge that explores the slow component of system dynamics in the presence of high-frequency oscillations in engineering, natural science and sociology systems.

The author is especially grateful to I. I. Blekhman for his cordial attitude [Blekhman, 2020] to the memory of Yu. I. Neymark, my teacher.

It was Yu. I. Neymark who performed the first work [Neimark, 1953] on the qualitative theory of vibro-impact systems. It is still referred to by researchers. Currently, researchers engaged in the study of vibro-impact systems, i.e. when impacts are added to the action of vibrations (the results of the impact of vibrations on technical systems are described in the monograph [Blekhman, 2018]), are busy:

methods for calculating vibro-impact systems (for example, [Astashev and Krupenin, 2016; Blekhman and Sorokin, 2016; Burd and Krupenin, 2016, etc.]);

development of models, methods of synthesis and analysis of dynamics vibro-impact systems of various types (for example, [Blekhman et al., 2018; Blekhman et al., 2021; Markeev and Sukhoruchkin, 2016, etc.]).

For a detailed presentation of the theory of vibro-impact systems and its current state, the interested reader is recommended, for example, the book [Dinamika, 2015].

From the point of view of studying local qualitative singularities of dynamical systems with impact interactions, the following is done. In [Denisov et al., 1973] (for a specific system) and in [Fedosenko, 1976] (for a non-autonomous dynamical system of a general type with a direct impact described by the Newton hypothesis), the structure of the phase space in the vicinity of a local singularity of a certain kind (as will be indicated below, of the fifth type) is established.

In [Gorbikov, 1987], six types of local qualitative singularities of dynamical systems with impact interactions of a general form are introduced.

In [Gorbikov and Neimark, 1981; Gorbikov, 1998; Gorbikov, 2020], a description of infinite-impact motions (for certain types) is given using smooth differential equations. *Infinite-impact motions are motions with an infinite number of impact interactions over a finite period of time* [McMillan V.A., 1951, p. 291; Feigin, M.I., 1967; Nagaev, R.F., 1985]

In [Gorbikov, 1998-2; Gorbikov, 2001], the topological equivalence of local qualitative singularities of the five selected types was established.

On the other hand, the conditions that make it possible to prove such an equivalence are only those conditions that should be studied. Therefore, in this paper, we analyze and present those conditions that allowed us to establish the topological equivalence of the last five types and another type (the topological equivalence of which has been proved, but not published).

2 The class of dynamical systems under consideration

It is assumed [Gorbikov and Neimark, 1981] that instantaneous impact interactions occur on the hypersurface $x_n = 0$, after reaching which the phase variables $x_1, x_2, \ldots, x_{n-1}$ changes abruptly (the variable x_n remains equal to zero) according to the formulas

$$\begin{cases} \bar{x}_1 = H_1 = x_1^- H_{11}(x_1^-, \ldots, x_{n-1}^-), \\ \bar{x}_i = H_i(x_1^-, \ldots, x_{n-1}^-) = x_i^- + \\ +x_1^- H_{1i}(x_1^-, \ldots, x_{n-1}^-), i = \overline{2, n-1}, \end{cases} \quad (1)$$

and for $x_n > 0$, the change in the phase variables obeys differential equations of the form

$$\begin{cases} \dfrac{dx_i}{dt} = \dot{x}_i = \Phi_i(x_1, \ldots, x_n), \ \ i = \overline{1, n-1}, \\ \dfrac{dx_n}{dt} = \dot{x}_n = \Phi_n(x_1, \ldots, x_n) = \\ = x_1 \Phi_{n1}(x_1, \ldots, x_n) + x_n \Phi_{nn}(x_1, \ldots, x_n). \end{cases} \quad (2)$$

The phase space of the system is points $(x_1, \ldots, x_{n-1}, x_n \geq 0)$. In the relations (1): $x_1^-, \ldots, x_{n-1}^-$ and $\bar{x}_1, \ldots, \bar{x}_{n-1}$ are the pre-impact and post-impact values of the variables, respectively.

