INTEGRODIFFERENTIAL APPROACH TO OPTIMAL CONTROL PROBLEMS OF ELASTIC BEAM MOTIONS

Kostin G.V., Saurin V.V.

Institute for Problems in Mechanics of the Russian Academy of Sciences

Abstract: An approach to modeling and optimization of controlled dynamical systems with distributed elastic and inertial parameters is considered. The general method of integrodifferential relations (IDR) for solving a wide class of boundary value problems is developed and criteria of solution quality are proposed. A numerical algorithm for discrete approximation of controlled motions has been worked out and applied to design the optimal control law moving an elastic system to the terminal position and minimizing the given objective function. The polynomial control for plane motions of a homogeneous cantilever beam is investigated. The optimal control problem of beam transportation from the initial rest position to given terminal states, in which the total mechanical energy of the system reaches its minimal value, is considered.

Keywords: Approximate analysis, Distributed-parameter systems, Optimal control.

1. INTRODUCTION

The elastic properties of the elements of structures affect their dynamical behavior substantially. Some parts of the mechanical systems with distributed parameters may be considered as elastic rods with given stiffness and inertia characteristics. Boundary-value problems of mathematical physics arise in simulation of the motions of these systems. One of the most widespread approaches to the solution of these problems is the method of separation of variables.

The investigation of systems with distributed parameters leads to a wide class of problems for which a large number of approaches is developed. The regular perturbation method (the small parameter method) for the investigation of the dynamics of weakly nonuniform thin rods with arbitrary distributed strain and different boundary-value conditions is proposed by Akulenko and Kostin (1992). Based on the classical Rayleigh–Ritz approach, a numerical-analytic method of fast convergence that allows one to obtain values of the desired quantities and functions with arbitrary bending stiffness and linear density of the rod that are sufficiently precise (Akulenko, *et al.* 1995). In modeling elastic systems, the methods of finite-dimensional approximation, which reduce a boundary value problem for partial differential equations to a system of ordinary differential equations, for example, the decomposition method and the regularization method, are widespread.

In this paper, the method of integro-differential relations, developed by Kostin and Saurin (2006); (2006a,b,c,d), is applied to finding an optimal control for the movement of elastic systems with distributed parameters. The algorithms of optimization for dynamical characteristics of uniform straight thin beam motions are constructed based upon the MIDR. Analysis and comparison of the results obtained by using this method for a polynomial control and a quadratic cost functional are performed.

2. STATEMENT OF THE PROBLEM

Consider the plane controlled motions of a homogeneous rectilinear elastic beam. One end of the beam is free, and the other is clamped on a truck that can move along a horizontal plane (see Fig.1). In the undeformed state, the beam is fixed in a vertical position. The control action on the beam is the horizontal acceleration u of the truck. Initially, the shape of the beam lateral deflection (displacement) w and its relative linear momentum density p are given in a coordinate system tied to the truck moved at the velocity v. The location of the truck is specified by x in a stationary coordinate system; hereinafter, $\dot{x} = v$ and $\dot{v} = u$. Without loss of generality, it can be assumed that the coordinate and velocity of the truck are initially zero.

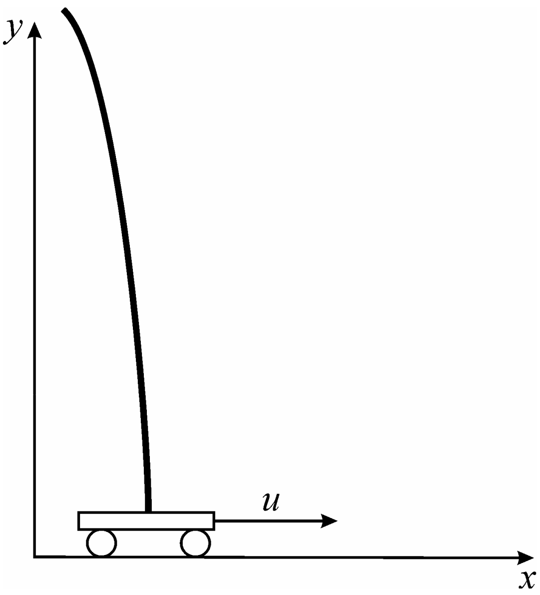

Fig. 1. Beam clamped on a truck

The equations of beam motions have the form

$$\dot{p} + m'' = -\rho u(t),\ y \in (0,l), \tag{1}$$

$$p = \rho\dot{w}, \quad m = EIw'', \quad t \in (0,T), \tag{2}$$

under the boundary conditions at $y = 0, l$

$$\begin{aligned} &w(t,0) = w'(t,0) = 0, \\ &m(t,l) = m'(t,l) = 0; \end{aligned} \tag{3}$$

