STABILITY ANALYSIS OF GYROSCOPIC SYSTEMS WITH DISCRETE AND DISTRIBUTED DELAYS

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Abstract
This paper provides stability analysis results for a linear mechanical system with a large parameter at the vector of gyroscopic forces and with delay in positional forces. Both cases of discrete and distributed delay are studied. Using the decomposition method and Lyapunov–Krasovskii functionals, conditions are found under which delay does not disturb the asymptotic stability of the considered system. The effectiveness of the obtained results is illustrated by a simulation example.

1 Introduction

The basic approach to the analysis of dynamics of complex and network systems is the decomposition method [Bullo, Cortes and Martinez, 2009; Frolov, Koronovskii, Makarov, Maksimenko, Goremyko and Hramov, 2017; Lakshmikantham, Leela and Martynyuk, 1989; Matrosov, 2001; Proskurnikov and Granichin, 2018; Siljak, 1991; Zubov, 1970]. The method is effectively used in various forms for the investigation of mechanical systems, see, for instance, [Andrievsky and Boikov, 2017; Alyshev, Dudarenko and Melnikov, 2018; Matrosov, 2001; Merkin, 1974] and the bibliography therein.

In [Zubov, 1970] and [Merkin, 1974], original approaches to the decomposition of linear time-invariant mechanical systems with a large parameter at the vectors of velocity and gyroscopic forces, respectively, have been proposed. The results of [Zubov, 1970] and [Merkin, 1974] permit us to reduce the stability problem for a considered second order system to that for two independent first order subsystems.

These approaches have got further development in [Kosov, 2005; Aleksandrov, Chen, Kosov and Zhang, 2011; Aleksandrov, Kosov and Chen, 2011; Aleksandrov and Aleksandrova, 2016], and new stability conditions were obtained not only for linear time-invariant systems, but also for systems with nonlinear and nonstationary force fields.

In [Kuptsov, 2000; Aleksandrov, Aleksandrova and Zhabko, 2014], such approaches were applied to the stability analysis of linear gyroscopic systems with a large parameter at the vector of gyroscopic forces and with delay in positional forces. It was proved that if auxiliary first order delay-free subsystems are asymptotically stable, then, for sufficiently large values of the parameter, one can guarantee asymptotic stability of an original second order time-delay system. However, it should be noted that the results of [Kuptsov, 2000; Aleksandrov, Aleksandrova and Zhabko, 2014] are delay-dependent. At the same time, in numerous applications, delay-independent stability conditions are required, see [Gu, Kharitonov and Chen, 2003].

In this paper, we propose another approach to the justification of possibility of decomposition for linear gyroscopic systems with time delay that permits to derive delay-independent stability conditions. In addition, we will show that the approach can be used for the stability investigation of gyroscopic systems with distributed delay, as well.

2 Statement of the Problem

Let motions of a mechanical system be described by the equations

$$A\ddot{x}(t)+(B+hG)\dot{x}(t)+Cx(t)+Dx(t-\tau) = 0. \quad (1)$$

Here $x(t), \dot{x}(t) \in \mathbb{R}^n$ are vectors of generalized coordinates and velocities, respectively; A, B, G, C, D are constant matrices; h is a positive parameter; τ is a constant nonnegative delay.

We assume that matrices A and B are symmetric and positive definite, and matrix G is skew-symmetric. Thus, the considered mechanical system is influenced by the dissipative forces $-B\dot{x}(t)$, gyroscopic forces $-hG\dot{x}(t)$ and positional forces $-Cx(t) - Dx(t - \tau)$.

Equations of the form (1) are widely used as linear approximations of models of gyroscopic systems (see [Merkin, 1974; Zubov, 1970]). As examples of such models, a gyrovertic with radial correction and force gyroscopic horizon with delay in feedback laws may be considered [Merkin, 1974]. In these systems, the parameter h can be treated as a frequency of a gyroscope rotation.

