# DELAYED FEEDBACK COHERENCE RESONANCE CHIMERAS

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**Abstract**
Coherence resonance chimeras are partial synchronization patterns made up of spatially separated domains of coherent and incoherent spiking. They have been recently discovered for nonlocally coupled networks of excitable elements in the presence of random fluctuations and demonstrate constructive role of noise for chimera states. These patterns are different from classical chimera states occurring in deterministic oscillatory systems and provide a link between two phenomena: coherence resonance and chimera states. A distinctive feature of this chimera type is its alternating behavior, i.e., periodic switching of the location of coherent and incoherent domains. Applying time-delayed feedback we demonstrate how to control coherence resonance chimeras by adjusting delay time. In particular, we show that the feedback increases the parameter intervals of existence of chimera states and has a significant impact on their alternating dynamics leading to the appearance of novel patterns, which we call *period-two coherence resonance chimera*.

**Key words**
Synchronization, Networks, Coherence resonance, Chimera state, Time-delayed feedback

## 1 Introduction

In real-world systems the occurrence of random fluctuations, i.e., noise, is unavoidable. It is known that noise can play a constructive role and give rise to new dynamic behavior, e.g., coherence resonance [Hu et al., 1993; Pikovsky and Kurths, 1997; Zakharova et al., 2010; Zakharova et al., 2013]. This counter-intuitive phenomenon describes a non-monotonic behavior of the regularity of noise-induced oscillations in the excitable regime. It has been previously shown that coherence resonance can be modulated by applying time-delayed feedback. A novel type of coherence resonance, *coherence resonance chimera state*, has been found recently in networks of nonlocally coupled excitable elements [Semenova et al., 2016; Zakharova et al., 2017]. This regime provides a link between two phenomena: coherence resonance, and chimera state [Panaggio and Abrams, 2015; Schöll, 2016], i.e., coexistence of spatially coherent and incoherent domains in a network of identical elements. These states are distinct from classical chimeras, which occur in deterministic oscillatory elements [Kuramoto and Battogtokh, 2002; Abrams and Strogatz, 2004]. It is well-known that in the presence of time delay simple dynamical systems can exhibit complex behavior, such as delay-induced bifurcations, stabilization of unstable periodic orbits or stationary states, to name just a few examples. Chimera states have also been found in delayed-feedback systems. In particular, internal delayed feedback has been shown to induce chimeras in systems of globally coupled phase oscillators [Yeldesbay et al., 2014] and laser networks [Böhm et al., 2015]. Chimera states in the presence of both delayed feedback and noise have been investigated in [Semenov et al., 2016].

In the present work we study the role of time-delayed feedback for coherence resonance chimeras. A distinctive feature of this noise-induced patterns in that they occur in a certain restricted interval of systems parameters. The question we address here is whether this intervals can be increased by introducing time-delayed feedback. While exploring the impact of time delay we uncover the mechanisms to control coherence resonance chimeras by time-delayed feedback. Our results show that applying feedback promotes the occurrence of coherence resonance chimeras and induces novel chimera patterns.

## 2 Model

We consider a ring of $N$ identical nonlocally coupled FitzHugh-Nagumo (FHN) systems with time-delayed feedback in the presence of Gaussian white noise:

$$\begin{aligned} \varepsilon\frac{du_i}{dt} &= u_i - \frac{u_i^3}{3} - v_i + \frac{\sigma}{2R}\sum_{j=i-R}^{i+R}[b_{uu}(u_j - u_i)+ \\ &+b_{uv}(v_j - v_i)] + \gamma(u_i(t) - u_i(t-\tau)), \\ \frac{dv_i}{dt} &= u_i + a + \frac{\sigma}{2R}\sum_{j=i-R}^{i+R}[b_{vu}(u_j - u_i)+ \\ &+b_{vv}(v_j - v_i)] + \sqrt{2D}\xi_i(t), \end{aligned} \tag{1}$$

