

# ATTRACTOR LOCALIZATION OF DYNAMICAL “SOLAR WIND-MAGNETOSPHERE-IONOSPHERE” MODEL

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## Abstract

The problem of localization of attractor of “solar wind-magnetosphere-ionosphere” (WINDMI) three-dimensional model has stimulated the development of method of conical nets. In the paper the development of this method is carried out and analytical localization of the attractor of this model are obtained.

## Key words

frequency method of positively invariant cone grids, attractor localization, WINDMI model, solar wind-magnetosphere-ionosphere.

## 1 Introduction

In the present paper we consider the localization problem of attractor of 3-dimensional simplified model of 6-dimensional system, describing the dynamics of energy flow through “solar wind-magnetosphere-ionosphere” (WINDMI) system. The problem of attractor localization has stimulated the development of method of positively invariant cone grids (which is often used for study of control systems) [Leonov et al.(1996<sup>1</sup>); Yakubovich et al.(2004); Leonov et al.(1996<sup>2</sup>); Leonov(2006); Leonov et al.(2009)] for its study. Further development of this method is carried out and analytical estimates of the attractor of the model are obtained.

## 2 Development of the method of positively invariant cone grids

Consider a system

$$\frac{dx}{dt} = Px + q\varphi(r^*x), \quad x \in \mathbb{R}^n. \quad (1)$$

Here  $P$  is a constant degenerate  $(n \times n)$ -matrix,  $q$  and  $r$  are  $n$ -dimensional vectors,  $*$  is an operation of transposition, and  $\varphi(\sigma)$  is a differentiable scalar function satisfying the following sector conditions

$$\varphi(\sigma) < \mu(\sigma - \alpha), \quad \forall \sigma \geq \alpha, \quad (2)$$

$$\mu(\sigma - \beta) < \varphi(\sigma), \quad \forall \sigma \leq \beta, \quad (3)$$

where  $\mu$  is a certain positive number,  $\alpha < \beta$ .

Let the pair  $(P, q)$  be totally controllable, the pair  $(P, r)$  be totally observable, and system (1) have a unique equilibrium.

**Theorem.** *Suppose,  $r^*q \leq 0$ , there exists a number  $\lambda > 0$  such that the matrix  $P + \lambda T$  has  $(n - 1)$  eigenvalues with negative real parts, and the following inequality*

$$\operatorname{Re} W(i\omega - \lambda) + \mu|W(i\omega - \lambda)|^2 \leq 0, \quad \forall \omega \in \mathbb{R}^1 \quad (4)$$

*is satisfied. Then for any solution  $x(t)$  of system (1) there exists a number  $T$  such that*

$$r^*x(t) \in (\alpha, \beta), \quad \forall t > T. \quad (5)$$

Here  $W(p) = r^*(P - pI)^{-1}q$  is a transfer function of system (1),  $I$  is a unit  $(n \times n)$ -matrix.

We give a scheme of the proof of Theorem 1. Condition (4) implies the existence of symmetric  $n \times n$ -matrix  $H$  such that for it there are valid the following conditions (the detailed proof of this fact can be found in [Leonov et al.(1996<sup>1</sup>)]):

1) the matrix  $H$  has one negative and  $(n - 1)$  positive eigenvalues,

2) for all  $z \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}^1$  the inequality

$$2z^*H[(P + \lambda I)z + q\xi] + r^*z(r^*z - \mu^{-1}\xi) \leq 0 \quad (6)$$

is satisfied. Note that relation (6) yields the relation

$$2Hq = \mu^{-1}r.$$

Then from Theorem 1 it follows that  $r^*H^{-1}r = 2\mu r^*q \leq 0$  and, therefore, [Leonov et al.(1996<sup>1</sup>)] we have

$$z^*Hz \geq 0, \quad \forall z \in \{z^*r = 0\}. \quad (7)$$

Let be  $d \in \mathbb{R}^n$  such that  $d \neq 0$  and  $Pd = 0$ ,  $r^*d = 1$ . Then relation (6) implies that  $d^*Hd < 0$ .

Consider now the Lyapunov-type function

$$\begin{aligned} V_1(x) &= V(x - \alpha d) = (x - \alpha d)^*H(x - \alpha d), \\ V_2(x) &= V(x - \beta d) = (x - \beta d)^*H(x - \beta d). \end{aligned}$$

From relations (2), (3), (6) it follows that

$$\begin{aligned} \dot{V}_1(x(t)) + 2\lambda V_1(x(t)) &< 0 \quad \text{for } r^*x(t) > \alpha, \\ \dot{V}_2(x(t)) + 2\lambda V_2(x(t)) &< 0 \quad \text{for } r^*x(t) < \beta. \end{aligned}$$

In this case relation (7) implies positive invariance of the sets [Leonov et al.(1996<sup>1</sup>)]

$$\begin{aligned} \Omega(\alpha) &= \{(x - \alpha d)^*H(x - \alpha d) < 0, \quad r^*x \geq \alpha\}, \\ \Omega(\beta) &= \{(x - \beta d)^*H(x - \beta d) < 0, \quad r^*x \leq \beta\}. \end{aligned}$$

Then it is easily seen that the closures  $\bar{\Omega}(\alpha)$ ,  $\bar{\Omega}(\beta)$  are also positively invariant. In this case the boundaries  $\partial\Omega(\alpha)$  and  $\partial\Omega(\beta)$  do not involve whole trajectories and they are almost everywhere transverse to vector field of system (1). In the phase space of system (1) these boundaries make up a continuum set of surfaces (conical net), which is shown in Fig. 1.

In increasing time  $t$  the structure constructed "huddles" any solution in the set  $\Omega(\alpha) \cap \Omega(\beta)$ . The latter proves the assertion of theorem.

