TRANSFORMATION OF EQUATIONS OF COUPLED ROTATORS TO THE STANDARD FORM AND STUDY OF THEIR DYNAMICAL PROPERTIES USING THE METHOD OF AVERAGING

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Abstract

An effective procedure to study systems of coupled rotators and oscillators that govern dynamics of different mechanical systems, electric-mechanical systems, superconductive systems etc. is discussed here. The procedure is based on special transformations of systems of coupled rotators to the standard form suitable for application of the method of averaging. Averaging of obtained systems leads to the equations in a form that allows for obtaining the full information about dynamics of the systems. In particular, the full knowledge about periodic, quasi-periodic and chaotic oscillatory regimes can be obtained. This information is used to obtain torque-speed curves. Proposed procedure is thoroughly illustrated by different examples. This technique can be applied for systems with any number of degrees-of-freedom.

Key words

Method of averaging, phase space, Lorenz attractor, torque-speed curve

1 Introduction

Since the time of van der Pol [1927], the method of averaging remains one of the main analytical methods used to study dynamical systems. In earlier studies, this method was used for approximate engineering calculations, but after the studies of Krylov and Bogolyubov [1947], Hale [1963], Mitropolsky and Lykova, [1973], Neimark [1972], and others, this method became one of the main methods in the qualitative theory of dynamical systems. This method provides initial information for qualitative and quantitative studies of system dynamics. This is important because with knowledge of the initial information of the structure of the phase space of a dynamical system one can proceed directly with a numerical study.

A significant contribution to the study of the dynamics of vibrational systems has been made by the school of I. I. Blekhman [1965, 1988, 1994].

The effectiveness of the method of averaging depends on the simplicity of the systems obtained. This is defined by the form of the transformation of the initial systems to a standard form (either in terms of Krylov-Bogolyubov or systems with fast-spinning phases), i.e., by a choice of the form of changes of variables. For systems of coupled slightly nonlinear oscillators, such changes are known. These are van der Pol changes and changes of the “amplitude-phase” form. With coupled rotators, it is not possible because of the cylindrical phase space of these systems.

2 The model

Here, we propose algorithms to transform systems of coupled rotators to the standard form [Verichev, 1986] and illustrate it by real examples of the dynamical systems. Consider dynamical systems of the form

$$\begin{aligned} & I\ddot{\varphi}_i+\lambda_i\left(1+f_{1i}\left(\varphi_i\right)\right)\dot{\varphi}_i+f_{2i}\left(\varphi_i\right) \\ & \quad =\gamma_i+F_i\left(\varphi_j,\dot{\varphi}_j,\psi,\boldsymbol{x},\dot{\boldsymbol{x}}\right), \\ & \dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\mu\boldsymbol{X}\left(\boldsymbol{x},\varphi,\psi\right), \\ & \dot{\psi}=\omega_0 . \end{aligned} \qquad (1)$$

Here: $i=\overline{1,n}$; j is any number from the interval $[1,n]$ for a given i; $\varphi \in S$; $\psi \in S$; $\boldsymbol{x} \in R^m$; λ_i and γ_i are the constant parameters; $I^{-1}=\mu$ is a small parameter; $\boldsymbol{A}$ is a constant (stable) Gurvitz matrix of dimension $(m\times m)$; and F_i and $\boldsymbol{X}$ are the coupling functions. All functions are periodic by phase variables. System (1) is defined in toroidal phase space $G(\varphi,\dot{\varphi},\psi,\boldsymbol{x})=T^{n+1}\times R^{n+m}$.

2.1 Example 1

The most simple dynamical system of the form (1) is a non-autonomous rotator governed by equations of the form

$$\begin{gathered} I\ddot{\varphi}+\dot{\varphi}+\sin\varphi=\gamma+A\sin\psi, \\ \dot{\psi}=\omega_0. \end{gathered} \tag{2}$$

Using this well-known system, let us demonstrate the algorithm of transformation to the standard form with a fast-spinning phase and study it using the method of averaging.

Consider the quasi-linear case, $I^{-1}=\mu \ll 1$. We study the dynamics of system (2) only in the zone of the main resonance.

Our goal is to transform system (2) to the equivalent system of the form

$$\begin{aligned} &\dot{x}=\mu X(x,\varphi,\psi), \\ &\dot{\varphi}=\omega_0+\mu\Phi(x,\varphi,\psi), \\ &\dot{\psi}=\omega_0. \end{aligned} \tag{3}$$

Because $\mu \ll 1$, system (3) has a standard form with fast-spinning phases φ and ψ. The objective is to determine functions $\Phi(x,\varphi,\psi)$ and $X(x,\varphi,\psi)$.

