ESTIMATES OF TRAJECTORY TUBES IN CONTROL PROBLEMS UNDER UNCERTAINTY

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Abstract
The paper deals with the problems of control and state estimation for nonlinear dynamical control system described by differential equations with unknown but bounded initial states. The nonlinear function in the right-hand part of a differential system is assumed to be of quadratic type with respect to state variable. Basing on the well-known results of ellipsoidal calculus developed for linear uncertain systems we present the modified state estimation approaches which use the special structure of the dynamical system.

Key words
Reachable sets, trajectory tubes, ellipsoidal estimates.

1 Introduction

The topics of this paper come from the control theory for systems with unknown but bounded uncertainties related to the case of set-membership description of uncertainties which are taken to be unknown but bounded with given bounds (e.g., the model may contain unpredictable errors without their statistical description) [Krasovskii and Subbotin, 1988; Kurzhanski, 1977; Kostousova and Kurzhanski, 1996; Kurzhanski and Veliov, 1990; Kurzhanski and Filippova, 1993; Kurzhanski and Valyi, 1997; Chernousko, 1994; Filippova, 2005; Filippova and Vzdornova, 2005].

The motivations for these studies come from applied areas ranged from engineering problems in physics to economics as well as to ecological modelling. The paper presents recent results in the theory of tubes of solutions (trajectory tubes) to differential control systems modelled by nonlinear differential inclusions with uncertain parameters or functions.

We will start by introducing the following basic notations. Let R^n be the n–dimensional Euclidean space and $x'y$ be the usual inner product of $x, y \in R^n$ with the prime as a transpose, with $\| x \| = (x'x)^{1/2}$. Denote comp R^n to be the variety of all compact subsets $A \subseteq R^n$ and conv R^n to be the variety of all compact convex subsets $A \subseteq R^n$.

Consider the control system described by the ordinary differential equation

$$\dot{x} = f(t, x, u(t)), \;\; t \in [t_0, T] \qquad (1)$$

with function $f : T \times R^n \times R^n \to R^m$ measurable in t and continuous in other variables. Here x stands for the state vector, t stands for time and control $u(\cdot)$ is a measurable function satisfying the constraints

$$u(\cdot) \in U = \{u(\cdot) : \; u(t) \in U_0, \;\; t \in [t_0, T]\}, \qquad (2)$$

where $U_0 \in \text{comp}R^m$. Let us assume that the initial condition $x(t_0)$ to the system (1) is unknown but bounded

$$x(t_0) = x_0, \;\; x_0 \in X_0 \in \text{comp}R^n. \qquad (3)$$

Let absolutely continuous function $x(t) = x(t, u(\cdot), t_0, x_0)$ be a solution to (1) with initial state x_0 satisfying (3) and with control function $u(t)$ satisfying (2). The differential system (1)–(3) is studied here in the framework of the theory of uncertain dynamical systems (differential inclusions [Filippov, 1985; Aubin and Frankowska, 1990]) through the techniques of trajectory tubes [Kurzhanski and Filippova, 1993]

$$X(\cdot) = \bigcup \{ \, x(\cdot) = x(\cdot, u(\cdot), t_0, x_0) \mid \\ x_0 \in X_0, \; u(\cdot) \in U \, \} \qquad (4)$$

which combine all solutions $x(\cdot, u(\cdot), t_0, x_0)$ to (1)-(3).

One of the main problems of the theory of uncertain systems consists in describing and estimating the trajectory tube $X(\cdot)$ of the nonlinear system (1)- (3). The point of special interest is to find the t – cross-section $X(t)$ of $X(\cdot)$ which is actually the attainability domain (reachable set) of the control system (1)–(3) at the instant t. The set $X(t)$ may be considered also as the set-valued estimate of the unknown state $x(t)$ of the uncertain dynamical system if we will treat the control functions in (1)- (3) as unknown but bounded disturbances.