Let initial functions for solutions of (1) belong to the space $C^1([-\tau, 0], \mathbb{R}^n)$ of continuously differentiable functions $\varphi(\theta) : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_\tau = \max_{\theta \in [-\tau, 0]} (\|\varphi(\theta)\| + \|\dot{\varphi}(\theta)\|).$$

Here $\|\cdot\|$ denotes the Euclidean norm of a vector. Let x_t stand for the restriction of a solution $x(t)$ of (1) to the segment $[t - \tau, t]$, i.e., $x_t : \theta \to x(t + \theta)$, $\theta \in [-\tau, 0]$.

In what follows we assume that n is an even number and $\det G \neq 0$.

We will look for asymptotic stability conditions for the system (1).

In [Kuptsov, 2000; Aleksandrov, Aleksandrova and Zhabko, 2014] it was proved that if the auxiliary delay-free subsystem

$$G\dot{y}(t) + (C + D)y(t) = 0 \qquad (2)$$

is asymptotically stable, then, for every $\tau \geq 0$, there exists $h_0 > 0$ such that the system (1) is asymptotically stable for any $h \geq h_0$. To derive such a result, in [Kuptsov, 2000], the first Lyapunov method was used, whereas, in [Aleksandrov, Aleksandrova and Zhabko, 2014], the Lyapunov direct method and the Razumikhin approach were applied. However, it should be noted that lower bounds for admissible values of parameter h obtained in [Kuptsov, 2000] and [Aleksandrov, Aleksandrova and Zhabko, 2014] depend on the magnitude of delay.

In the present paper, we will use another approach to the stability analysis of the system (1) that permits us to derive delay-independent asymptotic stability conditions. The approach is based on the decomposition method and a special construction of Lyapunov–Krasovskii functional for (1).

In addition, we will show that with the aid of the approach, sufficient conditions of asymptotic stability can be obtained for linear gyroscopic systems with distributed delay.

3 A System with Discrete Delay

Instead of (2), construct the auxiliary time-delay subsystem

$$hG\dot{y}(t) + Cy(t) + Dy(t - \tau) = 0. \qquad (3)$$

Denote $M = -G^{-1}C$, $N = -G^{-1}D$.

Theorem 1. *Assume that there exist constant symmetric positive definite matrices P and Q such that the matrix*

$$\begin{pmatrix} M^\top P + PM + Q & PN \\ N^\top P & -Q \end{pmatrix}$$

is negative definite. Then one can choose a number $h_0 > 0$ such that if $h \geq h_0$, then the system (1) is asymptotically stable for any nonnegative delay τ.

Proof. Let

$$y(t) = x(t) + (B + hG)^{-1} A\dot{x}(t), \quad z(t) = \dot{x}(t). \quad (4)$$

The substitution (4) transforms (1) to the system

$$\dot{y}(t) = \tfrac{1}{h} My(t) + \tfrac{1}{h} Ny(t - \tau)$$
$$+ \tfrac{1}{h}(B + hG)^{-1} BG^{-1}(Cy(t) + Dy(t - \tau))$$
$$+ (B + hG)^{-1} C(B + hG)^{-1} Az(t)$$
$$+ (B + hG)^{-1} D(B + hG)^{-1} Az(t - \tau), \quad (5)$$
$$A\dot{z}(t) = -(B + hG)z(t) - Cy(t)$$
$$+ C(B + hG)^{-1} Az(t) - Dy(t - \tau)$$
$$+ D(B + hG)^{-1} Az(t - \tau).$$

The system (5) can be treated as a complex system describing interaction of the subsystem (3) and the subsystem

$$A\dot{z}(t) = -(B + hG)z(t). \qquad (6)$$

It is known [Gu, Kharitonov and Chen, 2003], that, under the conditions of Theorem 1, the subsystem (3) is asymptotically stable for any $\tau \geq 0$, and a Lyapunov–Krasovskii functional for (3) can be chosen in the form

$$V_1(y_t) = hy^\top(t)Py(t) + \int_{t-\tau}^{t} y^\top(s)Qy(s)ds. \quad (7)$$

Moreover, the subsystem (6) is asymptotically stable and admits the Lyapunov function

$$V_2(z) = \frac{1}{2}z^\top Az. \qquad (8)$$

Let

$$V(y_t, z_t) = V_1(y_t) + \frac{1}{h}V_2(z(t)) + \frac{\lambda}{h}\int_{t-\tau}^{t}\|z(s)\|^2 ds,$$