Substituting the second equation of system (3) into system (2), we obtain

$$\begin{aligned} &\frac{\partial\Phi}{\partial\varphi}(\omega_0+\mu\Phi)+\frac{\partial\Phi}{\partial\psi}\omega_0+\frac{\partial\Phi}{\partial x}\dot{x} \\ &+\omega_0+\mu\Phi+\sin\varphi=\gamma+A\sin\psi. \end{aligned} \tag{4}$$

Demanding functions Φ to be bounded by and ψ, and using the first equation of system (3), we obtain that $\gamma-\omega_0=\mu\Delta$ (zone of main resonance) and obtain an equation that determines function $\Phi(x,\varphi,\psi)$:

$$\frac{\partial\Phi}{\partial\varphi}\omega_0+\frac{\partial\Phi}{\partial\psi}\omega_0=-\sin\varphi+A\sin\psi. \tag{5}$$

This equation has the solution

$$\Phi=\frac{1}{\omega_0}\cos\varphi+\frac{A}{\omega_0}\cos\psi+x. \tag{6}$$

Remaining terms in equation (4) determine the function $X(x,\varphi,\psi)$:

$$X(x,\varphi,\psi)=\Delta-\Phi\left(1+\frac{\partial\Phi}{\partial\varphi}\right). \tag{7}$$

Thus, the required functions are obtained. Having introduced phase mistuning $\eta=\varphi-\psi$, we reduce system (3) to the system with one fast-spinning phase:

$$\begin{aligned} &\dot{x}=\mu X(x,\varphi,\psi), \\ &\dot{\eta}=\mu\Phi(x,\varphi,\psi), \\ &\dot{\varphi}=\omega_0+\mu\Phi(x,\varphi,\psi). \end{aligned} \tag{8}$$

The theory of averaging of systems of the form (8) is known [Volosov and Morgunov, 1971, Sanders and Verhulst 1985]. Averaging this system and performing some simple transformations, we obtain the well-known equation of a pendulum:

$$\ddot{\eta}+\lambda^r\dot{\eta}+\sin\eta=\gamma^r, \tag{9}$$

where $\lambda^r=\sqrt{\dfrac{2\omega_0^2}{A}}$, $\gamma^r=\Delta\dfrac{2\omega_0^2}{A}$.

Definition

The function $\Omega=\lim\limits_{T\to\infty}\dfrac{1}{T}\int\limits_0^T\dot{\varphi}(\tau,t_0)\,d\tau$ is defined under the parameter space of system (1), and the space of its initial conditions is called the rotation characteristic of the rotor (torque-speed curve).

Evidently, any motion of the system under consideration has the following form: $\dot{\varphi}(\tau,\tau_0)=\tilde{\dot{\varphi}}(\tau,\tau_0)+\dot{\varphi}^*(\tau,\tau_0)$, where $\tilde{\dot{\varphi}}$ is a solution corresponding to the transient process of the finite solution $\dot{\varphi}^*$. For an autonomous pendulum, such solutions are equilibrium and a limit cycle. Clearly, average values of averaged processes are equal to zero; therefore,

$$\Omega=\lim_{T\to\infty}\frac{1}{T}\int_0^T\dot{\varphi}^*(\tau,\tau_0)\,d\tau . \qquad (10)$$

Thus, the problem of obtaining the different qualitative forms of torque-speed curves for different system parameters is related to a classical problem of the decomposition of the parameters' space into the domains corresponding to the qualitatively different structures of trajectories in the phase of the system. If we have known qualitative forms of phase trajectories and dependence of limit sets versus a certain parameter, it is not difficult to obtain the dependency of the torque-speed curve versus this parameter. We are interested in the dependency $\gamma=\gamma(\Omega)$. Using the aforementioned definition and method of averaging, we obtain

$$\left\langle\dot{\varphi}^*\right\rangle_t=\omega_0+\left\langle\dot{\eta}^*\right\rangle_t . \qquad (11)$$

For the parameters γ^r and λ^r from the domain for which there exists a limit cycle (synchronization regime) and invariant torus (regime of regular beating), the torque-speed curve of a non-autonomous rotator has the form shown in Figure 1.

