ON THE THEORY OF ONE CLASS OF VIBRO-RAMS

Vladimir S. Metrikin
Research Institute for Applied Mathematics and Cybernetics
Nizhny Novgorod
Russia

1 Introduction

In the past decades, along with the vibro-impact machines that use disbalanced exciters, the excentric vibro-impact machines (EIVM) with a crank vibration stimulator (CVS) are widely used in construction [Bobylev, 1980]. The EIVM utilize the principle of an "inversed vibrator", the working organ of which is situated on an excentric shaft.

The EIVM are widely used for soil densification and pile driving. The efficiency of these processes obviously depends on the amount of energy transferred from the vibrator to the soil or pile. Also of significant importance for the efficiency is the dynamic signature of the load produced by the machine. It is known that to obtain a dense and stable soil structure the pressure on the ground produced by the working organ of the machine should increase gradually. The lower boundary of this pressure is determined by the soil material properties, whereas the upper boundary is mainly dictated by the soil stability limit and/or by the technological conditions. Correspondingly, the working regime of the vibro-impact machines should provide the quasi-plastic soil behavior. It is also of importance that the frequency of the impacts does not allow the development of the elastic post-action between the impacts. Such multi-impulse loading may be realized with the help of EIVM with CVS, whose construction allows to easily tune the working regime by means of the kinematic connections.

In this paper, a mathematical model of multi-impact EIVM with CVS is presented that, accounts for soil flexibility. The model first analyzed in the phase space. Then, based on the features of the phase trajectories, the method of point mapping is applied in order to identify the domains of existence and stability of periodic regimes in the parameter space. The relationships between the model parameters are found that allow to tune the EIVM to its main working regime. The general methodology of investigation of the periodic regimes of the vibro-impact machine is illustrated by numerical examples of practical use.

2 Equations of motion

Consider a vibro-imact machine schematized in Figure 1. It consists of a machine body 1 that contains an excentric shaft 2 to the ends of which the fly-wheels are attached. On the shaft, the excentric mechanisms 3 are mounted. Each of these mechanisms consists of two excentrics, one inside the other. By moving the excentrics relative to each other, the excentricity values r_i and the phase shifts φ_i can be varied. On the free ends of the cranks, the crawlers-impactors (CI) 4 are mounted. The excentric mechanisms together with the cranks and CI convert the uniform rotation (with the frequency ω) of the fly wheels and the shaft to the oscillatory motion of the machine body about the pillars 5. The CIs are situated one inside the other and each impacts its own anvil.

Soil is represented as an elastically supported mass M_1. The spring coefficient of the support is C. The energy loss in the machine and in the soil is assumed to be of the viscous type and is characterized by damping coefficients b and b_1 respectively.

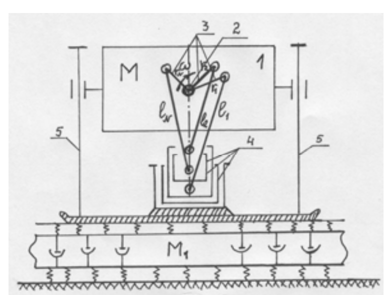

Fig 1.

The oscillations of the machine body take place relative to one of the excentric mechanisms as these mechanisms vibrate with a different phase and, generally speaking, have different lengths. The impacts on the anvil take place either when the active CUI changes or when the machine body detaches from it.

Assuming that the masses of the crank and CIs are negligible, the equations of independent motion of the machine and of the oscillator that represents the soil (the motion when the soil and the machine are not in contact) can be written as

$$M\frac{d^2y}{dt^2}+b\frac{dy}{dt}=-Mg$$
$$M_1\frac{d^2y_s}{dt^2}+b_1\frac{dy_s}{dt}+Cy_s=-M_1g \tag{1}$$

where $y(t)$ is the directed upward coordinate of the centre of rotation of the machine body referenced to the equilibrium of mass M_1, $y_s(t)$ is the directed upward coordinate of mass M_1 referenced to the same equilibrium, M is the mass of the machine body, g is the gravity acceleration.