where λ is a positive parameter, and $V_1(y_t)$, $V_2(z)$ are defined by the formulae (7) and (8), respectively. Then

$$a_1\left(h\|y(t)\|^2 + \frac{1}{h}\|z(t)\|^2\right) \leq V(y_t, z_t)$$

$$\leq a_2\left(h\|y(t)\|^2 + \frac{1}{h}\|z(t)\|^2\right.$$

$$\left. + \int_{t-\tau}^{t}\|y(s)\|^2 ds + \frac{\lambda}{h}\int_{t-\tau}^{t}\|z(s)\|^2 ds\right),$$

$$\dot{V}\big|_{(5)} = 2y^\top(t)PMy(t) + 2y^\top(t)PNy(t-\tau)$$

$$+ y^\top(t)Qy(t) - 2y^\top(t-\tau)Qy(t-\tau)$$

$$- \frac{1}{h}z^\top(t)Bz(t) + \frac{\lambda}{h}\|z(t)\|^2 - \frac{\lambda}{h}\|z(t-\tau)\|^2$$

$$+ 2y^\top(t)P(B+hG)^{-1}\left(BG^{-1}Cy(t)\right.$$

$$+ BG^{-1}Dy(t-\tau)) + hC(B+hG)^{-1}Az(t)$$

$$\left. + hD(B+hG)^{-1}Az(t-\tau)\right)$$

$$+ \frac{1}{h}z^\top(t)\left(-Cy(t) + C(B+hG)^{-1}Az(t)\right.$$

$$\left. - Dy(t-\tau) + D(B+hG)^{-1}Az(t-\tau)\right)$$

$$\leq -a_3\left(\|y(t)\|^2 + \|y(t-\tau)\|^2\right) - \frac{1}{h}(a_4 - \lambda)\|z(t)\|^2$$

$$+ \frac{a_5\|y(t)\|}{h}\left(\|y(t)\| + \|y(t-\tau)\| + \|z(t)\| + \|z(t-\tau)\|\right)$$

$$+ \frac{a_6\|z(t)\|}{h}\left(\|y(t)\| + \|y(t-\tau)\| + \frac{1}{h}\|z(t)\|\right.$$

$$\left. + \frac{1}{h}\|z(t-\tau)\|\right) - \frac{\lambda}{h}\|z(t-\tau)\|^2.$$

Here $a_1, a_2, a_3, a_4, a_5, a_6$ are positive constants independent of h, λ and τ.
Let $\lambda = 1/\sqrt{h}$. Then there exists $h_0 > 0$ such that

$$\dot{V}\big|_{(5)} \leq -\frac{a_3}{2}\left(\|y(t)\|^2 + \|y(t-\tau)\|^2\right)$$

$$- \frac{a_4}{2h}\|z(t)\|^2 - \frac{\lambda}{2h}\|z(t-\tau)\|^2$$

for $h \geq h_0$. Hence [Gu, Kharitonov and Chen, 2003], the system (5) is asymptotically stable for $h \geq h_0$ and for any value of delay. Then, from the properties of the transformation (4), it follows delay-independent asymptotic stability of (1). This completes the proof.

4 A System with Distributed Delay
Next, consider the system with distributed delay

$$A\ddot{x}(t) + (B+hG)\dot{x}(t) + Cx(t)$$

$$+ D\int_{t-\tau}^{t} x(s)ds = 0. \qquad (9)$$

Here all notations are the same as for (1).

It is worth noting that systems with distributed delays arise in traffic flow models, in network control systems, in PID controller design and in other engineering applications [Bullo, Cortes and Martinez, 2009; Gu, Kharitonov and Chen, 2003; Fridman, 2014; Solomon and Fridman, 2013]. Some conditions of stability and stabilization of mechanical systems with distributed delays were obtained, for instance, in [Anan'evskii and Kolmanovskii, 1989; Pavlikov, 2007].

Let us show that the approach proposed in the previous section can be used for the stability analysis of the system (9), as well.

