Impulse Switching over Synchronous States in Delay Coupled Semiconductor Lasers

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***Abstract*— We derive the discrete maps to describe the dynamics of coupled laser diodes. The maps allow us to find analytically regions of parameters and initial conditions in the functional phase space corresponding spiking with desired stable (or nearly stable) phase shift. The method developed is promising for further discussion of controlled switching between periodic states by a injection signal.**

I. INTRODUCTION

Synchronization of delay-coupled oscillators are under permanent studying in networks of various configurations. Phase synchronization for which stable temporal shift exists between oscillations of the elements has been experimentally observed in biological networks as well as in artificial networks for communications systems [1]. In laser systems complete synchronization, lag and anticipate synchronization [2]- [4], antiphase and splay states [5] have been discussed for semiconductor lasers coupled through uni- or bi-directional optoelectronic feedback.

In this paper we concentrate on synchronization of relaxation oscillations of two identical laser diodes coupled through pump injection current with finite time of signal propagation. We demonstrate numerically delay induced multistability of various spiking regimes and describe analytically the initial conditions and the regions of parameters providing phase synchronization. Additional impulse variation of the pumping rate provides switching between coexisting attractors and we propose the method of calculation of the optimal parameters of such an external force.

II. MODEL

Consider the system of single-mode rate equations for two coupled semiconductor lasers taking into account the delay time τ of signals in coupling circuit:

$$\begin{aligned} \frac{du_i}{dt} &= vu_i(y-1), \\ \frac{dy_i}{dt} &= q-y_i(1-u_i)+\alpha_j u_j(t-\tau), \end{aligned} \qquad (1)$$

where u_i and y_i are proportional to the photon density in the cavity and the population inversion in lasers $i=1,2$, v is the ratio of photon damping time in the cavity to the relaxation time of inversion of population. The pumping rate has the constant part q which is determined by constant injection current and has the part modulated by the intensity of another laser $j=2,1$ in the delayed moment $\alpha_j u_j(t-\tau)$. In general case the coupling coefficients are not equal, i.e. $\alpha_1 \neq \alpha_2$.

The peculiarity of the system is that for laser diodes as well as for class B lasers the parameter v is the large parameter while other parameters are of the order of unit. As a result, relaxation oscillations in the form of short width spikes are observed in wide parameter regions. Numerical integration of equations (1) shows that spiking is realized in various forms including periodic, quasi-periodic and chaotic ones. Constant phase shift between pulses of different lasers is observed mainly for periodic regimes. Below we find conditions for such regimes.

Equations (1) with a delayed argument compound a dynamical system in infinite dimensional (functional) phase space. In addition, Eqs.(1) are singular perturbed as $v \gg 1$. To study relaxation oscillations in the system we apply the asymptotic method developed in [6]. Let choose the set $S(\xi)$ depending on the vector parameter ξ as the set of initial conditions. It is possible then to construct uniform asymptotic approximations of solutions taking into account the large parameter and to show that after a certain time the solution again falls within $S(\xi)$. Thus, the Poincare operator of the shifting along the trajectories which maps ξ from S onto ξ that is also from S, is thereby analytically defined, $\xi = f(\xi)$. To an attractor of the operator there corresponds a stable spiking solution of a similar structure to the original system. In particular, a fixed point in the map corresponds to the periodic spiking and so on. Thus for each type of synchronization one can determine special initial conditions and construct the map with attractors (if exist) guarantee synchronized (in some sense) dynamics.

III. SLOW OSCILLATING REGIMES

Note, first of all, that complete synchronized solution is unstable under $v \to \infty$. That is why we consider further spikes follow each other in finite time intervals T_i.

We call the solution slow (fast) oscillating if spikes of two lasers alternate in time intervals T_i larger (smaller) than the time delay τ. The examples of slow oscillating regimes are given in Fig. 1. Three solutions (a) - (c) are the results of numerical integration of the Eqs.1 with the same parameters but under different initial conditions, hence, multistability of spiking takes place. In Fig. 1(a) slow oscillating spikes of two laser diodes are nearly in anti-phase (SO anti-phase solution), in Fig. 1(b) slow oscillating spikes are nearly in phase (SO in-phase solution), in Fig. 1(c) slow oscillating spikes of the first laser support fast oscillating spikes of the