Let us mention here the well-known result [Filippov, 1985] from the theory of differential inclusion that the trajectory tube $X(\cdot)$ coincides with the set of all solutions $\{x(\cdot) = x(\cdot, t_0, x_0)\}$ to the following differential inclusions

$$\dot{x} \in F(t,x) = \bigcup \{f(t,x,u) \mid u \in U_0\}, \quad t_0 \le t \le T, \tag{5}$$

with the initial condition similar to (3)

$$x(t_0) = x_0, \;\; x_0 \in X_0. \tag{6}$$

So we will use further the same notation $X(\cdot)$ for both trajectory tubes either for the control system (1)–(3) or for the differential inclusion (5)–(6).

It should be noted that the exact description of reachable sets $X(t)$ of a control system is a difficult problem even in the case of linear dynamics [Kurzhanski, 1977]. The estimation theory and related algorithms basing on ideas of construction outer and inner set-valued estimates of reachable sets have been developed in [Kurzhanski and Valyi, 1997; Chernousko, 1994] for linear control systems.

In this paper the modified state estimation approaches which use the special quadratic structure of nonlinearity of studied control system and use also the advantages of ellipsoidal calculus [Kurzhanski and Valyi, 1997; Chernousko, 1994] are presented.

The special type of nonlinearity chosen here for investigations is motivated by two reasons. First, we have found that it is very convenient in the theoretical analysis to correlate the nonlinear quadratic structure of system dynamics with the ellipsoidal assumptions on uncertain system data and therefore to extend the scope of ellipsoidal analysis for nonlinear control problems. Second, there exist some applied models which may be described by such nonlinear dynamical systems (e.g., [Apreutesei, 2009]). Therefore researches in this direction may present interest for both the theory and applications.

We develop here new ellipsoidal techniques related to constructing external and internal set-valued estimates of reachable sets and trajectory tubes of the nonlinear system. Some estimation algorithms basing on combination of discrete-time versions of evolution funnel equations and ellipsoidal calculus [Kurzhanski and Valyi, 1997; Chernousko, 1994] are given. Algorithms and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented.

The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

2 Problem Statement

Let us describe the modified state estimation approaches for the special class of nonlinear uncertain control systems.

Consider the case when the right-hand side $f(t, x, u(t))$ in (1) is quadratic in state variable x and is linear in control variable u. So we will study the problems of control and state estimation for a dynamical control system of the following type

$$\dot{x}(t) = A(t)x(t) + g(x(t)) + G(t)u(t), \quad t_0 \le t \le T, \tag{7}$$

with unknown but bounded initial condition

$$x(t_0) = x_0, \;\; x_0 \in X_0, \tag{8}$$

and with control constraint

$$u(t) \in U_0, \;\; t \in [t_0, T]. \tag{9}$$

Here matrices $A(t)$ and $G(t)$ (of dimensions $n \times n$ and $n \times m$, respectively) are assumed to be continuous on $t \in [t_0, T]$, X_0 and U_0 are compact and convex. The nonlinear n-vector function $g(x)$ in (7) is assumed to be of quadratic type

$$g(x) = (g_1(x), \dots, g_n(x)), \quad g_i(x) = x' B_i x, \; i = 1, \dots, n, \tag{10}$$

where B_i is a constant $n \times n$ - matrix ($i = 1, \dots, n$).

Consider the following differential inclusion [Filippov, 1985] related to (7)–(9) (with $P(t) = G(t)U_0$)

$$\dot{x}(t) \in A(t)x(t) + f(x(t)) + P(t), \quad t \in [t_0, T], \;\; x(t_0) = x_0 \in X_0. \tag{11}$$

We introduce here the following additional notations. Denote as $B(a, r)$ the ball in R^n, $B(a, r) = \{x \in R^n : \| x - a \| \le r\}$, I is the identity $n \times n$-matrix. Denote by $E(a, Q)$ the ellipsoid in R^n,

$$E(a, Q) = \{x \in R^n : \; (Q^{-1}(x - a), (x - a)) \le 1\}$$

with a center $a \in R^n$ and a symmetric positive definite $n \times n$–matrix Q.