In this case, we consider the following auxiliary subsystem:

$$hG\dot{y}(t) + Cy(t) + D \int_{t-\tau}^{t} y(s)ds = 0. \qquad (10)$$

Denote $M = -G^{-1}C$, $N = -G^{-1}D$.

Theorem 2. *Assume that, for a given $\tau \geq 0$, there exist constant symmetric positive definite matrices P and Q such that the matrix*

$$\begin{pmatrix} \frac{1}{\tau}(M^\top P + PM) + Q & PN \\ N^\top P & -Q \end{pmatrix}$$

is negative definite. Then one can choose a number $h_0 > 0$ such that if $h \geq h_0$, then the system (9) is asymptotically stable.

Proof. Using the substitution (4), we arrive at the system

$$\dot{y}(t) = \frac{1}{h}My(t) + \frac{1}{h}Ny(t-\tau)$$
$$+ \frac{1}{h}(B + hG)^{-1}BG^{-1}Cy(t)$$
$$+ \frac{1}{h}(B + hG)^{-1}BG^{-1}D \int_{t-\tau}^{t} y(s)ds$$
$$+ (B + hG)^{-1}C(B + hG)^{-1}Az(t)$$
$$+ (B + hG)^{-1}D(B + hG)^{-1}A \int_{t-\tau}^{t} z(s)ds, \qquad (11)$$
$$A\dot{z}(t) = -(B + hG)z(t) - Cy(t)$$
$$+ C(B + hG)^{-1}Az(t) - D \int_{t-\tau}^{t} y(s)ds$$
$$+ D(B + hG)^{-1}A \int_{t-\tau}^{t} z(s)ds.$$

From the conditions of Theorem 2 it follows (see [Gu, Kharitonov and Chen, 2003; Fridman, 2014]) that the subsystem (10) is asymptotically stable and admits the

Lyapunov–Krasovskii functional

$$\widetilde{V}_1(y_t) = hy^\top(t)Py(t)$$
$$+ \int_{-\tau}^{0} \int_{t+\theta}^{t} y^\top(s)Qy(s)dsd\theta. \qquad (12)$$

Choose a candidate Lyapunov–Krasovskii functional for (11) in the form

$$\widetilde{V}(y_t, z_t) = \widetilde{V}_1(y_t) + \frac{1}{h}V_2(z(t))$$
$$+ \frac{\lambda}{h} \int_{t-\tau}^{t} (s + \tau - t)\|z(s)\|^2 ds,$$

where λ is a positive parameter, and $\widetilde{V}_1(y_t)$, $V_2(z)$ are defined by the formulae (12) and (8), respectively.

We obtain

$$a_1\left(h\|y(t)\|^2 + \frac{1}{h}\|z(t)\|^2\right) \leq \widetilde{V}(y_t, z_t)$$

$$\leq a_2\left(h\|y(t)\|^2 + \frac{1}{h}\|z(t)\|^2\right.$$

$$+ \int_{-\tau}^{0} \int_{t+\theta}^{t} \|y(s)\|^2 ds + \frac{\lambda}{h} \int_{t-\tau}^{t} \|z(s)\|^2 ds\bigg),$$

$$\dot{\widetilde{V}}\Big|_{(11)} \leq -a_3\left(\|y(t)\|^2 + \int_{t-\tau}^{t} \|y(s)\|^2 ds\right)$$

$$- \frac{1}{h}(a_4 - \lambda\tau)\|z(t)\|^2 - \frac{\lambda}{h} \int_{t-\tau}^{t} \|z(s)\|^2 ds$$

$$+ \frac{a_5\|y(t)\|}{h}\left(\|y(t)\| + \int_{t-\tau}^{t} \|y(s)\|ds\right.$$

$$+ \|z(t)\| + \int_{t-\tau}^{t} \|z(s)\|ds\bigg)$$

$$+\frac{a_6\|z(t)\|}{h}\left(\|y(t)\|+\int_{t-\tau}^{t}\|y(s)\|ds\right.$$

$$\left.+\frac{1}{h}\|z(t)\|+\frac{1}{h}\int_{t-\tau}^{t}\|z(s)\|ds\right).$$

Here $a_1, a_2, a_3, a_4, a_5, a_6$ are positive constants independent of h and λ.