where $u_i$ and $v_i$ are the activator and inhibitor variables, respectively, $i = 1, ..., N$ and all indices are modulo $N$, $\sigma$ is the coupling strength, $R$ is the number of nearest neighbours in each direction on a ring. We also introduce the coupling range which is the normalized number of nearest neighbours $r = R/N$, where $N$ is the total number of elements in the network. Further, $\xi_i(t) \in \mathbb{R}$ is Gaussian white noise, i.e., $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$, $\forall i, j$, and $D$ is the noise intensity. The feedback term is characterized by time delay $\tau$ and strength $\gamma$. A small parameter responsible for the time scale separation of the fast activator and slow inhibitor is given by $\varepsilon > 0$, and $a_i$ defines the excitability threshold. For an individual FHN element it determines whether the system is excitable ($|a_i| > 1$), or oscillatory ($|a_i| < 1$). In the present study we assume that all elements are in the excitable regime close to the threshold ($a_i \equiv a = 1.001$, except Figs.6 and 7). Eq. (1) contains not only direct, but also cross couplings between activator ($u$) and inhibitor ($v$) variables, which is modeled by a rotational coupling matrix:

$$B = \begin{pmatrix} b_{\rm uu} & b_{\rm uv} \\ b_{\rm vu} & b_{\rm vv} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{2}$$

where $\phi \in [-\pi; \pi)$. Here we fix the parameter $\phi = \pi/2 - 0.1$. In the absence of time delay $\tau = 0$ chimera states have been found for this value of $\phi$ in both the deterministic oscillatory [Omelchenko et al., 2013] and the noisy excitable regime [Semenova et al., 2016; Zakharova et al., 2017]. In the presence of Gaussian white noise a special type of chimera state called *coherence resonance chimera* appears in a ring of $N$ nonlocally coupled excitable FHN systems (Fig. 1).

## 3 Coherence resonance chimeras in the presence of time-delayed feedback

In the present work to control coherence resonance chimeras we introduce time-delayed feedback to the first equation in system Eqs. (1). For that purpose we fix all the parameters of the system in the regime of coherence resonance chimeras and vary those characterizing the feedback term: $\gamma$ and $\tau$. For $\gamma = 0$ the system Eqs. (1) demonstrates coherence-resonance chimeras with the period $T \approx 4.76$. This regime can also be observed in the presence of time-delayed feedback for $\gamma = 0.2$, $\tau = 1.0$ and is shown as a space-time plot color-coded by the variable $u_i$ in Fig. 1(a,b). One can clearly distinguish the regions of coherent and incoherent spiking. To characterize spatial coherence and in-

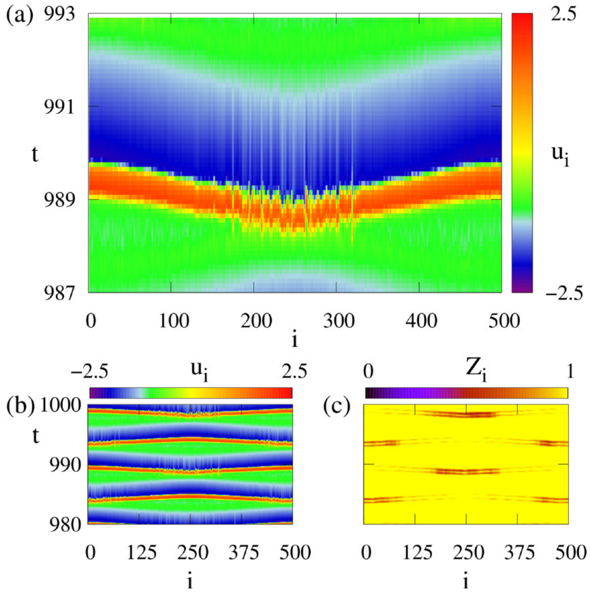


Figure 1. (a),(b) Space-time plots and (c) local order parameter for the coherence-resonance chimera. Initial conditions: randomly distributed on the circle $u^2 + v^2 = 4$. Parameters: $N = 500$, $\varepsilon = 0.05$, $\phi = \pi/2 - 0.1$, $a = 1.001$, $\sigma = 0.4$, $r = 0.2$, $D = 0.0002$, $\gamma = 0.2$, $\tau = 1.0$.