This work was not supported by any organization

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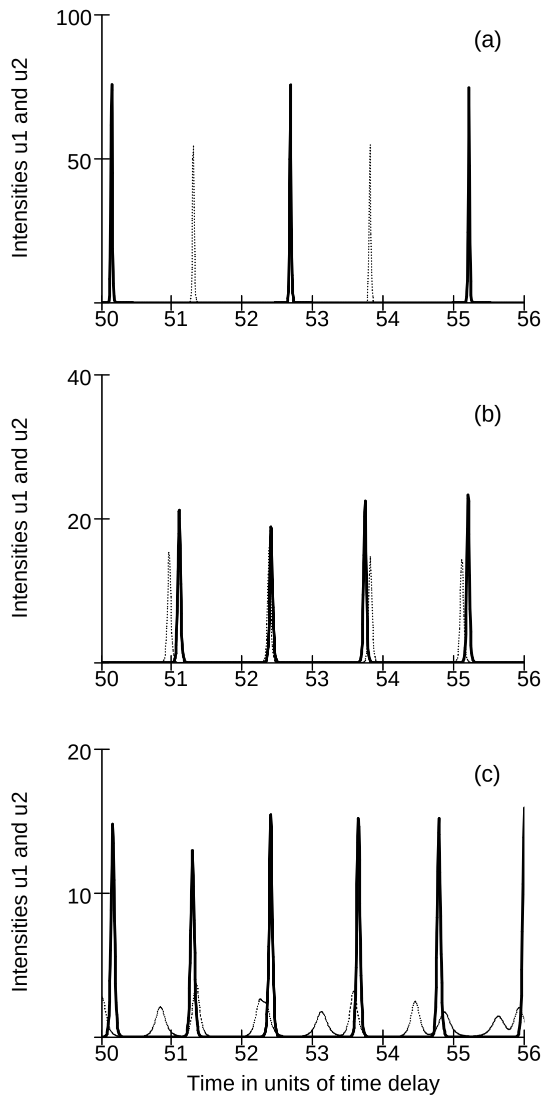

Fig. 1. Coexisting spiking in system (1) for the pumping rate $q = 1.5$, the time delay in the coupling circuit $\tau = 0.4$, coefficients of the coupling are $\alpha_1 = 0.2,\ \alpha_2 = 0.4$, normalized photon damping rate $\nu = 10^3$ and different initial conditions: (a) slow oscillating spikes of two laser diodes are nearly in anti-phase, (b) slow oscillating spikes are nearly in phase, (c) slow and fast oscillating spikes.

second laser (frequecy locking solution). Below we present the maps responsible for solutions of different types.

A. Map for SO anti-phase solution

To derive the map let us fix the moment $t = 0$ at the moment of the starting laser numbered $i = 1$ and determine the set of initial conditions as follows:

$$y_1(0) = c_1,\ c_1 > 1,\ y_2(0) = c_2,\ c_2 > 0, \tag{2}$$

and we choose initial functions $u_i(s) = h_i(s)$ for $s \in [-\tau, 0)$ from relatively wide class of functions with properties:

$$\int_{-\tau}^{0} h_i(s) ds \le \nu^{-1/2}, \tag{3}$$

and for the moment $t = 0$

$$u_1(0) = 1,\ u_2(0) = \exp(\nu m) \ll 1,\ m < 0. \tag{4}$$

The initial conditions given mean the light intensities of both lasers are supposed to be (asymptotically) small on the interval of the delay $s \in [-\tau, 0)$, $0 < h_1(s) \ll 1,\ 0 < h_2(s) \ll 1$, but not small for 1-laser at $s = 0$. It is possible to prove that the width of spike $\delta = O(\nu^{-1})$, hence, $\delta \to 0$ under $\nu \to \infty$. Hereafter all formulae are valid with an accuracy $O(\nu^{-1})$.

Integrating asymptotically under $\nu \to \infty$ system (1) with initial conditions (2)-(4) we get that in time interval T the original situation (2)-(4) appears again with replacing the laser numbers (indexes i) $1 \leftrightarrow 2$ and change vector parameter $\xi = (c_1, c_2, m) \to \bar{\xi} = (\bar{c_1}, \bar{c_2}, \bar{m})$ determined by the three dimensional map

$$\begin{aligned} \bar{c_1} &= q + (c_2 - q + p\alpha e^{\tau}) e^{-T}, \\ \bar{c_2} &= q + (c_1 - q - p) e^{-T}, \\ \bar{m} &= (q-1)T + (c_1 - q - p)\left(1 - e^{-T}\right), \end{aligned} \tag{5}$$

with $\alpha = \alpha_1$ and $\alpha = \alpha_2$ for odd and even iterations, respectively. The function $p = p(c_1)$ characterizes the energy of the pulse and it's value can be found as positive root of the equation

$$c_1 - p = c_1 e^{-p}. \tag{6}$$

The function $T = T(c_1, c_2, m)$ characterizes time interval between two consequent spikes of two lasers, it's value can be found as the first positive root of the equation

$$\begin{aligned} m + (q-1)T + (c_2 - q)\left(1 - e^{-T}\right) + \\ + p\alpha\left(1 - e^{\tau - T}\right) = 0. \end{aligned} \tag{7}$$