and the initial conditions at $t = 0$

$$w(0,y) = f(y), \quad p(0,y) = g(y). \tag{4}$$

Here, m is the bending moment in the beam cross section; l and ρ are the length and linear density of the beam, respectively; EI is its flexural rigidity; and T is the terminal time instant of the control process. The dotted symbols denote the partial derivatives with respect to t, and the primed symbols stand for the partial derivatives with respect to y. It is worth noting that the initial conditions (4) and boundary conditions (3) should be compatible (for example, $f(0)=0$, $f'(0)=0$, $g''(l)=0$, $g'''(l)=0$).

The problem is to find an optimal control $u(t)$ that moves the truck from its initial to terminal states in the given time T

$$x(T) = x^f, \quad v(T) = v^f, \tag{5}$$

and minimizes a objective function $J[u]$ in the class U of admissible controls:

$$J[u] \to \min_{u \in U}. \tag{6}$$

To solve the boundary value problem (1)–(4), we apply the method of IDR, described by Kostin and Saurin 2005, in which some strict local equalities are replaced by an integral relation. In this case, it is possible to reduce problem (1)–(4) to a variational problem. If a weak solution p^*, m^*, and w^* exists then the following functional Φ under local constraints (1), (3), (4) reaches its absolute minimum on this solution

$$\begin{aligned} &\Phi(p^*,m^*,w^*) = \min_{p,m,w} \Phi(p,m,w) = 0 \\ &\Phi = \int_0^T \int_0^l \varphi(p,m,w)\,dy\,dt, \\ &\varphi = \frac{(p-\rho\dot{w})^2}{2\rho} + \frac{(m-EIw'')^2}{2EI}. \end{aligned} \tag{7}$$

Note that the integrand φ in (7) has the dimension of the energy density and is nonnegative. Hence, the corresponding integral is nonnegative for any arbitrary functions p, m, and w ($\Phi \ge 0$).

3. AN APPROXIMATION ALGORITHM

To find an approximate solution of the optimization problem defined by Eqs. (1), (3)–(7) we use a polynomial representation of the unknown functions. The functions p, m, and w are approximated by bivariate polynomials

$$\begin{aligned} &\tilde{p} = \sum_{i+j=0}^{N_p} p_{ij} t^i x^j, \quad \tilde{m} = \sum_{i+j=0}^{N_m} m_{ij} t^i x^j, \\ &\tilde{w} = \sum_{i+j=0}^{N_w} w_{ij} t^i x^j; \end{aligned} \tag{8}$$

The control u is restricted to a set of time polynomials

$$U = \left\{ u: \quad u = \sum_{i=0}^{N_u} u_i t^i \right\}. \tag{9}$$

Here p_{ij}, m_{ij}, w_{ij}, and u_i are unknown real coefficients.

The basis functions are chosen so that the approximations can exactly satisfy the boundary conditions (3), initial polynomial conditions (4), and the equation of motion (1) by suitably selected integers N_p, N_m, N_w, and N_u.

The resulting finite-dimensional unconstrained minimization problem (7) yields an approximate solution $\tilde{p}^*(t,y,u)$, $\tilde{m}^*(t,y,u)$, $\tilde{w}^*(t,y,u)$ for an arbitrary control $u \in U$, where U is the set of time polynomial functions with a given degree N_u. The optimal control $u^*(t)$ is determined from condition (6). Let us consider a functional $J[u]$ quadratic with respect to the parameters u_i of the polynomial control law (total mechanical energy of the beam at the terminal time T)

$$J = \int_0^l \eta(T, y)dy,$$
$$\eta(t, y) = \frac{\rho\dot{w}^2}{2} + \frac{EI(w'')^2}{2}. \qquad (10)$$

The corresponding optimization problem is reduced to a system of linear equations.

4. NUMERICAL EXAMPLE

For numerical modeling, the following dimensionless parameters $l = 1$, $\rho = 1$, $EI = 1$, $T = 2$, $v^f = 0$, and $x^f = 1$ are used. An optimal control (in the sense of the functional J in (10)) for the beam motion is analytically constructed for the following integers in (8): $N_p = 20$, $N_m = 21$, and $N_w = 23$. In the case when $N_u > 1$, the control u in (9) contains $N_u - 1$ unknown parameters, which are used for minimizing J. The optimal controls obtained by the IDR method for $N_u = 3, 5$ are shown in Fig. 2 (dashed and solid lines). The optimal values of J are equal to 2.24×10^{-3}, and 2.79×10^{-5}, respectively.