The following conditions are met: $-1 < H_{11}(0, x_2^-, \ldots, x_{n-1}^-) \leq 0$; $H_{11}(x_1^-, x_2^-, \ldots, x_{n-1}^-) \leq 0$; $\Phi_{n1}(x_1, \ldots, x_{n-1}, 0) > 0$; $t-$ time. Functions $H_{1j}, j = \overline{1, n-1}$, are defined and are smooth of class C^m, $m \geq 5$, in small neighborhoods of points $(x_1^- \leq 0, x_2^-, \ldots, x_{n-1}^-)$ spaces R^{n-1}, and functions Φ_j, $j = \overline{1, n-1}$, Φ_{n1}, Φ_{nn} are defined and are smooth of class C^m in small neighborhoods of points $(x_1, \ldots, x_{n-1}, x_n \geq 0)$ of the space R^n.

It should be noted that the specific form of equations (2) and the condition of the inequality type on the function Φ_{n1} only means that:

1) on the hypersurface $x_n = 0$, according to (2),

$$\dot{x}_n = x_1 \Phi_{n1}(x_1, \ldots, x_{n-1}, 0); \quad (3)$$

2) therefore, the phase trajectories of the system (2) at $x_1 = 0$ they touch the hypersurface $x_n = 0$, as the time t increases, they leave the points $(x_1 > 0, x_2, \ldots, x_{n-1}, x_n = 0)$, and as the time t decreases, they leave the points $(x_1 > 0, x_2, \ldots, x_{n-1}, x_n = 0)$. $(x_1 < 0, x_2, \ldots, x_{n-1}, 0)$ (Figure 1). In Figure 1, solid lines the trajectories of the system are indicated (2), and the dotted lines connect the points and their images when displayed (1).

Here we have replaced: the impact hypersurface $S = 0$ with the condition $x_n = 0$.

The specified form (1) of impact interactions implies only that when the hypersurface $x_n = 0$ is reached by the phase trajectory with the rate of change the last variable equal to $\dot{x}_n = 0$ impact interactions do not change the values of the phase variables (because by virtue of (3) the condition $\dot{x}_n = 0$ implies the equality $x_1 = 0$), and the conditions in the form of inequalities on the function H_{11} mean the loss of the absolute value of the rate of change of the variable x_n after impact interactions .

Therefore, the equations of motion of many mechanical systems with a single impact pair are presented in this way.

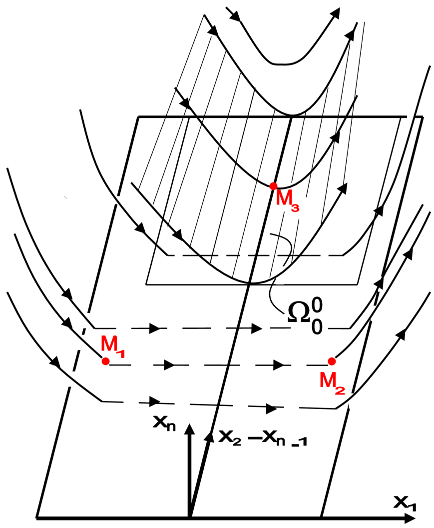

Figure 1. Possible behavior of phase trajectories in the vicinity of the hypersurface $x_n = 0$.

In the future, the following types [Gorbikov, 1987] of local qualitative singularities of M^* dynamical systems of the form (1) - (2).

The first type. *At the point M^* takes place $x_n = 0$, $\dot{x}_n << 0$ (here $\dot{x}_n = \Phi_n(x_1, \ldots, x_n)$).*

The second type. *At the point M^* takes place $x_n = 0$, $\dot{x}_n > 0$.*

The third type. *At the point M^* is valid $x_n = 0$, $\dot{x}_n = = 0$, $\ddot{x}_n > 0$ (here $\ddot{x}_n = \sum_{k=1}^{n} \dfrac{\partial \Phi_n}{\partial x_k} \Phi_k$).*