Note that the estimate obtained cannot be improved in the considered class of nonlinearities since if for all  $\sigma \in (\alpha, \beta)$   $\varphi(\sigma) = 0$ , then for  $\nu \in (\alpha, \beta)$   $x = \nu d$  is a stationary solution of system.

### 3 WINDMI system

Consider a system

$$\begin{aligned} \ddot{x} + b\dot{x} + c_1\dot{x} + \varphi(x) &= 0, \\ \varphi(x) &= (c_2 + c_3 \tanh(x)), \end{aligned} \quad (8)$$

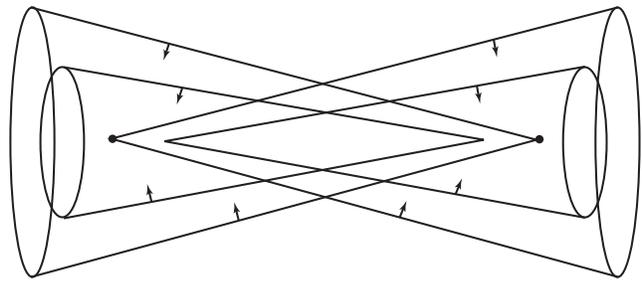


Figure 1. Cone grids

where

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

System (8) is 3-dimensional simplified model of 6-dimensional system (obtained in [Horton et al.(2001)]) used for analysis of geomagnetic storms and substorms [Spencer et al.(2006)] and modeling the energy flow through the solar wind-magnetosphere-ionosphere system.

System (8) is called a simplified WINDMI model. As in [Horton et al.(2001)], the parameters of (8), by assumption, are the following

$$b > 0, c_1 > 0, c_3 > c_2 > 0.$$

These parameters are given in the dimensionless form and computed by the formulas from [Horton et al.(2001)]. Some dynamical features of model (8) (such as the graphs of the largest Lyapunov exponent; Lyapunov dimension versus the solar wind dynamo voltage; the bifurcation diagram) are shown in [Horton et al.(2001)].

Our aim is to study a location of invariant sets of simplified WINDMI model.

Let us apply the theory, developed above, to investigation of equation (8). In this case the maximal coefficient  $\mu$  can be computed [Yakubovich et al.(2004)] by the formula

$$\mu = \begin{cases} \frac{b}{3} \left( c_1 - \frac{2}{9}b^2 \right), & b^2 \leq 3c_1 \\ \frac{b}{3} \left( c_1 - \frac{2}{9}b^2 \right) + 2 \left( \frac{b^2}{9} - \frac{c_1}{3} \right)^{3/2}, & b^2 \geq 3c_1 \end{cases}$$

Determine a point  $x_0 > 0$  such that  $\varphi'(x_0) = \mu$ ,

$$x_0 = \operatorname{argcosh} \sqrt{\frac{c_3}{\mu}} \quad \text{for } \frac{c_3}{\mu} > 1.$$

Taking into account the relation  $\varphi'(x_0) = \varphi'(-x_0) = \mu$ , we obtain restrictions (2) and (3) on the nonlinearity

$\varphi(x)$  (Fig. 2). Denote

$$\alpha_0 = -\frac{\varphi(x_0)}{\mu} + x_0, \quad -\frac{\varphi(-x_0)}{\mu} - x_0 = \beta_0.$$

Then by (2), (3), and (5) we obtain

$$\alpha_0 \leq \liminf_{t \rightarrow +\infty} x(t), \quad \limsup_{t \rightarrow +\infty} x(t) \leq \beta_0. \quad (9)$$

For the estimation of  $\dot{x}$  and  $\ddot{x}$  we remark that by the conditions of positiveness of coefficients the characteristic polynomial of linear part of equation (8) has roots to the left of imaginary axis and the nonlinearity  $\varphi(x)$  is bounded:

$$|\varphi(x)| \leq c_2 + c_3.$$

In this case, using Cauchy formula, we can obtain [Cesari(1959); Leonov(2001); Leonov & Kuznetsov(2007)] the following estimates for  $|\dot{x}(t)|$ :

$$y_{att} = \begin{cases} \limsup_{t \rightarrow +\infty} |\dot{x}(t)| \leq \frac{1}{c_1}(c_2 + c_3), & b^2 \geq 4c_1, \\ \limsup_{t \rightarrow +\infty} |\dot{x}(t)| \leq \frac{2}{b\sqrt{c_1 - \frac{b^2}{4}}}(c_2 + c_3), & b^2 < 4c_1, \end{cases}$$

and for  $|\ddot{x}(t)|$ :

$$\limsup_{t \rightarrow +\infty} |\ddot{x}(t)| \leq \frac{(c_1 y_{att} + c_2 + c_3)}{b}.$$

These estimates together with estimate (9) for  $|x(t)|$  localize an attractor of equation (8).

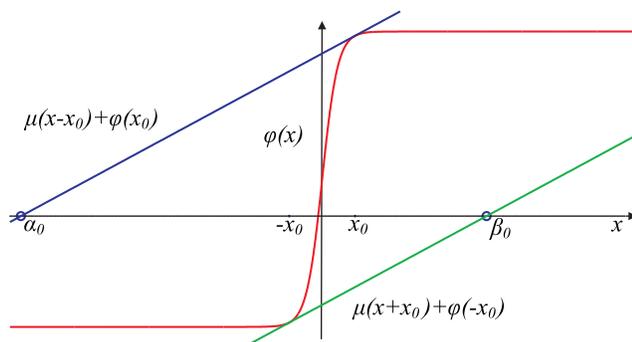


Figure 2. Estimation of nonlinearity

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