For any $n \times n$-matrix Q denote its track as $\mathrm{Tr}\, Q$ and its determinant as $|Q|$. Denote as $h(A, B)$ the Hausdorff distance for $A,\ B \subseteq R^n$, $h(A, B) = \max\{h^+(A, B),\ h^-(A, B)\}$, with $h^+(A, B)$ and $h^-(A, B)$ being the Hausdorff semidistances between A and B, $h^+(A, B) = \sup\{d(x, B) \mid x \in A\}$, $h^-(A, B) = h^+(B, A)$, $d(x, A) = \inf\ \{\|x - y\| \mid y \in A\}$.

Assume now that $P(t) = E(a, Q)$ in (11), matrices B_i $(i = 1, ..., n)$ are symmetric and positive definite, $A(t) \equiv A$. We may assume that all trajectories of the system (11) belong to a bounded domain $D = \{x \in R^n : \| x \| \leq K\}$ where the existence of such constant $K > 0$ follows from classical theorems of the theory of differential equations and differential inclusions [Filippov, 1985].

The main problem is to construct external and internal set-valued estimates of reachable sets $X(t)$ of the nonlinear system (7)–(9). The approach presented here uses the techniques of ellipsoidal calculus developed for linear control systems. It should be noted that external ellipsoidal approximations of trajectory tubes may be chosen in various ways and several minimization criteria are well-known. We consider here the ellipsoidal techniques related to construction of external estimates with minimal volume (details of this approach and motivations for linear control systems may be found in [Chernousko, 1994; Kurzhanski and Valyi, 1997]).

3 Auxiliary Results

The approach discussed here is related to evolution equations of the funnel type that describe the dynamics of set–valued system states $X(t)$ of the differential inclusion (5)–(6). The basic assumptions on set–valued map $F(t, x)$ for the following results to be true may be found in [Kurzhanski and Filippova, 1993; Panasyuk, 1990; Wolenski, 1990].

3.1 Funnel Equations

Let us consider the "equation" for set-valued function $X(t)$ (the funnel equation),

$$\lim_{\sigma \to +0} \sigma^{-1} h(\ X(t+\sigma),\ \bigcup_{x \in X(t)} (x + \sigma\ F(t, x))\ ) = 0,\ \ t \in [t_0, T], \tag{12}$$

with the initial set-valued condition

$$X(t_0) = X_0. \tag{13}$$

Theorem 1. ([Panasyuk, 1990; Wolenski, 1990]) *The trajectory tube $X(t)$ of the system (5)–(6) is the unique set-valued solution to the evolution equation (12)-(13).*

Other versions of funnel equations may be considered by substituting the Hausdorff distance h for a semidistance h^+ [Kurzhanski and Valyi, 1997]. The solution to the h^+-versions of the evolution equation may be not unique and the "maximal" one (with respect to inclusion) is studied in this case. Mention here also the second order analogies of funnel equations for differential inclusions and control systems based on ideas of Runge-Kutta scheme [Dontchev and Lempio, 1992; Veliov, 1989; Veliov, 1992]. Discrete approximations for differential inclusions based on set-valued Euler's method were developed in [Dontchev and Lempio, 1992; Dontchev and Farkhi, 1989].

Let us discuss the estimation approach based on techniques of evolution funnel equations.

Consider the following uncertain system

$$\begin{gathered} \dot{x} \in Ax + h(\tilde{f}(x))d + E(\hat{a}, \hat{Q}), \\ x_0 \in X_0 = E(a_0, Q_0),\ t_0 \leq t \leq T, \end{gathered} \tag{14}$$

where $x \in R^n$, $\|x\| \leq K$, d is a given n-vector and a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$ with a symmetric and positive definite matrix B, a scalar nonnegative function $h(\cdot)$ is continuously differentiable and bounded.

In this case the funnel equation (12)-(13) of Theorem 1 takes the following form

$$\lim_{\sigma \to +0} \sigma^{-1} h(\ X(t+\sigma),\ \bigcup_{x \in X(t)} ((I + \sigma A)x + \sigma h(\tilde{f}(x))d + \sigma E(\hat{a}, \hat{Q}))\ ) = 0,\ \ t \in [t_0, T], \tag{15}$$

with the initial set-valued condition

$$X(t_0) = X_0. \tag{16}$$

Note that the direct use of funnel equations (15)-(16) for finding trajectory tubes $X(t)$ is very difficult because it takes a huge amount of computations based on grid techniques.