The fourth type. *At the point M^* is valid $x_n = 0$, $\dot{x}_n = 0$, $\ddot{x}_n < 0$.*

The fifth type. *At the point M^* takes place $x_n = = 0$, $\dot{x}_n = 0$, $\ddot{x}_n = 0$, $x_n^{'''} > 0$ (here $x_n^{'''} = = \sum_{j=1}^{n} \dfrac{\partial}{\partial x_j} (\sum_{k=1}^{n} \dfrac{\partial \Phi_n}{\partial x_k} \Phi_k) \Phi_j$).*

The sixth type. *At the point M^* takes place $x_n = 0$, $\dot{x}_n = = 0$, $\ddot{x}_n = 0$, $x_n^{'''} < 0$.*

More degenerate cases (when at the point M^* the first nonzero derivative of the function x_n describing the impact manifold is the fourth or even higher) are not considered here.

3 The first three types of local qualitative singularities

Consideration of the first three types of local singularities [Gorbikov, 1998-2] is not difficult.

To prove the topological equivalence of local qualitative singularities of First type, it is only necessary to know that:

1) the same value of n (this is necessary for all types of local qualitative singularities, so this requirement will be implied everywhere else, but not mentioned);

2) *exists a sufficiently small neighborhood Ω of this point (in the phase space of the system (1) - (2)). Through any point Ω at $x_n > 0$ passes a phase trajectory that leads (in a fairly short period of time) a phase point to the impact manifold $x_n = 0$. Due to the impact interactions that then act, the phase point leaves the selected small neighborhood Ω.*

(In Figure 1, a singularity of this type corresponds, for example, to the point M_1).

To prove the topological equivalence of local qualitative singularities of Second type, it is only necessary to know that:

there is a sufficiently small neighborhood Ω of a point M^* in the phase space of the system (1) - (2), such that at the points of the set $\Omega_0 = \Omega \cap \{(x_1, \dots, x_n) | x_n = 0\}$ the phase points fall from the points of the set $\dot{x}_n < 0, x_n = 0$. At the points of the set Ω_0, the phase trajectories of the system (2) go to the region $x_n > 0$. They make up the entire set Ω.

(In Figure 1, a singularity of this type corresponds, for example, to the point M_2).

To prove the topological equivalence of local qualitative singularities of Third type, it is only necessary to know that:

there is such a sufficiently small neighborhood Ω of a point M^* in the phase space of system (1) - (2), which is the set of trajectories of system (2) that exit with increasing and decreasing t from the points of the set $\{x_n = 0, \dot{x}_n = \Phi_n(x_1, \dots, x_n) = 0 (x_1 = 0)\} = \Omega_0^0$, is divided into two parts: the phase trajectories from the 1st part pass in the region of Ω without crossing the hypersurface of the impact, and the phase trajectories from the 2nd part are sections of the phase trajectories of the system (2), which lead the phase points to the hypersurface of the impact at $\dot{x}_n < 0, x_n = 0$, then the strength of impact interactions (1) phase points go into points of the set $x_n = 0, \dot{x}_n > 0$, then the phase point overlook in the area of $x_n > 0$ along the trajectories of system (2).

(In Figure 1, a singularity of this type corresponds, for example, to the point M_3).

4 The fourth type of local qualitative singularities

As it was established in [Gorbikov, 1998-2]: the entire sufficiently small neighborhood Ω of the point M^* (the local qualitative singularity of the fourth type) is filled with infinite-impact motions.

The trajectory of an infinite-impact motion exiting at $x_n = 0$ from a point $M(x_1 > 0, x_2, \dots, x_{n-1})$, leaves a " trace " of an infinite number of points on the manifold $x_n = 0, x_1 > 0$

$$M_j = T^j(M),\ j = 1, 2, 3, \dots, \quad (4)$$

Here is a point map $T = T_2 T_1$ of a part of the manifold $x_n = 0, x_1 \geq 0$ into yourself. The mapping T_1 translates the point $(x_1 \geq 0,\ x_2, \dots, x_{n-1}, 0)$ to the point $(x_1 \leq 0, x_2, \dots, x_{n-1}, 0)$ along the trajectories of the system (2); T_2 is the mapping given by the formulas (1) of impact interactions.