The independent motion takes place when $y_{pi}>y_c$, where y_{pi} are the coordinates of the CIs. When one of the CIs impacts the soil, i.e. when $y_{pi}=y_c$ an instantaneous interaction is assumed to take place such that

$$\dot{y}_{pi}^{+}=\frac{(\mu_0-R)\dot{y}_{pi}^{-}+(1+R)\dot{y}_c^{-}}{1+\mu_0}$$
$$\dot{y}_c^{+}=\frac{(\mu_0+R)\dot{y}_{pi}^{-}+(1-\mu_0R)\dot{y}_c^{-}}{1+\mu_0} \tag{2}$$

where $\dot{y}_{pi}^{-}$ and $\dot{y}_c^{-}$ are the velocities of the i-th impactor and of the soil just before the impact, $\dot{y}_{pi}^{+}$ and $\dot{y}_c^{+}$ are the respective velocities right after the impact, $0\le R\le 1$ is the coefficient of velocity restoration, $\mu_0=M/M_1$.

If the impact is perfectly inelastic, i.e. if $R=0$, the machine body and the soil move together after the impact. This regime is described by the following equation of motion:

$$(M_1+M)\frac{d^2y_s}{dt^2}+(b_1+b)\frac{dy_s}{dt}+Cy_s=-(M_1+M)g \tag{3}$$

The coordinates y_{pi} of the CIs are given by the following expression:

$$y_{pi}=y-s_i+r_i\cos(\omega t-\varphi_i)-\sqrt{l_i^2-r_i^2\sin^2(\omega t-\varphi_i)} \tag{4}$$

where s_i is the distance between the point of fixation (a hinge) of the i-th crank to the head of the i-th CI, r_i and l_i are the excentricity and the length of the i-th CI.

For further analysis, it is customary to reformulate the above problem statement in a dimensionless form. To this end, we introduce the dimensionless time $\tau=\omega t$, the dimensionless coordinate of the machine body $x=(y-s_2-l)/l$ and the dimensional coordinate of the soil $x_1=y_c/l$. Additionally, the following dimensionless parameters are introduced:

$$\mu=r_1/l,\ \gamma_i=r_i/r_1,\ \varepsilon_i=(s_i-s_2)/l,$$
$$p=g/\omega^2 l,\ \lambda^2=C/M_1\omega^2,\ 2h_1=b_1/M_1\omega \tag{5}$$

Using the above-introduced notations, introducing $f_i(\tau)=\varepsilon_i-\mu\gamma_i\cos(\tau-\varphi_i)$ and assuming $r_i<<l_i$, the following dimensionless problem statement is obtained

$$\text{if } x-x_1>f(\tau):$$
$$\ddot{x}+2h\dot{x}=-p \tag{6}$$
$$\ddot{x}_1+2h_1\dot{x}_1+\lambda^2x_1=-p$$

$$\text{if } x-x_1=f(\tau):$$
$$(1+\mu_0)\ddot{x}+2(h_1+\mu_0h)\dot{x}+\lambda^2x=-(1+\mu_0)p+F(\tau) \tag{7}$$

$$\dot{x}^{+}=\frac{(\mu_0-R)\dot{x}^{-}+(1+R)\dot{x}_1^{-}+(1+r-\mu_0)\dot{f}(\tau)}{1+\mu_0}$$
$$\dot{x}_1^{+}=\frac{\mu_0(1+R)\dot{x}^{-}+(1-R\mu_0)\dot{x}_1^{-}-\mu_0(1+R)\dot{f}(\tau)}{1+\mu_0} \tag{8}$$

where the overdot denotes derivative with respect to the dimensionless time. A periodic in τ function $F(\tau)$ in equation (7) is given as

$$F(\tau)=\varepsilon_i+(\lambda^2-1)f(\tau)+2h_1\dot{f}(\tau) \tag{3}$$

and $f(\tau)=\max\{f_1(\tau), f_2(\tau),..., f_N(\tau)\}$.