If attractors of the map (5)-(7) exist and for each iteration of the map the conditions

$$T > \tau,\ m < 0,\ c_1 > 1,\ c_2 < c_1 \tag{8}$$

are valid then to the attractors there correspond the slow oscillating regimes with the time interval $T > \tau$ between spikes of different lasers and with the time interval between spikes of a given laser more than 2τ. Note, the conditions (8) bound the region of the initial conditions leading to SO anti-phase solutions.

B. Fixed point of the map for SO anti-phase solution

In the region discussed the twice iterated map (5)-(7) has the stable fixed point, $\bar{\bar{\xi}} = \xi = \xi^*$. To this attractor there corresponds the limit cycle - the periodic solution of spiking type. The period of oscillations for each laser $T_0 = T_1^* + T_2^*$ is more than 2τ. The lasers are (nearly) in anti-phase, i.e. spikes of the different lasers are half-period shifted.

In the case of symmetrical coupling, $\alpha_1 = \alpha_2 = \alpha$, the upper boundary $\tau_{up}(\alpha)$ of the region where the symmetrical slow oscillating solution exists can be found analytically the transcendental equations (8). It is presented in Fig.2 as the curve for $a = 1$. As the coefficient of asymmetry $a = \alpha_2/\alpha_1$ decreases the region of slow oscillation regimes decreases too.

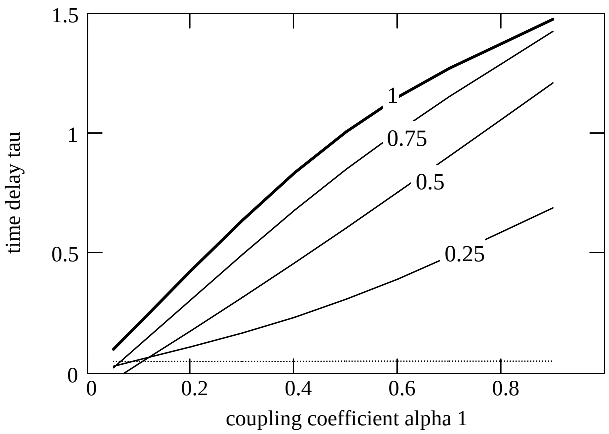

Fig. 2. Regions of the parameters α_1, τ where slow oscillating regimes are realized in the cases of values of the asymmetry coefficient $a = \alpha_2/\alpha_1 = 1; 0.75; 0.5; 0.25$. The normalized pumping rate $q = 1.5$.

For the fixed point we get simple relation between the energy spike and the inter-spike intervals for each laser:

$$\frac{p_i^*}{T_1^* + T_2^*} = (q-1)\frac{(1+\alpha_i)}{1-\alpha_1\alpha_2} \tag{9}$$

independent on τ although both p_i^* and T_i^* depend on the time delay. From the physical point of view it means the energy average over the full period of oscillations increases as the product of the coupling coefficients increase. The ratio of energies is given by

$$\frac{p_1^*}{p_2^*} = \frac{1+\alpha_2}{1+\alpha_1}, \tag{10}$$

that indicate differentiation of energies due to the asymmetrical coupling.

For the time delay comparable with inversion damping time, $\tau <\sim 1$, we get additionally $p_1^* T_1^* \approx p_2^* T_2^*$ and estimate the phase shift of spikes as

$$\frac{T_1^*}{T_1^* + T_2^*} \approx \frac{1}{2+\alpha_1-\alpha_2}. \tag{11}$$

Hence, the phase shift of spikes varies from $1/3$ to $1/2$ as the coefficients of coupling varies from $\alpha_2 = 0, \alpha_1 = 1$ (in the strong asymmetrical case) to $\alpha_2 = \alpha_1$ (in the symmetrical case).

