TIME-DELAYED FEEDBACK CONTROL: QUALITATIVE PROMISE AND QUANTITATIVE CONSTRAINTS

Bernold Fiedler
Institute of Mathematics
Free University of Berlin
Germany
fiedler@math.fu-berlin.de

We explore stabilization of unstable periodic orbits $x(t) \in \Gamma$ of ordinary differential equations

$$\dot{x}(t) = f_0(x(t)). \tag{1}$$

Stabilization of Γ is attempted via time-delayed feedback systems of the general form

$$\dot{x}(t) = f(x(t), g(x(t), x(t-\tau))) \tag{2}$$

with $f(x, 0) = f_0(x)$ and $g(x, x) = 0$ for all x.

If we choose the delay $\tau = np$ to be an integer multiple $n = 1, 2, 3, \ldots$ of the minimal period p, then Γ remains a periodic orbit of (2) because the *time-delayed feedback control* g vanishes. Therefore the feedback control (2) is noninvasive on Γ. Nevertheless the linearized stability properties of Γ in the differential delay system (2) may differ markedly from (1), and in fact some unstable periodic orbits Γ of (1) may be stabilized by suitable choices of f and g. This idea of time-delayed feedback control goes back to Pyragas [Pyragas, 1992]. Applications to problems from physics, chemistry, biology, and medicine can be found in [Pyragas and Tamaševičius, 1993; Bielawski, Derozier, and Glorieux, 1994; Pierre, Bonhomme and Atipo, 1996; Hall, Christini, Tremblay, Collins, Glass and Billette, 1997; Sukow, Bleich, Gauthier and Socolar, 1997; Lüthje, Wolff and Pfister, 2001; Parmananda, Madrigal, Rivera, Nyikos, Kiss and Gáspár, 1999; Krodkiewski and Faragher, 2000; Fukuyama, Shirahama and Kawai, 2002; Loewenich, Benner and Just, 2004; Rosenblum and Pikovsky, 2004; Popovych, Hauptmann and Tass, 2005; Schikora, Hövel, Wünsche, Schöll and Henneberger, 2006; Schöll, Hizanidis, Hövel and Stegemann]; see also [Schimansky-Geier, Fiedler, Kurths and Schöll, 2007; Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007b; Fiedler, Flunkert, Georgi, Hövel and Schöll, 2008].

It has been claimed by some authors, and quoted by many more, that delay stabilization of periodic orbits is not possible when the number of real Floquet multipliers $\mu > 1$ of Γ is odd; see [Socolar, Sukow and Gauthier, 1994; Just, Bernard, Ostheimer, Reibold and Benner, 1997; Nakajima, 1997; Nakajima and Ueda, 1998; Harrington and Socolar, 2001; Pyragas, Pyragas and Benner, 2004; Pyragas and Pyragas, 2006]. We have refuted such a *qualitative constraint*, which had been proliferated under the name of "odd number limitation". Our examples of successful delay stabilization are planar, $\dim x = 2$, with a single unstable Floquet multiplier $\mu > 1$ near 1; see [Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007a; Just, Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007; Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007b; Fiedler, Flunkert, Georgi, Hövel and Schöll, 2008; Fiedler, Yanchuk, Flunkert, Hövel, Wünsche and Schöll, 2008]. To stabilize Γ we have used the special form

$$\dot{x}(t) = f_0(x(t)) + B(x(t-\tau) - x(t)) \tag{3}$$

where the nonlinearity f_0 and the matrix B both commute with rotations of x. By these results noninvasive time-delayed feedback control of periodic orbits Γ has become a more promising way to go.

Conversely, and as a caveat, we present a *quantitative constraint* on time-delayed feedback stabilization of the quite general form (2). Again we consider f, g which commute with rotations, and periodic orbits Γ which rotate at constant speed. As before let p denote the minimal period of Γ and $\mu > 1$ the uncontrolled unstable Floquet multiplier in (1).

Theorem. *Noninvasive delay stabilization by a multiple period time delay $\tau = np$ is possible if, and only if, the unstable Floquet multiplier μ of the periodic orbit* *satisfies the quantitative constraint*

$$\mu < \exp(9/n). \qquad (4)$$

Our previous counterexamples to the "odd number limitation" have been obtained near bifurcations, only, where $\mu \approx 1$ and the constraint (4) becomes void. For the case of Hopf bifurcation from equilibria to rotating waves see [Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007a; Just, Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007; Fiedler, Flunkert, Georgi, Hövel and Schöll, 2007b; Fiedler, Flunkert, Georgi, Hövel and Schöll, 2008]. The case of saddle-node bifurcations of rotating waves is considered in [Fiedler, Yanchuk, Flunkert, Hövel, Wünsche and Schöll, 2008].

The constraint $\mu < \exp(9) = 8103.1\ldots$, beyond which time-delayed feedback stabilization fails entirely, is large when a delay $\tau = p$ equal to the minimal period p can be realized. Delayed feedback control at minimal period will therefore fail for violently unstable planar rotating waves, only. Even a moderately unstable Floquet multiplier $\mu = 2$ cannot be stabilized, on the other hand, when the experimentally feasible delay $\tau = np$ exceeds $n = 12$ minimal periods. Such a quantitative constraint becomes relevant, for example, in laser optics where Γ oscillates at very high frequencies.