In [Gorbikov and Neimark, 1981], the following theorem is proved on the description of these motions using smooth differential equations.

Theorem. *All points of the " trace " (4) lie on the integral curve of the system of differential equations passing through M*

$$\frac{dx_i}{dx_1} = f_i(x_1, \dots, x_{n-1}),\ i = \overline{2, n-1}, \quad (5)$$

where the functions f_i is defined when $x_1 \geq 0$, $f_i \in C^{m-2}$.

The solutions of the system (5) give an idea of the " trace" infinite-impact motions on the manifold $x_n = 0$ at $x_1 \geq 0$. Therefore, the integral curves of the system (5) are further called *auxiliary sliding movements*.

In Figure 2, a singularity of such a type corresponds, for example, to the points M^* and M_∞. (The point M_∞ is the limit point of the trace (4), which belongs to the manifold $x_n = 0, x_1 = 0$).

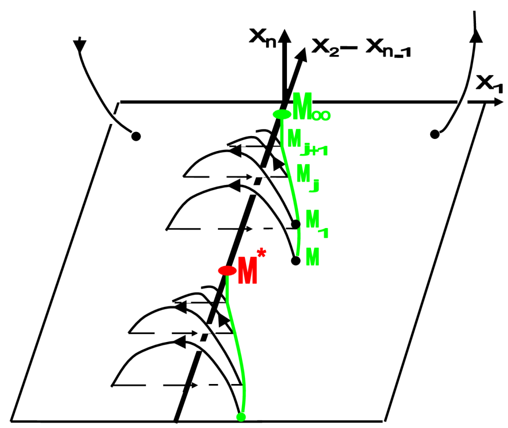

Figure 2. The behavior of phase trajectories in the vicinity of local qualitative singularities of the fourth type. The phase trajectories of auxiliary sliding movements are highlighted in green

The arguments of [Gorbikov, 1998-2] lead to the following conclusion: the fourth type for **the proof of topological equivalence requires only the representation of infinite-impact motions by means of smooth differential equations (5).**

5 The sixth type of local qualitative singularities

To simplify the consideration, you can move the point M^* (a local qualitative singularity of the sixth type) to the origin. In addition, the points of the manifold $x_n = 0, x_1 = 0, x_2 > 0$ are points of the third type, and the points of the manifold $x_n = 0, x_1 = 0, x_2 < 0$ are points of the fourth type.

As shown in [Gorbikov, 2020], the following is true.

Statement 1.

1) The entire sufficiently small neighborhood of the Ω point M^* (a local qualitative singularity of the sixth type) is filled with infinite-impact motions, and all phase points fall on the manifold $x_n = 0, x_1 \geq 0$ (therefore, the behavior of the phase points of the set Ω can be characterized by considering the behavior of the points of the set $x_n = 0, x_1 \geq 0$);

2) all points lying between the points of the set $x_n = 0, x_1 = 0, x_2 \geq 0$ and the points of the set ν_1 fall under the action of mapping T to the points of the set lying between the points of the sets ν_1 and $\nu_2 = T(\nu_1)$.

Here the set ν_1 is the image of the set $x_n = 0, x_1 = 0, x_2 \geq 0$ under the mapping action T (Figure 3). The form of the sets ν_1, ν_2 is specified in [Gorbikov, 2020] and is clear from Figure 3.

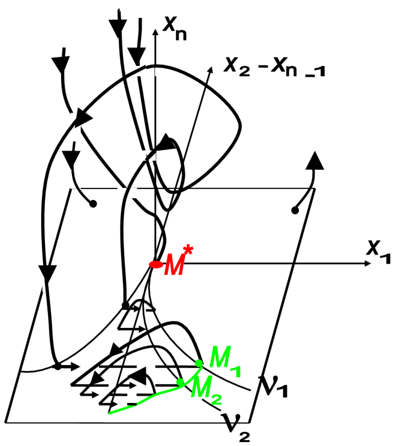

Figure 3. The behavior of phase trajectories in the vicinity of a local qualitative singularity of the sixth type. The phase trajectory of the auxiliary sliding movements is highlighted in green

Further, the equations of auxiliary motions act on the manifold $x_n = 0, x_1 \geq 0, x_2 < 0$. Their appearance changes in comparison with (5), and is given in [Gorbikov, 2020].