3 Dynamics of the system assuming the soil to be immovable

Assuming $x_1=0$, $M_1=\infty$ equations (6)-(8) are reduced to [Feigin, 1994]

$$
\begin{aligned}
&\ddot{x}+2h\dot{x}=-p, \quad x>f(\tau)\\
&\dot{x}^{+}=-R\dot{x}^{-}+(1+R)\dot{f}(\tau), \; x=f(\tau)
\end{aligned}
\tag{9}
$$

The phase space of system (4) in coordinates $x, \dot{x}, \tau$ is limited in x such that $x \geq f(\tau), \dot{x} < +\infty$, see Fig.2. All phase trajectories are located either on the surface $x = f(\tau)$ or above it. Generally speaking, this surface consists of N (the number of CIs) intersecting surfaces.

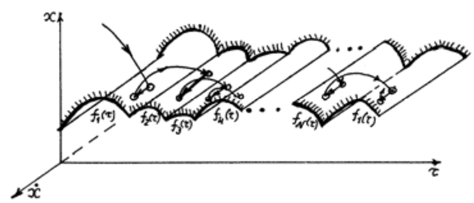

Fig. 2.

In the following, main attention will be paid to the periodic regimes in which the impactors nock on the anvils one after the other within the period Γ. It is obvious, that such regimes are possible only under the condition of intersection in pairs of the surfaces $f_k(\tau)$ and $f_{k+1}(\tau)$. To satisfy this condition the following constraint should hold

$$
\frac{(\varepsilon_{k+1}-\varepsilon_k)^2}{\mu^2} \leq \gamma_{k+1}^2+\gamma_k^2-2\gamma_k\gamma_{k+1}\cos(\varphi_{k+1}-\varphi_k) \tag{10}
$$

If for any k the above inequality is not fulfilled, one of the CIs would be inactive, which is undesirable.

Looking at the phase space, it becomes clear that the system dynamics may be carried out by studying the properties of a point mapping T of the surface $x = f(\tau)$ onto itself. Let us consider the basic point mappings T_{k+1} that map points of the surface $x = f_{k+1}(\tau)$ onto points of the surface $x = f_{k+2}(\tau)$. These mappings in the case $h = 0$ can be written as

$$
\begin{aligned}
&f_{k+2}(\tau_{k+1})=\Delta\tau_{k+1}(\dot{x}_k-p\Delta\tau_{k+1}/2)+f_{k+1}(\tau_k)\\
&\dot{x}_{k+1}=R(p\Delta\tau_{k+1}-\dot{x}_k)+(1+R)\dot{f}(\tau_{k+1})\\
&\Delta\tau_{k+1}=\tau_{k+1}-\tau_k, \quad k=0,1,\ldots,N-1
\end{aligned}
\tag{11}
$$

Parameters in (11) have to satisfy constraint (10), as well as to ensure the existence of the mapping, i.e.

$$
\begin{aligned}
&f_{k+1}(\tau_k)\geq f(\tau_k), \quad k=0,1,\ldots,N-1\\
&f_{k+2}(\tau_{k+1})\geq f_l(\tau_{k+1}), \quad l\neq k+2, l=0,1,\ldots,N-1\\
&x(\tau)>f(\tau), \quad \tau_k<\tau<\tau_{k+1}
\end{aligned}
\tag{12}
$$

The position of a fixed point $\mathrm{M}^*(\tau^*, \dot{\mathrm{X}}^*)$ that corresponds to the periodic regime under consideration is determined by the system of 2(N+1) equations (9) supplemented by the following conditions that ensure that this point is fixed [Feigin, 1994]:

$$
\dot{\mathrm{X}}_{N+1}=\dot{\mathrm{X}}_1=\dot{\mathrm{X}}^*, \tau_{N+1}=\tau_1+n\Gamma=\tau^* \tag{13}
$$