The vertical line $\Omega=\omega_0$ on the torque-speed curve corresponds to the synchronization regime. Its length satisfies the inequality $|\gamma^r|\le 1$, or $\omega_0-\mu\frac{A}{2\omega_0^2}\le\gamma\le\omega_0+\mu\frac{A}{2\omega_0^2}$. For $|\gamma^r|=1+0$, the limit cycle disappears and the rotator jumps from the synchronization regime to the regime of quasi-periodic beating, which corresponds to the torus T^2. In this case, $\left\langle\dot{\eta}^*\right\rangle_t\ne 0$, and the value $\left\langle\dot{\eta}^*\right\rangle_t$ increases (decreases) as $\gamma^r\sim\gamma$ increases (decreases). If $\gamma^r>1$, then with the decrease of this parameter, the regime of regular beating becomes a regime of chaotic beating. The chaotic limit set of the type "torus-attractor" is related to the bifurcation of destruction of a two-dimensional invariant torus T^2 [Afraimovich and Shilnikov, 1991]. During the motion of the rotator along the torus-attractor, the value of $\left\langle\dot{\eta}^*\right\rangle_t$ crucially depends on those initial conditions at which system enters the chaotic regime. During the numerous repetitions of the experiment, transition to the regime of synchronization occurs from different branches of the torque-speed curve. In the shaded domains shown in Figure 1, there exist an infinite number of branches "growing" from the branch corresponding to stationary beating. This is called an effect of the scattering of the torque-speed curve [Belykh, Pedersen and Soerensen, 1988].

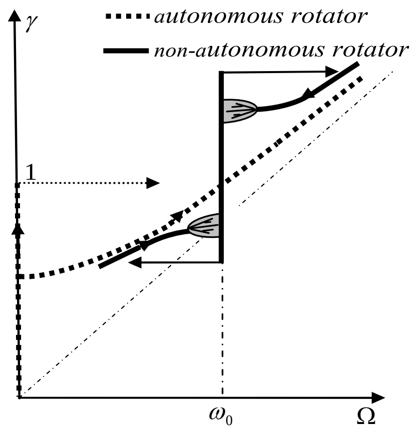


Figure 1. Torque-speed curve of non-autonomous rotator in a zone of main resonance.

2.2 Example 2

Consider a dynamical system of the form [Verichev, Verichev, Erofeyev, 2007]

$$
\begin{aligned}
&\ddot{x}+\omega_0^2 x=\frac{c_1 r_1}{m}\sin\varphi+\frac{qr}{m}\dot{\varphi}-\frac{q+k}{m}\dot{x},\\
&I\ddot{\varphi}=\widetilde{M_d}\left(\dot{\varphi}\right)-rq\left(r\dot{\varphi}-\dot{x}\right)\\
&\quad+c_1 r_1\left(x-r_1\sin\varphi\right)\cos\varphi\\
&\quad-M_0\cos\left(\varphi+\varphi_0\right)
\end{aligned}
\tag{12}
$$

as defined in the cylindrical phase space $G\left(\varphi,\dot{\varphi},x,\dot{x}\right)=T^1\times R^3$.

This system governs the dynamics of a vibrational system with energy supply of limited power of the form of an asynchronous electric motor with imbalanced rotor [Verichev, Verichev and Erofeyev, 2007; Alifov and Frolov, 1990]. We assume that $I^{-1}=\mu\ll 1$, μ is a small parameter, $\left(q+k\right)/m=2\mu h$ (dissipation in the "oscillatory" part of the system is small enough), $c_1 r_1/m=2\mu\lambda\omega_0$, and $c_1 r_1=2\mu b\omega_0$. All other combinations of the parameters are not treated as small. Also, it is assumed that the torque of an asynchronous motor represents a linear function of the form $\widetilde{M_d}\left(\dot{\varphi}\right)=M_d-\delta\dot{\varphi}$, where M_d is a constant component (for a DC motor, its value is governed by the electric current in the drive circuit), and δ is a coefficient describing the moment of resistance forces acting on the rotor [Levitsky, 2001].

With these assumptions, following the aforementioned procedure, system (12) is reduced to a Lorenz-type system [Lorenz, 1963] of the form

$$
\begin{aligned}
&\dot{x}=-\sigma\left(x-y\right)+\rho,\\
&\dot{y}=-y+Rx-xz,\\
&\dot{z}=-z+xy+\Lambda x.
\end{aligned}
\tag{13}
$$