If $\lambda = 1/\sqrt{h}$, then the estimate

$$\dot{\tilde{V}}\Big|_{(11)} \leq -\frac{a_3}{2}\left(\|y(t)\|^2 + \int_{t-\tau}^{t}\|y(s)\|^2ds\right)$$

$$-\frac{a_4}{2h}\|z(t)\|^2 - \frac{\lambda}{2h}\int_{t-\tau}^{t}\|z(s)\|^2ds$$

holds for sufficiently large values of h. This completes the proof.

Remark 1. *Unlike Theorem 1, Theorem 2 provides us delay-dependent asymptotic stability conditions.*

5 Results of a Numerical Simulation

To illustrate the effectiveness of the obtained results, consider the system

$$\ddot{x}(t) + \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + h\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\dot{x}(t)$$

$$+ \begin{pmatrix} 1 & -4 \\ 2 & -1 \end{pmatrix}x(t) - \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}x(t-\tau) = 0. \quad (13)$$

Here $x(t), \dot{x}(t) \in \mathbb{R}^2$; h is a positive parameter; τ is a nonnegative delay.

Construct the auxiliary subsystem (3) for (13). We obtain

$$h\dot{y}(t) = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}y(t) + \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}y(t-\tau). \quad (14)$$

With the aid of the results of [Aleksandrov and Mason, 2016], it can be proved that the subsystem (14) admits a Lyapunov–Krasovskii functional of the form (7). Hence (see Theorem 1), there exists a number $h_0 > 0$ such that if $h \geq h_0$, then the system (13) is asymptotically stable for any nonnegative delay τ.
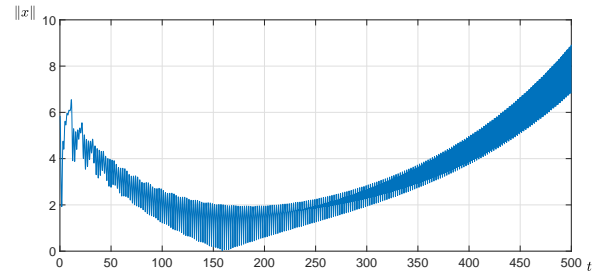

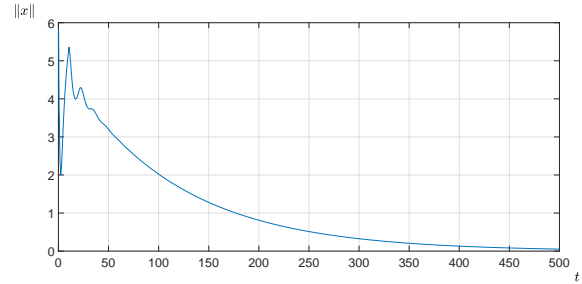Figure 1. The results of simulation for $h = 3.3$

Figure 2. The results of simulation for $h = 8$

For simulation, we choose $\tau = 10$ and $x(t) = (-3,5)^{\top}$ for $t \in [-10, 0]$. In Figs. 1 and 2 the dependence of $\|x\|$ on t is presented.

First, the case is considered where $h = 3.3$. Figure 1 shows that, for this value of h, the system (13) is unstable.

Next, we take $h = 8$ and repeat calculations with the same delay and initial conditions. Figure 2 demonstrates the asymptotic stability of the corresponding system.

6 Conclusion

In this contribution, with the aid of the decomposition method and special constructions of Lyapunov–Krasovskii functionals, sufficient conditions of the asymptotic stability are found for linear gyroscopic systems with discrete and distributed delays. In is worth noting that, for the case of discrete delay, these conditions are delay-independent, whereas, for systems with distributed delay, admissible values of parameter h depend on delay.

We outline some suggestions for further research. First, using the constructed Lyapunov–Krasovskii functionals, estimates of the convergence rate of solutions can be derived for considered systems. Next, the developed approaches can be applied to the stability analysis of mechanical systems with switched force fields. In addition, we believe the approaches in this paper can be useful for the study the consensus problem for multi-agent systems described by double integrators with communication delays.

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