C. Map for SO in-phase solution

In order to get the map for nearly in-phase SO spiking represented in Fig. 1(b) we start again with initial conditions (2)-(4), i.e. determine the vector parameter $\xi = (c_1, c_2, c_3)$. Integrating asymptotically under $\nu \to \infty$ system (1) we get that in time interval $T = min\{T_1, T_2\}$ the original situation (2)-(4) appears again with replacing the vector parameter $\xi \to \bar{\xi}$, where $\bar{\xi}$ is determined by the three dimensional map

$$\begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{m}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} f_1(T_1) \\ f_2(T_1) \\ f_4(T_1) \end{pmatrix}, & \mathrm{T}_1 < T_2 \\ \begin{pmatrix} f_2(T_2) \\ f_1(T_2) \\ f_3(T_2) \end{pmatrix}, & \mathrm{T}_2 < T_1 \end{cases} \tag{12}$$

where

$$\begin{aligned} f_1(T_i) &= q + (c_1 - p_1 - q)e^{-T_i} + \alpha_2 p_2 e^{-T_i+\tau+\theta}, \\ f_2(T_i) &= q + (c_2 - p_2 e^{\theta} - q)e^{-T_i} + \alpha_1 p_1 e^{-T_i+\tau}, \\ f_3(T_i) &= (q-1)T_i + (c_1 - p_1 - q)(1 - e^{-T_i}) \\ &\quad + \alpha_2 p_2 (1 - e^{-T_i+\tau+\theta}), \\ f_4(T_i) &= (q-1)T_i + (c_2 - q)(1 - e^{-T_i}) \\ &\quad - p_1(1 - e^{-T_i+\theta}) + \alpha_1 p_1 (1 - e^{-T_i+\tau}), \end{aligned}$$

and p_1 is positive root of Eq.(6), θ is positive root the equation $m + (q-1)\theta + (c_2 - q)(1 - \exp(-\theta)) = 0$, T_1 is the positive root of the equation $f_3(T_1) = 0$, T_2 is the positive root of the equation $f_4(T_2) = 0$ and p_2 is the positive root of the equation $(1 - e^{-p_2})[m + c_2 + (q-1)\theta] - p_2 = 0$.

SO solution exists if the conditions

$$T > \tau, \ \theta < \tau, \ m < 0, \ c_1 > 1, \ c_2 > 1 \tag{13}$$

are valid for each iteration of the map (14) and any attractor of the map (12) exists. These conditions bound the region of initial conditions for regimes with $\theta < \tau$. In the case $\theta \ll \tau$ the solutions can be characterized as nearly in-phase state.

IV. SWITCHING BETWEEN SPIKING

It has been already mentioned that in addition to slow oscillating solutions there exist fast oscillating solutions in the system (1). One can derive the corresponding maps following the method described in [6]. The dimension of the map increases as the number of spikes increases in the delay interval. The maps obtained allows us to investigate analytically stability of the fixed points and, in this way, stability of spiking oscillations. Also the conditions of the type (8) and (13) bound the regions of the parameters and initial conditions for desirable regimes.

In view of possible applications it would be useful to describe the method of controlled switching over variously synchronized states in the coupled lasers. It means in the framework of the dynamical theory that an external force applied has to change the state of the system in the phase space, in instance, from the attractor of the map (5) into the basin of the attractor of the map (12). One can formulate the problem of optimal external force providing minimal energy or/and minimal time of switching. In general case, however, the problem is nontrivial as the delayed system (1) is of the infinite dimension similar to spatially extended systems.

In special case switching can be achieved by additional impulse pumping P_i applied for each lasers at the time moments x_i. Fig.3 shows impulse induced switching from

SO anti-phase spikes to SO in-phase spikes and to frequency locking spikes through long (Fig.3a and 3b) and short (Fig.3c) transient processes. In order to realize such a fast switching we calculate the values P_i and x in the following way.

Let the pumping rate in Eqs.(1) to be

$$q = q_0 + q_i$$

where $q_i = 0$ everywhere but $q_i \neq 0$ for $t \in (x, x+\delta)$. The width $\delta \ll 1$ but $P_i = q_i\delta \sim 1$. Denote the fixed point of the map (5) as $(c^*_{1a}, c^*_{2a}, m^*_a)$ (anti-phase solution) and the fixed point of the map (12) as $(c^*_{1i}, c^*_{2i}, m^*_1)$ (in-phase solution). Then we integrate asymptotically the system (1) with boundary conditions $(c^*_{1a}, c^*_{2a}, m^*_a)$ at the moment $t = 0$ and $(c^*_{1i}, c^*_{2i}, m^*_1)$ at the moment $t = T > \tau$. Doing so we get the system of transcendental equations