coherence of chimera states one can use the local order parameter:

$$Z_k = \left|\frac{1}{2\delta_Z}\sum_{|j-k|\le\delta_Z} e^{i\Theta_j}\right|, \quad k = 1, \dots N \tag{3}$$

where the geometric phase of the $j$-th element is defined by $\Theta_j = arctan(v_j/u_j)$ and $Z_k = 1$ and $Z_k < 1$ indicate coherence and incoherence, respectively. Figure 1(c) represents a space-time plot color-coded by $Z_i$ and illustrates coexistence of coherent and incoherent domains with the latter characterized by values of $Z_i$ noticeably below unity (dark regions). One of the main features of these noise-induced chimera states is their alternating behavior which is absent in the oscillatory regime without noise. In more detail, the incoherent domain of the chimera pattern switches periodically its position on the ring, although its width remains fixed (Fig. 1(b,c)). This property has been previously described in [Semenova et al., 2016] and the explanation based on the time evolution of the coupling term has been provided in [Zakharova et al., 2017]. Taking into account that the system Eqs. (1) involves both direct and cross couplings between activator $u$ and inhibitor $v$ variables, in total we have four coupling terms. It turns out that the coupling terms form patterns shown as space-time plots in Fig. 2(a)–(d). The crucial point

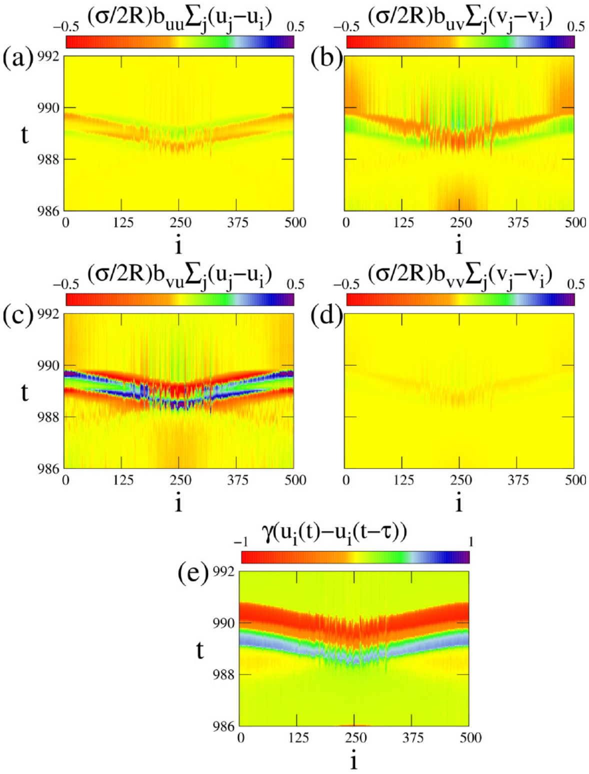


Figure 2. Space-time plots of coupling terms for $u$ and $v$ variables in the coherence-resonance chimera regime: (a) direct coupling for the $u$ variable, (b) cross coupling for the $u$ variable, (c) cross coupling for the $v$ variable, (d) direct coupling for the $v$ variable. (e) Space-time plot of the delay term. Parameters as in Fig. 1.

is that the coupling acts as an additional term and shifts the nullclines of every individual element of the network. The coupling term with the strongest impact corresponds to cross coupling for the $v$ variable (Fig. 2(c)). It means that the coupling significantly influences the $\dot{u} = 0$ nullcline and shifts the threshold parameter $a$ which is responsible for the excitation. As a result for a certain group of nodes the threshold becomes lower due to coupling and the probability of being excited by noise increases. Therefore, the elements of this group are the first to start the large excursion in the phase space and experience random spiking. The elements constituting the rest of the network spike coherently since they are pulled by already excited nodes and are, therefore, excited by coupling and not by noise. This scenario can also be obtained for the system Eq. (1) in the presence of time-delayed feedback (Fig. 2). Due to feedback an additional term appears in Eq. (1) and should be taken into account. Its evolution in time for all nodes of the network is shown in Fig. 2(e). The color-code bar clearly indicates that the values of the feedback term are larger than those of the coupling terms. However, for the chosen value of delay time $\tau = 1.0$ the feedback does not have any essential impact on the behavior of coherence resonance chimeras since it is less than the intrinsic period of oscillations $T = 4.76$ (Figs 1, 2).