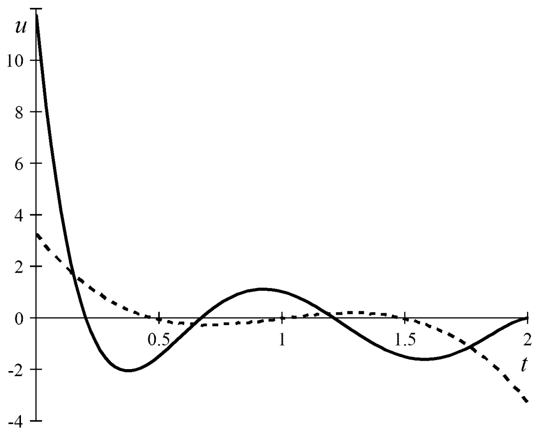

Fig. 2. Optimal control u.

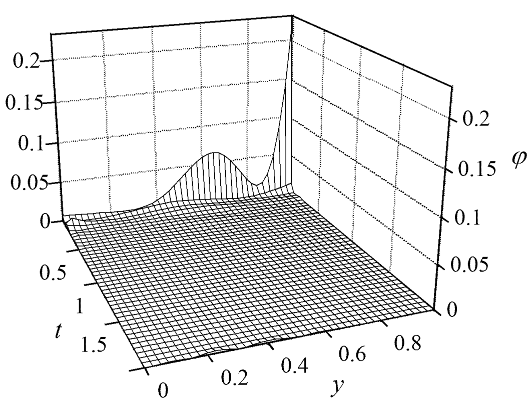

Fig. 3. Distribution of local solution error φ.

The value of the functional Φ can be considered as an integral performance criterion for the optimal solution whereas the integrand φ in (7) is a local quality characteristic. Figure 3 shows the distribution of the function $\varphi(t, y)$ for $N_u = 5$. It can be seen that its value is small almost everywhere, except for the vicinity of $t = 0$ with its maximum at the point $y = 1$. For the defined parameters the value of the functional is equal to $\Phi = 5{,}34 \times 10^{-7}$. As the number of free parameters of the polynomial control in the optimization problem (1), (3)–(7) increases, the total energy of the beam at the terminal time reduces considerably. In the case of four free control parameters, the value of J can be reduced by more than 5000 times.

5. CONCLUSIONS

In this paper, the problem of constructing the controlled motions of a uniform straight elastic rod mounted on a moving vehicle was considered. The algorithm of optimization based on a regular method of integro-differential relations for constructing the control, which steers the system to the state of minimum total energy at a final time instant, was developed. For two-dimensional motions of an elastic rod, the case of a polynomial control was considered. The results obtained by using the integro-differential approaches are analyzed.

ACKNOWLEDGMENTS

This research was supported by the Russian Foundation for Basic Research (grants Nos. 05-01-00563 and 05-08-18094) and the Program “State Support for Leading Scientific Schools” (NSh-1245.2006.1, NSh-9831.2006.1).

REFERENCES

Akulenko, L.D., G.V. Kostin (1992). The Perturbation Method in Problems of the Dynamics of Inhomogeneous Elastic Rods. J. of Applied Mathematics and Mechanics. **56**(3).

Akulenko,L.D., G.V. Kostin, S.V. Nesterov (1995). Numerical-Analytic Method for Investigation of Free Oscillations on Nonhomogeneous Beams. *Mechanics of Solids*, **30**(5).

Kostin, G.V. and V.V. Saurin (2005). Itegro-differential approach to solving problems of linear elasticity. *Doklady Physics*, **50**(10), pp. 535–538.

Kostin, G.V. and V.V. Saurin (2006a). The method of integrodifferential relations for linear elasticity problems. *Archive of Applied Mechanics*. **76**(7–8), pp. 391–402.

Kostin, G.V. and V.V. Saurin. Modeling of Controlled Motions of an Elastic Rod by the Method of Integro-Differential Relations. *J. of Comp. and Sys. Sci. Int.*, **45**(1), pp. 56–63 (2006b)

Kostin, G.V. and V.V. Saurin (2006c). The Optimization of the Motion of an Elastic Rod by the Method of Integro-Differential Relations.

J. of Comp. and Sys. Sci. Int., **45**(2), pp. 217–225

Kostin, G.V. and V.V. Saurin (2006d). Modeling and Optimization of Elastic System Motions by the Method of Integro-Differential Relations. *Doklady Mathematics.* **73**(3), pp. 469–472