$$\begin{aligned} c_{1i} &= c^*_{1a} + P_2 e^{x_1 - T}, \\ c_{2i} &= c^*_{2a} + P_1 e^{x_2 - T}, \\ m^*_i &= m^*_a + P_1(1 - e^{x_1 - T}), \\ m^*_a &= (q-1)T + (c^*_{2a} - q)(1 - e^{-T}) + \\ & \quad + \alpha p_1(1 - e^{\tau - T}) + P_2(1 - e^{x_2 - T}) \end{aligned} \tag{14}$$

from which T can be found as $T(P_1, P_2, x_1, x_2)$ and can be minimized for optimal values of arguments, $T_{min} = T(\bar{P_1}, \bar{P_2}, \bar{x_1}, \bar{x_2}) > \tau$.

V. Conclusion

In this paper spiking regimes in coupled laser diodes have been classified as slow and fast oscillating ones in the scale of the time delay. We have reduced the original system with delayed argument to the finite-dimensional maps responsible for each type of the solutions, namely, for the slow oscillating nearly anti-phase solution, for slow oscillating nearly in-phase solution and for the fast oscillating phase-shift solution. The order of the map increases as a number of spikes on the delay interval increases.

The discrete maps obtained allow us to determine analytically regions of parameters and initial conditions to get spiking in coupled laser diodes with desired stable phase shift.

We also find analytically conservation laws in the form of relations between energies and inter-spike intervals in systems with symmetrical and asymmetrical coupling. At the same time redistribution of the energy between spikes of different lasers depends on the time delay as well as on the asymmetrical coefficient.

Delay-induced multistability has been demonstrated in the infinite dimensional phase space as coexistence of various spiking. The method developed is promising for further discussion of controlled switching between different coexisting periodic states by a small injection signal.

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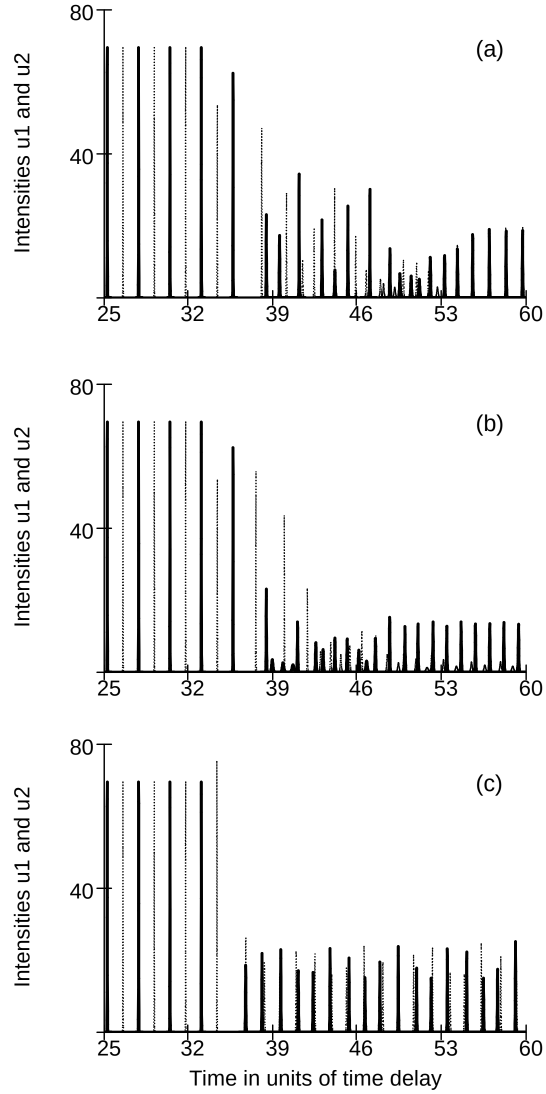

Fig. 3. Switching from slow oscillating anti-phase spikes to (a) slow oscillating in-phase spikes and to (b) frequency locking oscillating spikes represented in Fig.1. Switching is induced by additional impulse pumping in 1-laser at the moment $t = 34\tau$. (c) Switching through short transient process is induced by optimal impulses in both lasers at the moment $t = 34\tau$. The pumping rate $q = 1.5$, the time delay in the coupling circuit $\tau = 0.4$, coefficients of the coupling are $\alpha_1 = \alpha_2 = 0.3$, normalized photon damping rate $v = 10^3$.

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