Since our main goal is to study the impact of time-delayed feedback we now choose the parameters of the system in the regime of coherence resonance chimera and vary only the feedback parameters $\gamma$ and $\tau$. For the fixed feedback strength $\gamma = 0.4$ we observe the change of dynamic regimes by tuning the delay time $\tau$. In more detail, for $\tau = 3.6$ all the nodes of the network spike coherently, i.e, in-phase synchronization occurs (Fig. 3,a). The feedback with $\tau = 2.2$ shifts the system into the regime which is incoherent in space and periodic in time: all the nodes demonstrate spiking behavior, but the spiking events of the neighboring nodes are not correlated (Fig. 3,b).

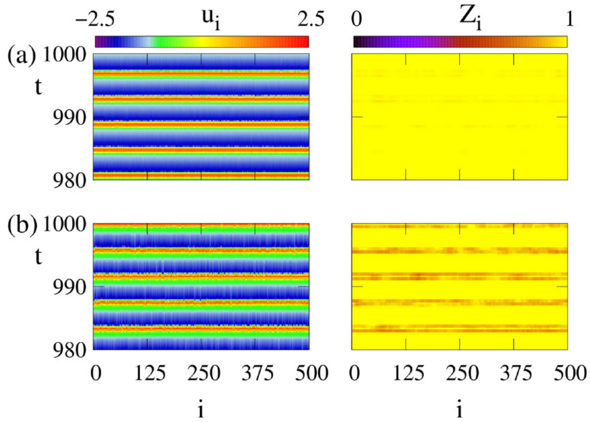


Figure 3. Space-time plots for the variable $u_i$ (left panels) and local order parameter $Z_i$ in the regime of (a) spatial synchronization for $\gamma = 0.4$, $\tau = 3.6$ and (b) spatial incoherence for $\gamma = 0.4$, $\tau = 2.2$. Other parameters as in Fig. 1.

## 4 Noise intensity interval of existence

Without feedback as previously reported, coherence resonance chimeras are observed for a certain restricted interval of noise intensity $0.000062 \leq D \leq 0.000325$ for the following parameters of the system: $N = 500$, $\varepsilon = 0.05$, $a = 1.001$, $\phi = \pi/2 - 0.1$, $r = 0.2$, $\sigma = 0.4$ (this set of parameters is fixed throughout the manuscript). Time-delayed feedback modifies this interval. To illustrate this effect we consider two cases: $\gamma < 0.5$ and $\gamma > 0.5$ which allows for a better understanding of the impact of feedback strength on this interval. Also for the two values of parameter $\gamma$ we choose different delay times $\tau$. Time-delayed feedback slightly changes the range of noise intensity values where chimera states occur in the system Eqs.(1) for both considered values of feedback strength: $\gamma = 0.2$ (Fig. 4) and $\gamma = 0.6$ (Fig. 5).

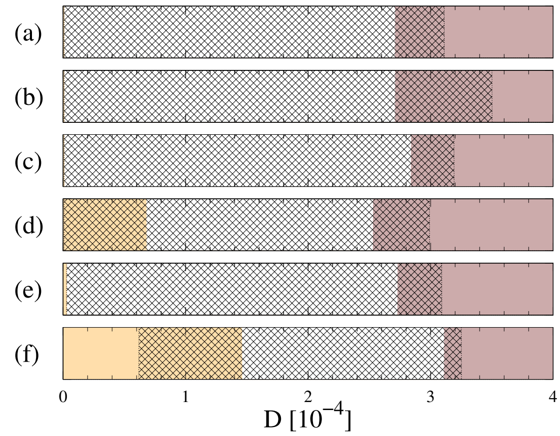


Figure 4. Dynamic regimes depending on the noise intensity $D$ for feedback strength $\gamma = 0.2$ and different values of delay time: (a) $\tau = 9.52$, (b) $\tau = 6.0$, (c) $\tau = 4.76$, (d) $\tau = 1.8$, (e) $\tau = 0.8$, (f) $\tau = 0$. Dynamic regimes: steady state (yellow/light grey); spatially incoherent spiking (pink/dark grey); coherence resonance chimeras (hatching). Other parameters as in Fig. 1.