Example 1. We illustrate the application of this approach and find the trajectory tube $X(t; t_0, X_0)$ of the following nonlinear system

$$\begin{cases} \dot{x}_1 = \ x_1, \\ \dot{x}_2 = \ 2x_2 + x_1^2 + 0.125x_2^2 \end{cases},\ \ 0 \leq t \leq T. \tag{17}$$

The uncertainty in the system is defined by uncertain initial states x_0 that are unknown but belong to the following ellipsoid

$$X_0 = E(0, Q_0),\ Q_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

The results of computer simulations for this system are shown at Fig.1. Here the precise trajectory tube $X(t) = X(t, t_0, X_0)$ is shown in black color and the blue grid corresponds to the Euler set-valued approximation for solution of the funnel equation (15)-(16).

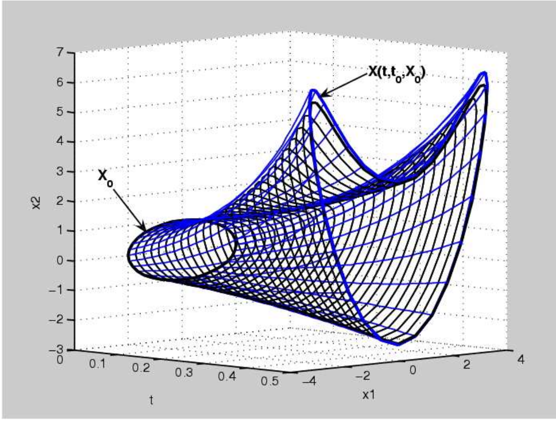

Figure 1. The precise trajectory tube $X(t)$ of system (17) and its discrete approximation through funnel equations (here $0 \leq t \leq 0.5$).

3.2 Ellipsoidal Techniques

For a simpler case of system nonlinearities when $h(\tilde{f}(x)) = \tilde{f}(x)$ (and in this case $h(\cdot)$ is not bounded) we presented the ellipsoidal techniques in [Filippova and Berezina, 2008; Filippova, 2009]. The approach was based on the following theorems which give easy computational tools to find external and internal ellipsoidal estimates of $X(t)$ by step-by-step procedures.

Theorem 2. ([Filippova, 2009]) *Let $X_0 = E(a, k^2B^{-1})$ with $k \neq 0$. Then for the trajectory tube $X(t)$ of the system (14) and for all $\sigma > 0$ the following inclusion holds*

$$X(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0,1), \ \lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0, \tag{19}$$

where

$$a^+(\sigma) = a(\sigma) + \sigma\hat{a}, \quad Q^+(\sigma) = (p^{-1}+1)Q(\sigma) + (p+1)\sigma^2\hat{Q}, \tag{20}$$

$$a(\sigma) = a + \sigma(Aa + a'Ba \cdot d + k^2 d), \quad Q(\sigma) = k^2(I + \sigma R)B^{-1}(I + \sigma R)', \quad R = A + 2da'B, \tag{21}$$

p is the unique positive solution of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \lambda_i} = \frac{n}{p(p+1)}, \tag{22}$$

and $\lambda_i \geq 0$ are the roots of the equation $|Q(\sigma) - \lambda\sigma^2\hat{Q}| = 0$.

Theorem 3. ([Filippova, 2009]) *Let $X_0 = E(a, k^2B^{-1})$ with $k \neq 0$. Then for the trajectory tube $X(t)$ of the system (14) and for all $\sigma > 0$ the following inclusion holds*

$$E(a^-(\sigma), Q^-(\sigma)) \subseteq X(t_0 + \sigma) + o(\sigma)B(0,1), \ \lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0 \tag{23}$$

where

$$a^-(\sigma) = a(\sigma) + \sigma\hat{a}, \quad Q^-(\sigma) = Q(\sigma) + \sigma^2\hat{Q} + 2\sigma Q(\sigma)^{1/2}\hat{Q}(\sigma)^{1/2}Q(\sigma)^{1/2}, \quad \hat{Q}(\sigma) = Q(\sigma)^{-1/2}\hat{Q}Q(\sigma)^{-1/2} \tag{24}$$

and $a(\sigma)$, $Q(\sigma)$ are defined in (21).