A detailed proof of the theorem will be given in [Fiedler, Flunkert, Georgi, Hövel and Schöll, 2008]. We can only sketch the argument here. Because (1),(2) commute with rotations the periodic orbit is harmonic, $x(t) = x(0)\exp(2\pi i t/p)$. Passing to co-rotating coordinates freezes the rotating wave to become a circle of equilibria, and retains the autonomous form of (1),(2). Linearization of (2) provides a characteristic equation for Floquet exponents η of the periodic orbit Γ. Scaling and eliminating $\tau = np$ by linear scaling we obtain

$$0 = \eta - a + b_0 w + b_1 w\eta + b_2 w^2 \eta. \qquad (5)$$

Here $w = (\exp(-\eta) - 1)/\eta$ and $\tau = np$. The original Floquet multiplier μ of the periodic orbit Γ of (1) is given by

$$\mu = \exp((a/\tau)p) = \exp(a/n). \qquad (6)$$

To prove the theorem we have to discuss unstable solutions η of the characteristic equation (5) with positive real part $\Re\eta > 0$. Their total number u, including algebraic multiplicities, denotes the *unstable dimension* of the periodic orbit Γ. Thus the theorem reduces to showing that

$$u = 0 \quad \Longleftrightarrow \quad a \leq 9 \qquad (7)$$

for suitable choices of τ, b_0, b_1, b_2.

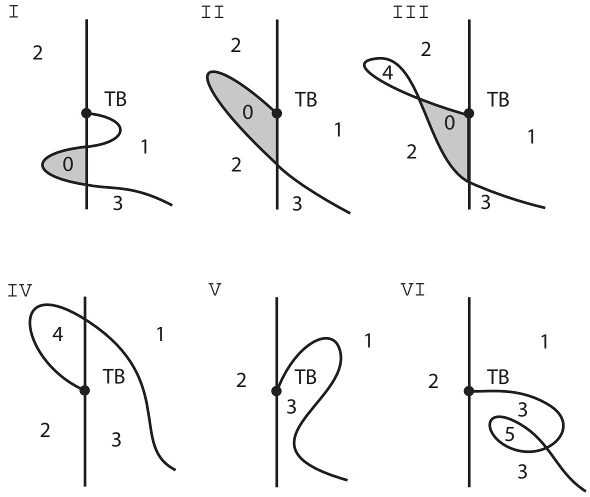

Figure 1. Qualitative bifurcation diagrams I – VI in the (b_0, b_1)-plane. Vertical b_1-axis at $b_0 = -a$ coincides with zero eigenvalues η. Takens-Bogdanov point TB, emanating Hopf bifurcation curve, and resulting unstable dimensions u of slowly oscillating eigenvalues are indicated. Stability region $u = 0$ in gray.

Even after elimination of τ the characteristic equation (5) contains a cumbersome four remaining real parameters $a > 0, b_0, b_1, b_2$. Figure 1 shows six resulting qualitative bifurcation diagrams I – VI of (a, b_2) in the (b_0, b_1)-plane, not drawn to scale. Figure 2 indicates corresponding regions I – VI of (a, b_2) where these bifurcation diagrams hold. All curves which define the bifurcation diagrams in this paper can be parametrized explicitly, due to linearity of the characteristic equation (5) in the four parameters. Together the bifurcation diagrams describe the behavior of the characteristic equation in these four parameters, and prove the theorem as follows.

We first address necessity of the multiplier bound $a = 9$ and show that $u \geq 1$ for $a > 9$. For (a, b_2) in regions IV–VI of figure 2 we indeed observe absence of stability regions $u = 0$ in figure 1. We always have $u = 1$ at $b_0 = b_1 = b_2 = 0 > -a$, because the rotating wave of the ODE (1) is unstable by assumption. Stability changes can only occur by eigenvalues $\eta = iy$ which cross the imaginary axis at an imaginary Hopf pair $\pm y \neq 0$, or at $y = 0$. At $y = 0$ eigenvalues cross the imaginary axis only as b_0 crosses through $b_0 = -a$. This accounts for odd parity of u, and hence for instability, whenever $b_0 > -a$ in figure 1.

As the top solid curve is crossed from region V into region I, by (a, b_2) in figure 2, the Hopf curve (b_0, b_1) of slow imaginary eigenvalues with $0 < y < 2\pi$ touches the vertical b_1-axis at $b_0 = -a$ and enters the left side, in figure 1, where the parity of u is even. A stability region is thus created where we had $u = 2$ before.