This classical system, for some values of the parameters $\sigma=\sigma^*$, $R=R^*>R_c$, and $R_c=\left(\sigma^2+4\sigma\right)/\left(\sigma-2\right)$, has a unique attracting limit set in the phase space—strange attractor (Lorenz attractor); see Figure 2. The statements about the existence of the chaotic attractors of a certain type in the initial system (12) have been made on the basis of the existence of the corresponding attractors in the averaged system (13). To confirm that, the Poincare mapping has been plotted using the secant hyperplane $\varphi=const$ at the period 2π; i.e., $\left(\theta,\eta,\xi\right)_{\varphi=\varphi_0}\rightarrow\left(\bar{\theta},\bar{\eta},\bar{\xi}\right)_{\varphi=\varphi_0+2\pi}$. For rotational motions, this secant is global. The corresponding parameters' set of system (12)is $qr/m=1$, $qr^2=0.5$, $rq=2.11$, $c_1 r_1=0.14$, $\mu=0.1$, $\omega_0=1.05$, $\delta=0.49$, $\lambda=0.2$, $h=1$, $M_d=0.9674$, $M_0=0.671$, $r_1=0.1$, and $\varphi_0=\pi/2$. These parameters result in $\sigma=9.9$, $R=27.39$, and $\Lambda=-0.87$.

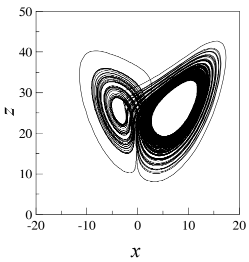
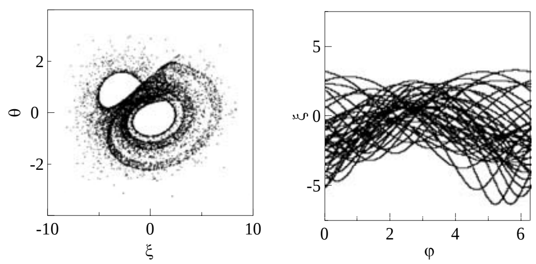


Figure 2. Lorenz attractor: $\sigma=9.7,\ R=27,\ \Lambda=-1,\ \rho=28.53$.



a) b)

Figure 3. a) Lorenz attractor in the Poincare plane; b) involute of the phase cylinder of the initial system.

Fig. 3a shows the asymmetrical Lorenz attractor in the Poincare domain. The slight difference between this attractor and that shown in Fig. 2 is caused by the change of the variables made to reduce system (12) to system (13). This change of variables provides parallel displacement and rotation of the system of coordinates. Thus, attractors depicted in Figures 3a and 2 represent projections made at different angles. To make them identical, one had to perform an extra change of the variables at the Poincare plane. However, this is not necessary because the qualitative forms of attractors are quite recognizable.

Figure 4 shows the torque curve for $R > R_c$. In this case, for any value of the parameter ρ/σ from the shaded domain, there exists a certain chaotic attractor in the phase space of system (13) that is the unique attracting limit set. In other words, in the aforementioned interval, there exist an infinite number of chaotic attractors, every one of which has individual spatio-temporal properties. For each point, bifurcations of the homoclinic trajectories and of the corresponding saddle periodic motions occur. The temporal average value $\left\langle x\left(t,t_0\right)\right\rangle_t$ for each attractor is different. Moreover, owing to the strong dependence of the trajectories on the initial conditions and the finiteness of the real averaging interval, this value will strongly depend on the initial time t_0.

With respect to a torque curve in the shaded domain, one can conclude the following: a) the torque curve in the shaded domain is irreproducible—during the quasi-stationary increase of the parameter ρ/σ (constant part of the motor torque), one obtains one curve (branch), but for an opposite change (arbitrarily small), one obtains a completely different curve; b) the torque curve in the shaded domain has an infinite number of mixed branches that start at the frequency jump points corresponding to the ends of solid bold lines. For this reason, the torque curve in this zone is not presented.

Such behavior of the torque curve is an aforementioned effect of the scattering of the torque curve of the rotator. In particular, such effect is known to take place for the synchronization of the superconductive junction by a microwave field [Mintz, 1955].

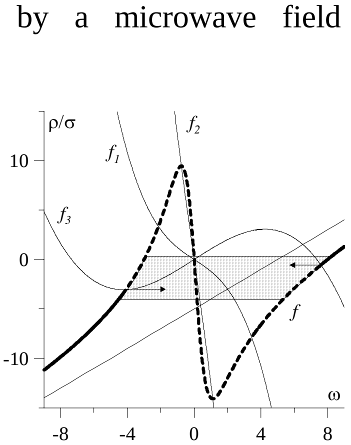


Figure 4. An example of torque-speed curve for the following parameters: $\sigma = 10$, $R = 25$, $\Lambda = -5$.

3 Conclusions

The proposed technique is quite helpful in studying systems with cylindrical and toroidal phase spaces. Other examples of study of the dynamics of coupled rotators following the proposed procedure can be found in [Belykh and Verichev, 1997, 1988a, 1988b; Belykh, Verichev and Belykh, 1997; Verichev, Verichev and Erofeyev, 2008].

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