Using (9) and (13), one can obtain

$$
\begin{aligned}
&\dot{\mathrm{X}}^*=\frac{b_N-R^N\sum_{k=1}^{N}(-1)^{k+1}b_{N-k}}{1+(-1)^{N-1}R^N}\\
&\dot{\mathrm{X}}_{k+1}=R^k[(-1)^{k+1}\dot{\mathrm{X}}^*+\sum_{i=0}^{N}(-1)^i b_{k-i}]
\end{aligned}
\tag{14}
$$

where the components of the N-dimensional vector-function $b(b_1,\ldots,b_N)$ are independent of $\dot{\mathrm{X}}^*, \dot{\mathrm{X}}_{k+1}$, but do depend upon τ^*, τ_{k+1} and the system parameters:

$$
b_j=Rp(\tau_{j+1}-\tau_j)+(1+R)\dot{f}_{j+1}(\tau_{j+1}) \tag{15}
$$

The time intervals spent on the motion along an individual part of the relevant phase trajectory can be found from the following system of nonlinear equations:

$$
\begin{aligned}
&f_{j+1}(\tau_{j+1}^*)+\Delta\tau_{j+1}^*(p\Delta\tau_j/2-\dot{\mathrm{X}}_j^*)=f_j(\tau_0^*)\\
&f_1(\tau^*)-f_N(\tau_N^*)+(\tau^*-\tau_N^*)\left(\frac{p(\tau^*-\tau_N^*)}{2}-\dot{\mathrm{X}}_N^*\right)=0\\
&j=1,2,\ldots,N-1
\end{aligned}
\tag{16}
$$

The local stability of the main periodic regime is determined by the roots of the characteristic equation $\chi(Z)\equiv a_0Z^2+a_1Z+a_2=0$, whose coefficients $a_l, l=0,1,2$ can be found by linearizing the equations of the around the fixed point [Feigin 1994]. After some evaluations, one can derive the following equation

$$
\chi(Z)=\begin{vmatrix} A(Z)\ldots B \\ C(Z)\ldots D(Z) \end{vmatrix}=0 \tag{17}
$$

in which $A(Z)$, $B(Z)$, $C(Z)$, $D(Z)$ are square matrixes whose non-zero elements are given as

$$\begin{aligned} &a_{i,j+1} = (-1)^j (p\Delta\tau_{i+1} - \dot{X}_i^*) + \dot{f}_{i+j}(\tau_{i+j}^*) \\ &a_{N,1} = Z[p(\tau_N^1 - \tau^*) + \dot{X}_N^* - \dot{f}_1(\tau^*)] \qquad (18) \\ &c_{i,i+j} = -R(-1)^j p + (1+R)j\,\ddot{f}_{i+1}(\tau_{i+1}^*) \\ &c_{N,1} = Z(Rp + (1+R)\ddot{f}_1(\tau^*)) \\ &b_{i,i} = \Delta\tau_{i+1}^*, \;\; d_{i,i+j} = (j-1)R - j \qquad (19) \\ &d_{N,1} = -Z, \quad j = 0,1; \;\; i = 1,2,...,N \end{aligned}$$

As an example, let us assume that $N = 2$. The coordinates of the fixed points can be found as follows: assuming a value for ξ (the time interval needed to travel between the two surfaces $x = f_i(\tau)$, i =1,2) and using equations (17)-(19), one can find the corresponding value of μ and the phase. The latter satisfies the following equation:

$$\tan\left(\tau^*\right) = \frac{a(\xi)B(\xi) - c(\xi)A(\xi)}{d(\xi)A(\xi) - b(\xi)B(\xi)} \qquad (20)$$

Then , using μ and $\tan\left(\tau^*\right)$ and employing (14), the impact velocities $\dot{X}_i^*$ can be calculated.