The arguments of [Gorbikov, 2001] lead to the following conclusion: the sixth type for **the proof of topological equivalence requires only the representation of infinite-impact motions by means of these smooth differential equations and statement 1.**

6 The fifth type of local qualitative singularities

To simplify the consideration, you can move the point M^* (a local qualitative singularity of the fifth type) to the origin. In addition, the points of the manifold $x_n = 0, x_1 = 0, x_2 > 0$ are points of the third type, and the points of the manifold $x_n = 0, x_1 = 0, x_2 < 0$ are points of the fourth type.

Consideration of a singularity of this type can be carried out by studying the behavior of phase points that lie on the manifold $x_n = 0, x_1 \geq 0$.

Statement 2. *All points that lie between the set $\{x_n = 0, x_1 = 0, x_2 > 0\}$ and the set γ_1 come from a small neighborhood Ω of the point M^* (a local qualitative singularity of the fifth type).*

(Here γ_1 is a set whose points fall into the points of the set $\{x_n = 0, x_1 = 0, x_2 \geq 0\}$ under the inverse mapping action T^{-1}).

Statement 3. *All points lying between the set γ_{n+1} and the set γ_n, after n impacts, come out of the small neighborhood Ω of the point M^*.*

The set γ^* is limiting for infinite iterations of the mapping T^{-1}.

Here $\gamma_{n+1} = T^{-n}(\gamma_1)$ - consecutive images of the set γ_1 under the mapping action T^{-1}.

The type of sets γ_1, γ_2,... is specified in [Gorbikov, 1998] and is clear from Figure 4.

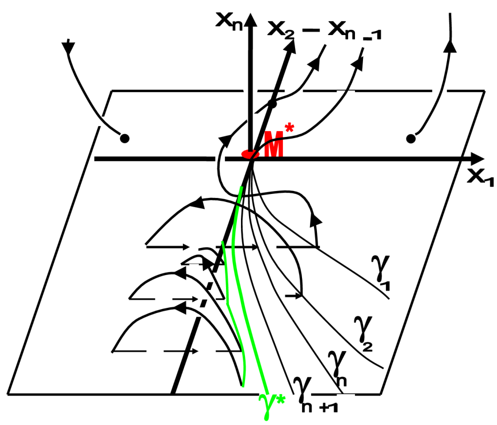

Figure 4. The behavior of phase trajectories in the vicinity of a local qualitative singularity of the fifth type. The phase trajectories of auxiliary sliding movements are highlighted in green

At the points of the set Ω that lie on the manifold $x_n = 0$ between points $x_n = 0, x_1 = 0, x_2 < 0$ and the points γ^* (including at points of the set γ^*) operate the auxiliary equation of motion. Their appearance changes in comparison with (5), and is given in [Gorbikov, 1998].

As proven (but not published): the fifth type of local qualitative singularities (for the proof of topological equivalence) **requires only the representation of infinite-impact motives by means of these smooth differential equations and statements 2 and 3.**

7 Conclusion

Next, we will formulate a number of unsolved problems that are related to the tasks studied here.

I. Study of more degenerate cases than those described here.

II. Classification and study of local features of dynamical systems with shock interactions, which are described by mathematical models other than those discussed here.

III. Classification and study of bifurcations of periodic motions, dynamical systems with shock interactions leading to the appearance of chaotic motions.

IV. Study of specific, frequently encountered dynamic systems with shock interactions, by identifying the main steady-state movements; indications of those of them that have significant areas of existence in the parameter space, as well as those on which the optimal operating modes of the original technical devices are achieved.

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