3.3 Related Estimating Algorithms

Basing on these results the algorithms which give the external and internal ellipsoidal estimates of the trajectory tube $X(t)$ of the system (14) with $h(\tilde{f}(x)) = \tilde{f}(x)$ may be constructed.

Consider first the algorithm which provides the external ellipsoidal estimate of the reachable set $X(T)$.

Algorithm 1. *Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ ($i = 1, \ldots, m$), $h = (T - t_0)/m$, $t_m = T$.*

Repeat consequently the following steps, at the end of the process we will get the external estimate $E(a^+(t), Q^+(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

1. *Given $X_0 = E(a, k_0^2B^{-1})$ with $k_0 \neq 0$, define $X_1 = E(a_1, Q_1)$ from Theorem 2 for $a_1 = a^+(\sigma)$, $Q_1 = Q^+(\sigma)$, $\sigma = h$.*
2. *Define k_1^2 as the maximal eigenvalue of the matrix $B^{1/2}Q_1B^{1/2}$.*
3. *Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2B^{-1})$ as the initial ellipsoid at instant t_1.*

Remark. Note that k_1 defined at the second step of the algorithm is the smallest positive constant such that $E(a_1, Q_1) \subset E(a_1, k_1^2B^{-1})$.

The next algorithm gives the internal (inner) ellipsoidal bound of the reachable set $X(T)$.

Algorithm 2. *Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ ($i = 1, \ldots, m$), $h = (T - t_0)/m$, $t_m = T$.*

Repeat consequently the following steps, at the end of the process we will get the external estimate $E(a^-(t), Q^-(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

1. *Given $X_0 = E(a, k_0^2 B^{-1})$ with $k_0 \neq 0$, define $X_1 = E(a_1, Q_1)$ from Theorem 3 for $a_1 = a^-(\sigma)$, $Q_1 = Q^-(\sigma)$, $\sigma = h$.*
2. *Define k_1^2 as the minimal eigenvalue of the matrix $B^{1/2} Q_1 B^{1/2}$.*
3. *Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2 B^{-1})$ as the initial ellipsoid at instant t_1.*

Remark. Note that k_1 defined at the second step of the algorithm is the largest positive constant such that $E(a_1, k_1^2 B^{-1}) \subset E(a_1, Q_1)$.

The following example illustrates these results.

Example 2. Consider the following system

$$\begin{cases} \dot{x}_1 = \ x_1 + x_1^2 + x_2^2 + u_1, \\ \dot{x}_2 = \ - x_2 + u_2, \end{cases} , \ 0 \leq t \leq T. \tag{25}$$

Here $t_0 = 0$, $T = 0.3$, $a_0 = \hat{a} = (0, 0)$, $Q_0 = \hat{Q} = I$. Results of computer simulations are shown at Fig. 2 where twelve iterations of both algorithms have been undertaken.

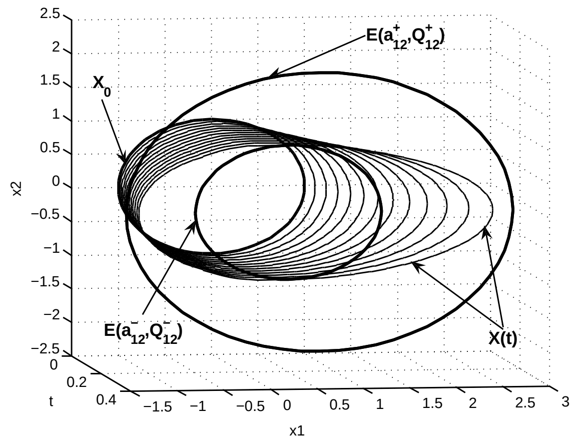

Figure 2. Trajectory tube $X(t)$ and its external and internal ellipsoidal estimates $E(a^+(t), Q^+(t)), E(a^-(t), Q^-(t))$.

It should be noted however that these outer and inner estimates $E(a^+(t), Q^+(t))$, $E(a^-(t), Q^-(t))$ of the tube $X(t)$ may be consistent only for small time intervals $[t_0, T]$ because trajectories of the system having quadratic nonlinearity in dynamics may tend to infinity when time goes to a finite instant.