Similarly consider (a, b_2) crossing the top solid curve from region V into region II between TBT and TB0, in figure 2. The tangent direction of the emanating Hopf branch (b_0, b_1) then switches from right to left in fig-

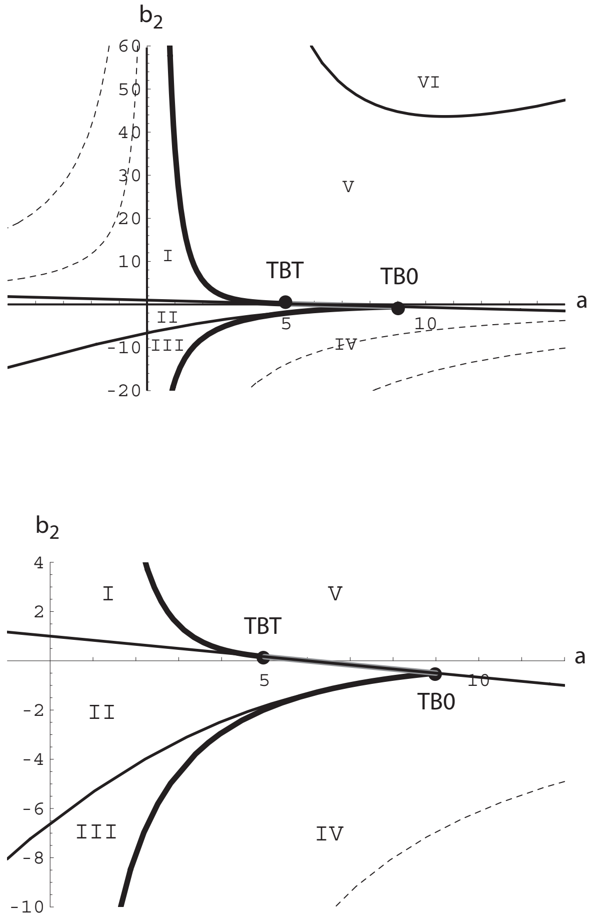

Figure 2. Numerical plots of regions I – VI in the (a, b_2)-plane where the (b_0, b_1) bifurcation diagrams of figure 1 hold. Thick solid curves emanating from $a = 9, b_2 = -1/2$ to the left bound regions I–III where stability $\Re\eta < 0$ holds for suitable choices of b_0, b_1 near the Takens-Bogdanov point. Top: global diagram. Bottom: zoom into stability region.

ure 1 and points vertically downwards along the line $b_2 = 1 - a/6$ between TBT and TB0. The switch again generates a stability region in the (b_0, b_1)-plane. At TBT a codimension 4 Takens-Bogdanov point occurs where the Hopf and zero curves emanate with identical tangent and curvature.

As (a, b_2) crosses the bottom solid curve from region IV into region III, in figure 2, the Hopf curve (b_0, b_1) of slow eigenvalues $0 < y < 2\pi$ crosses the Takens-Bogdanov point TB in figure 1 and forms a loop. The loop of the Hopf curve creates higher instability $u = 4$ inside the loop, but also produces a stability region $u = 0$ near TB. The higher instability loop of (b_0, b_1) terminates at a Hopf cusp curve of (a, b_2) which separates regions II and III in figure 2.

The (a, b_2) region VI is generated by an analogous cusp in the odd parity region $b_0 > -a$ of (b_0, b_1). However, stability is not recovered. Similarly, rapid Hopf bifurcations with $y > 2\pi$ can only increase instability further. Therefore Pyragas stabilization fails for $a > 9$.

Conversely we have to show that time-delayed feedback stabilization is possible for $a < 9$ and suitable choices of b_0, b_1, b_2. The choice $b_2 = -1/2$ keeps us in region II for all $0 < a < 9$. The dashed (a, b_2) curves in figure 2 indicate when the (b_0, b_1) curves of rapid Hopf bifurcations with $y > 2\pi$ move across the Takens-Bogdanov point TB at $b_0 = -a$, $b_1 = -a/2 + b_2 + 1 = (1 - a)/2$, in figure 1. None of these curves interferes with our choice of b_2 as long as we choose $b_0 = -a - \beta$, $b_1 = (1 - a)/2 - \beta$ for sufficiently small positive β. Thus stabilization remains unaffected by rapid Hopf bifurcation. This completes the sketch of proof of our theorem.

We conclude that stabilization by time-delayed feedback control is ruled entirely – granted as well as constrained – by the codimension 4 Takens Bogdanov point TB0 at

$$a = 9, \quad b_0 = -9, \quad b_1 = -4, \quad b_2 = -1/2. \quad (8)$$

At this point the Hopf branch (b_0, b_1) emanates at order y^4 from TB, rather than at the generic order y^2.

Stabilization of equilibria by linear rank-1 feedback matrices is known as *pole assignment* in classical control theory; see for example [Wonham, 1985]. That special case $\det B = 0$ corresponds to $b_2 = 0$ in (3). By figure 2 time-delayed feedback stabilization by rank-1 feedback matrices B is feasible if, and only if,

$$\mu < \exp(6/n). \quad (9)$$

Applicability is thus restricted by the aggravated constraint $\exp(6) = 403.4\ldots$ for the unstable Floquet multiplier μ. Rank-1 delayed feedback stabilization is therefore limited to multiples $n \leq 8$ of the minimal period p in the case of planar rotating waves, even for moderate Floquet multipliers $\mu = 2$.

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