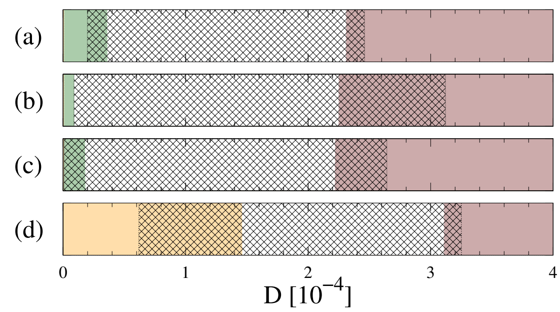


Figure 5. Dynamic regimes depending on the noise intensity $D$ for feedback strength $\gamma = 0.6$ and different values of delay time: (a) $\tau = 9.52$, (b) $\tau = 4.76$, (c) $\tau = 0.8$, (d) $\tau = 0$. Dynamic regimes: steady state (yellow/light grey); spatially incoherent spiking (pink/dark-grey); synchronization (green/grey); coherence resonance chimeras (hatching). Other parameters as in Fig. 1.

For rather weak feedback strength $\gamma = 0.2$ the interval of existence of chimera patterns is enlarged for all the considered delay times. Furthermore, due to feedback, chimera states appear for vanishing noise intensity (Fig. 4(a)–(d)). Therefore, time-delayed feedback promotes coherence resonance chimeras not only increasing the range of noise values where they exist, but also inducing these patterns in the case of almost no noise. The largest range of $D$ corresponds to $\tau = 6.0$ (Fig. 4,b). Large feedback strength $\gamma = 0.6$ can also shift the left boundary of the chimera interval to lower (Fig. 5a,b) and even zero (Fig. 5c) noise values. The right boundary of the interval strongly depends on $\tau$ and shifts into the direction of lower noise intensities (Fig. 5a,b,c). The largest detected interval for $\gamma = 0.6$ corresponds to $\tau = 4.76 \approx T$ (Fig. 4,b) and for $\tau = 9.52 \approx 2T$ we even observe the shrinking of the interval (Fig. 5a). If we compare the interval of chimera existence without time-delayed feedback $0.000062 \leq D \leq 0.000325$ (Fig. 4,f and Fig. 5,d) with the interval the most enlarged by the feedback $0.000001 \leq D \leq 0.00035$ it turns out that we achieve 33 per cent improvement rate.

## 5 Threshold parameter interval of existence

It has been previously shown that coherence resonance chimera can be obtained only in a small interval of $a$ values ($0.995 \leq a \leq 1.004$). To analyze the impact of time-delayed feedback we again consider two cases: $\gamma = 0.2$ and $\gamma = 0.6$ and different values of delay time. Figure 6 corresponds to the case of small feedback strength $\gamma = 0.2$ and Figure 7 illustrates the results for the case of larger delay strength $\gamma = 0.6$.

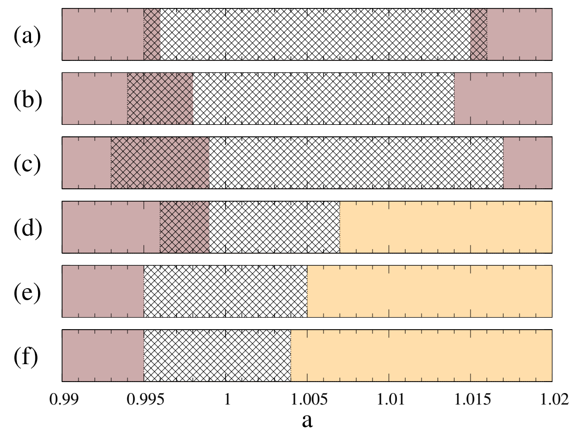


Figure 6. Dynamic regimes depending on the threshold parameter $a$ for feedback strength $\gamma = 0.2$ and different values of delay time: (a) $\tau = 9.52$, (b) $\tau = 6.0$, (c) $\tau = 4.76$, (d) $\tau = 1.8$, (e) $\tau = 0.8$, (f) $\tau = 0$. Dynamic regimes: steady state (yellow/light grey); spatially incoherent spiking (pink/dark grey); coherence resonance chimeras (hatching). Other parameters as in Fig. 1.