4 Main Result: External Estimate of Trajectory Tubes of a Control System with Bounded Nonlinearity

Consider now the common case, namely, consider the following uncertain system

$$\begin{aligned} &\dot{x} \in Ax + h(\tilde{f}(x))d + E(\hat{a}, \hat{Q}), \\ &x_0 \in X_0 = E(a_0, Q_0), \ t_0 \leq t \leq T, \end{aligned} \tag{26}$$

where $x \in R^n$, $\|x\| \leq K$, d is a given n-vector and a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$ with a symmetric and positive definite matrix B, a scalar nonnegative function $h(\cdot)$ is continuously differentiable and bounded. Denote by $L > 0$ the Lipschitz constant of the function $h(x)$ for $x \in R^n$, $\|x\| \leq K$.

The following new result presents an easy computational tool to find external estimates of $X(t)$ by step-by-step procedures.

Theorem 4. *Let $X_0 = E(a, k^2 B^{-1})$ with $k \neq 0$. Then for all $\sigma > 0$ the following inclusion holds*

$$\begin{aligned} &X(t_0 + \sigma, t_0, X_0) \subseteq E(a(\sigma), Q(\sigma)) + \\ &\sigma(rB(0,1) + E(\hat{a}, \hat{Q})) + o(\sigma)B(0,1), \\ &\lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0 \end{aligned} \tag{27}$$

where

$$a(\sigma) = a + \sigma(Aa + h(a'Ba + k^2) \cdot d), \tag{28}$$

$$Q(\sigma) = k^2 (I + \sigma A) B^{-1} (I + \sigma A)', \tag{29}$$

$$r = 2Lk||a'B^{1/2}|| \cdot ||d||. \tag{30}$$

Proof. The proof of this result may be done along the lines of the proof of Theorem 3 [Filippova and Berezina, 2008] with some minor modifications due to the presence here of a nonlinear Lipschitz function h.

Basing on this theorem we may formulate the following new scheme that gives the external estimate of trajectory tube $X(t)$ of the system (26) with given accuracy.

Algorithm 3. *Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ ($i = 1, \ldots, m$), $h = (T - t_0)/m$, $t_m = T$.*

1. *Given $X_0 = E(a, k_0^2 B^{-1})$ with $k_0 \neq 0$, define the ellipsoid $X_1 = E(a_1, Q_1)$ such that*

$$\begin{aligned} &E(a(\sigma), Q(\sigma)) + \sigma(rB(0,1) + \\ &+ E(\hat{a}, \hat{Q})) \subseteq E(a_1, Q_1) = X_1 \end{aligned}$$

for $a(\sigma)$, $Q(\sigma)$, r defined in Theorem 4 with $\sigma = h$.

2. *Find the smallest constant k_1 such that*

$$E(a_1, Q_1) \subset \tilde{X}_1 = E(a_1, k_1^2 B^{-1}),$$

and it is not difficult to prove that k_1^2 is the maximal eigenvalue of the matrix $B^{1/2} Q_1 B^{1/2}$.

3. *Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2 B^{-1})$ as the initial ellipsoid at instant t_1.*
4. *Next steps continue iterations 1-3. At the end of the process we will get the external estimate $E(a(t), Q(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.*

Remark. Here we have overcome the difficulties associated with the use of algorithms 1-2 for obtaining multi-valued estimates of trajectory tubes of systems with quadratic nonlinearity caused by the need to consider only small time intervals. Basing on the results of ellipsoidal calculus and using above techniques we present new state estimation approach for the system of type (26) which may be used for estimation of the reachable sets at any finite time interval.

5 Conclusion

The paper deals with the problems of state estimation for a dynamical control system described by differential inclusions with unknown but bounded initial state.

The solution to the differential system is studied through the techniques of trajectory tubes with their cross-sections $X(t)$ being the reachable sets at instant t to control system.

Basing on the results of ellipsoidal calculus developed for linear uncertain systems we present the modified state estimation approaches which use the special nonlinear structure of the control system and simplify calculations.

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