The elements of the matrices $A(Z)$, $B(Z)$ in the case of consideration can be found as

$$\begin{aligned} &a_0 = b_{11}b_{22}, \\ &a_1 = (a_{11}b_{22} + a_{22}b_{11} - a_{21}b_{12}), \qquad (21) \\ &a_2 = (a_{11}a_{22} - a_{12}a_{21}) \end{aligned}$$

where

$$\begin{aligned} &a_{11} = RR_2\mu F_3 + Rp + R^2(\dot{X}^* - -\mu F_4)\xi, \\ &a_{12} = R^2 L_0 \xi, \\ &a_{21} = L[R(\dot{X}^* - \mu F_4)/\xi + R_2\mu F_3 + p] \\ &\quad -\dot{X}_1^* - \mu F_4, \\ &a_{22} = LRL_0/\xi, \\ &b_{11} = (p\xi - \dot{X}^* + +\mu F_4)/\xi, \qquad (22) \\ &b_{22} = -Lp - L_1, \\ &b_{12} = -[(p\xi + +L_0)/\xi + Rp + R_2\mu\cos\tau_0^*], \\ &L_i = \mu\sin\tau_0^* - \dot{X}_i^*, \\ &F_3 = \gamma\cos(\tau_0^* + \xi - \varphi), \\ &F_4 = \gamma\sin(\tau_0^* + \xi - \varphi) \end{aligned}$$

Using the above expressions, the domains of existence and stability of the main periodic regime can be found in the parameter space. It has been done numerically and the calculations showed, that with increasing ε, φ the stability domains widen in μ, whereas with increasing R move towards large values of p that correspond to lower frequencies. It was also shown, that the values $\tau_1^*, \dot{X}_1^*$ depend only marginally on the frequency parameter p, whereas the phase does depend on it strongly. These results may be used for the preliminary mechanism tuning.

It is possible to show, that for $R = 0$ we obtain the point mapping of a circle into itself. To elaborate on this, let us examine point mapping

$$T\tau = \tau + f(\tau,\alpha,\beta), \qquad 0 < \tau < 1, \qquad (23)$$

in which a periodic nonlinear function with the period 1

$$0 < f(\tau,\alpha,\beta) < K(\alpha,\beta), \;\; K(\alpha,\beta) > 1 \qquad (24)$$

satisfies the conditions specified below.

A. The derivative $f_\tau(\tau,\alpha,\beta)$ exists everywhere (except maybe $\tau = a < 1$), and also one-sided derivatives existence is assumed, as in the points $\tau = 0 \vee 1$.

B.

$$\begin{aligned} &f_\tau(\tau,\alpha,\beta) < 0, \quad 0 < \tau \le b < a; \\ &f_\tau(\tau,\alpha,\beta) > 0, \quad a < \tau < 1; \qquad (25) \\ &f_{\tau\tau}(\tau,\alpha,\beta) > 0, \;\; 0 < \tau < a. \end{aligned}$$

C.

$$f_\tau(0,\alpha,\beta) > -2 \qquad (26)$$

As the function $f(\tau,\alpha,\beta)$ is non-monotonous, the mapping (23) over $1 < f(\tau,\alpha,\beta) < 2$ may have one-rotating $\overline{\tau} = \tau + 1$ fixed points $\tau^* < a$ and $\tau^{**} > a$, the first of which under the conditions B and C is always stable, whereas the second is always unstable.

Further, if it is not explicitly specified otherwise, we shall consider the case $1 < f(0,\alpha,\beta) < 2$. Let us note, that mapping (23) is monotonous when $f_\tau(0,\alpha,\beta) > -1$ (condition B) and, because $f_{\tau\tau}(\tau,\alpha,\beta) > 0, (0 < \tau < a)$, the derivative $f_\tau(\tau,\alpha,\beta)$ achieves its minimum value at the point $\tau = 0$. The equation $f_\tau(0,\alpha,\beta) = -1$ defines the boundary Γ_m of T_τ transformation monotonous character in the parameters plane α, β. Let us designate as G_m the parameters region α, β where T_τ is monotonous.

The coordinates of the ordinary fixed points τ^*, τ^{**} of point mapping (23) are determined from the condition $f(\tau, \alpha, \beta) = 1$, and the boundaries of its existence are defined by the conditions:

$$\begin{aligned} &\min f(\tau, \alpha, \beta) = 1 \\ &\max f(\tau, \alpha, \beta) = 1 \\ &\tau \in (0,1) \end{aligned} \tag{27}$$

It is also obvious that the equations $f(a, \alpha, \beta) = 1$, $f(0, \alpha, \beta) = 1$ define the boundaries of the appearance and disappearance of the above fixed points. Let us designate as G the region of existence of the fixed points of mapping (23) that satisfy equation (27).