For the two considered values of $\gamma$ time-delayed feedback significantly changes the range of threshold parameter $a$ where coherence resonance chimera exists. Moreover, in both cases this interval is increased the most when delay time is equal to the intrinsic period of the system $\tau = 4.76 \approx T$ (Fig. 6c and Fig. 7b). However, smaller feedback strength allows for a stronger enlargement of the interval: for $\gamma = 0.2$ and $\tau = 4.76$ it is $0.993 \leq a \leq 1.017$, and is more than doubled compared to the case without feedback $0.995 \leq a \leq 1.004$ (Fig. 6c).

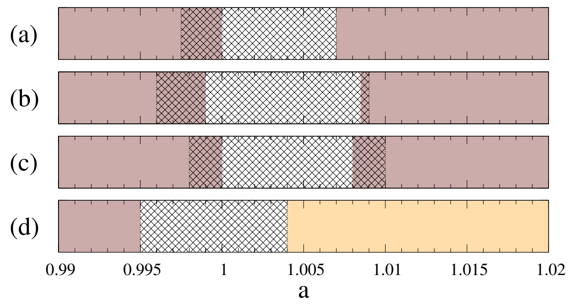


Figure 7. Dynamic regimes depending on the threshold parameter $a$ for feedback strength $\gamma = 0.6$ and different values of delay time: (a) $\tau = 9.52$, (b) $\tau = 4.76$, (c) $\tau = 0.8$, (d) $\tau = 0$. Dynamic regimes: steady state (yellow/light grey); spatially incoherent spiking (pink/dark grey); coherence resonance chimeras (hatching). Other parameters as in Fig. 1.

Interestingly, for $\tau \approx T$ and $a > 1.01$ we find a novel chimera regime which is induced by time-delayed feedback and has not been previously shown for the system Eq.(1) without delay. The space-time plot for the variable $u_i$ and the local order parameter indicate the coexistence in space of coherent and incoherent spiking as well as alternating behavior, typical features of coherence resonance chimera (Fig. 8). Furthermore, the alternation takes place periodically, and the incoherent domain switches its position on the ring. However, the switching events occur not for every spiking cycle as in the coherence resonance chimera state (Fig. 1b,c), but for every second spiking event (Fig. 8a,b). Due to this distinguishing feature we call the pattern *period-two coherence resonance chimera*.

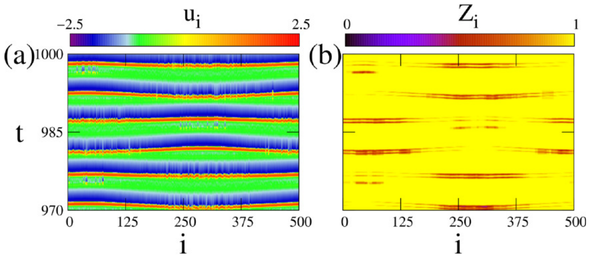


Figure 8. Space-time plot for the variable $u_i$ (a) and local order parameter $Z_i$ (b) in the regime of period-two coherence-resonance chimera. Initial conditions: randomly distributed on the circle $u^2 + v^2 = 4$. Parameters: $N = 500$, $\varepsilon = 0.05$, $a = 1.012$, $\sigma = 0.4$, $r = 0.2$, $D = 0.0002$, $\gamma = 0.2$, $\tau = 4.76$.

## 6 Conclusion

In conclusion, we demonstrate that time-delayed feedback in a network of nonlocally coupled noisy FitzHugh-Nagumo elements in the excitable regime promotes coherence resonance chimeras. It allows one to control the range of parameter values where they exist, and in most of the cases this range increases. Moreover, the feedback induces coherence resonance chimeras for vanishing noise intensities. Additionally, we show that the threshold parameter interval of chimera existence can be more than doubled by applying feedback with delay time close to the intrinsic period of the system. Compared to the case without feedback this provides an essential improvement relevant for the experimental realization of coherence resonance chimeras. Furthermore, when the feedback delay coincides with the intrinsic period of the network we find a novel feedback-induced regime which we call period-two coherence resonance chimera. Since the dynamics of every individual network element in our study is given by the FitzHugh-Nagumo system, a paradigmatic model for neurons in the excitable regime, we expect wide-range applications of our results to neural networks.


## Acknowledgments

This work was supported by Deutsche Forschungsgemeinschaft in the framework of Collaborative Research Center SFB 910.


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