Using the above-specified properties of function $f(\tau, \alpha, \beta)$ and mapping T_τ, the following can be shown:

1. When $\overline{\tau}(0) < \tau^{**}$ $\overline{\tau}(0) = f(0, \alpha, \beta) - 1$, in spite of a non-monotonous character of the function T_τ, all the points of the segment, with the exception of τ^{**}, are transformed into the stable point τ^*.
2. When $\overline{\tau}(0) > \tau^{**}$ the countable number of single segment points are transformed into non-stable fixed point.
3. The equation $\overline{\tau}(0) = \tau^{**}$ in the parameter plane defines the bifurcation curve

$$\tau^{**} = f(0, \alpha, \beta) - 1 \tag{28}$$

Distinguishing from G the region of stability "in large". The value of τ^* in (28) can be found from $f(\tau^{**}, \alpha, \beta) = 1$, where $a < \tau^{**} < 1$.

Let us show, that the boundary (28) that we will designate as Γ_* is non-attainable. It means that in any neighborhood of this boundary, there are bifurcation boundaries corresponding to the appearance of the point mapping T^n fixed points.

Let us define $\tau = \tau_1^0$ from the equation $0 = T\tau_1^0$, that is $T^{-1}(0) = \tau_1^0$ (under the A, B, C conditions the point mapping properties τ_1^0 are defined uniquely).

Let us look for the condition of existance of the reverse mappings $T^{-n}(\tau_1^0)$ with the values τ in the range (τ^{**}, τ_1^0). As the function $\overline{\tau}$ is monotonous in this interval, the condition $T^{-n}(\tau_1^0)$ is defined uniquely.

Let us mark $T^{-n}(0) = \tau_n^0$. From this definition it follows, that $\tau^{**} < \tau_n^0 < \tau_1^0$, and

$$\begin{aligned} &T^{-(n-1)}(\tau_1^0) = T^{-n}(0), \\ &T^{-1}(\tau_n^0) = \tau_{n+1}^0, \\ &T(\tau_n^0) = \tau_{n-1}^0, \\ &\lim \tau_n^0 = \tau^{**}, n \to \infty \end{aligned} \tag{29}$$

Using equations (29) and the A, B, C conditions it will be shown that the following lemma is correct.

Lemma. If $T(0) = \tau_{n-1}^0, n \ge 1$, then

$$\begin{aligned} &T^n(\tau_i^0) = \tau_i^0, \\ &i = 0,1,\ldots,n-1, \tau_0^0 = 0 \end{aligned} \tag{30}$$

Consequence. The condition $T(0) = \tau_{n-1}^0$ defines on the parameter plane a bifurcation boundary that corresponds to appearance of n fixed points $\tau = \tau_i^0, i = 0,1,\ldots,n-1$ for point mapping $T^n(\tau)$, that is the cycle of *n*-divisible points of the mapping $T\tau$.

The bifurcation curve, corresponding to the appearance of the n fixed points of mapping $T^n(\tau)$ in the parameter plane we will designate as Γ_n. The equation of this curve is

$$\tau_{n-1}^0 = f(0, \alpha, \beta) - 1 \tag{31}$$

Comparing the bifurcation curves Γ_n with the curve Γ_*, we prove the lemma.

The following useful theorem can also be formulated.

Theorem. For each $\varepsilon > 0$ there exists an integer *N*, such that for all *n*>*N* the bifurcation curves in the parameter α, β plane are located in ε-neighborhood of Γ_*.

References

Bobylev, L. M. (1980) *Soil Densification of Inverse Fillings in Uneasy Conditions of Construction*, Moscow, Stroyizdat (in Russian).

Feigin, M.I. (1994) *Forced Vibrations of Systems with Discontinuous Nonlinearities*, Moscow, Nauka, (in Russian).