AGENTS INTERNAL MECHANISMS INDUCE CONSENSUS IN EVOLUTIONARY GAMES ON NETWORKS

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Abstract
Classical evolutionary game theory focuses on infinite well-mixed populations, where the dynamics is ruled by averaging the payoffs of all possible interactions between couples of players. Classical theory has been recently extended to account for the presence of networks of connections among a finite number of agents. Anyway, in both the above theories, agents are only allowed to interact with others players. In this paper, we introduce the concept of internal mechanisms, represented by games that agents play against themselves (self games), and we study their impact on the dynamics at the level of single agents and of the whole population. The main findings concern with the onset of mixed Nash equilibria, which can be stable and, in some cases, represent consensus solutions. Surprisingly, the internal mechanisms can drive games with only dominant strategies, such as the prisoner's dilemma game, towards globally stable mixed Nash equilibria. The results have been obtained on the basis of theoretical reasonings as well as extensive numerical experiments.

Key words
Evolutionary game theory. Consensus. Internal steady states. Stability and Control.

1 Evolutionary game theory on networks

Evolutionary game theory has been developed to model the mechanisms ruling the time evolution of strategies in a well-mixed population of agents (hereafter also called individuals or players). Basically, this is achieved by dynamically comparing the strategy fitness of each player involved in games; individuals which use the fittest strategy are favored with respect to others, thus changing in time the strategy distribution within the population.

Evolutionary game theory is grounded on the well known replicator equation [13; 4; 9], here reported in the simplest version in which only two strategies are allowed:

$$\begin{cases} \dot{x}_1 = x_1(p_1 - \phi) \\ \dot{x}_2 = x_2(p_2 - \phi). \end{cases} \tag{1}$$

In equation (1), x_1 and x_2 represent the distributions of strategies 1 and 2 in the population ($x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 = 1$), while p_1 and p_2 are the fitnesses of the two strategies, respectively. $\phi = x_1p_1 + x_2p_2$ is the average fitness of the whole population. Since the population is well-mixed, p_1 and p_2 depend on all the possible two-players games played by the infinitely many individuals within the population. The game outcomes are ruled by the so called payoff matrix $B = \{b_{s,r}\}$, where $b_{s,r}$ is the payoff that a player which uses strategy s earns against another player which uses strategy r. Then the fitnesses are defined as:

$$\begin{cases} p_1 = b_{1,1}x_1 + b_{1,2}x_2 \\ p_2 = b_{2,1}x_1 + b_{2,2}x_2. \end{cases} \tag{2}$$

The terms $(p_1 - \phi)$ and $(p_2 - \phi)$ in equations (1) can be seen as a sort of growth coefficients for x_1 and x_2; for instance, if $(p_1 - \phi)$ is positive, then $(p_2 - \phi) < 0$, and consequently x_1 will grow up, while x_2 will decrease (x_1 is fitter than x_2). Using the fact that $x_2 = 1 - x_1$, and substituting x_1 with x, equation (1) reduces to the following ordinary differential equation:

$$\begin{aligned} \dot{x} &= x(1-x)\Delta p = \\ &= x(1-x)(p_1 - p_2) = \\ &= x(1-x)[(\sigma_1 + \sigma_2)x - \sigma_2], \end{aligned} \tag{3}$$

where $\Delta p = p_1 - p_2$ is the fitness difference, and the parameters $\sigma_1 = b_{1,1} - b_{2,1}$ and $\sigma_2 = b_{2,2} - b_{1,2}$ fully characterize the game type described by the payoff matrix B. Indeed, in the context of evolutionary game theory it has been proven that any payoff matrix B is equivalent to a diagonal matrix $\text{diag}(\sigma_1, \sigma_2)$ [13; 4; 12], then in the rest of this work, we will refer only to diagonal payoff matrices $B = \text{diag}(\sigma_1, \sigma_2)$.

Recent works [6; 11; 10; 7; 5] showed that the replicator equation can be extended to finite populations of individuals arranged over a network of connections. In this case, the network is described by a graph with a finite set of vertices $\mathcal{V} = \{1, \ldots, N\}$, and adjacency matrix $A = \{a_{v,w}\}$. Games are played only between neighboring players, i.e., player v plays with player w only if there is an edge connecting them in the graph, or equivalently, if $a_{v,w} = 1$. The graph is directed: $a_{v,w} = 1$ encodes the influence of player w on player v, and, at the same time, $a_{w,v}$ can be null, if v does not influence w.

In this framework, each player has its own payoff matrix $B_v = \text{diag}(\sigma_{v,1}, \sigma_{v,2})$. Moreover, the strategy fitnesses, evaluated for each player, are indicated by $p_{v,1}$ and $p_{v,2}$ and are defined as the sum of the strategy fitnesses of all the interactions with neighboring players (accounted by the entries $a_{v,w}$ of the adjacency matrix A). Their difference $\Delta p_v = p_{v,1} - p_{v,2}$ reads as

$$\Delta p_v = \sum_{w=1}^{N} a_{v,w} \left[(\sigma_{v,1} + \sigma_{v,2}) x_w - \sigma_{v,2} \right]. \quad (4)$$

For each player, the variable $x_w \in [0, 1]$ in equation (4), denotes the propensity of player w to use strategy 1, and $1 - x_w$ is the propensity for strategy 2. Based on the fitness definition reported in (4), we can write the replicator equation on networked population, hereafter called EGN [7]:

$$\dot{x}_v = x_v(1 - x_v)\Delta p_v. \quad (5)$$

The state space of system (5) is the hypercube $[0, 1]^N$. Notice that equation (5) is significantly different from the standard replicator equation: while the state variable x in (1) denotes the distribution of strategy 1 over the whole population, in the networked context x_v specifies the state of the vth player. Moreover, while in the standard context a player is allowed only to choose pure strategies, here players can also use mixed strategies, thus increasing the dimension of their decision space.

Furthermore, equation (5) can be read as a model of distributed agents in a social context, thus providing a new approach to tackle consensus problems (see, for example, [2; 14]).

In this paper, we investigate the presence of pure and mixed Nash equilibria in the EGN depending, for each player, on the entries of both the payoff matrices of games played with neighbor players (external mechanisms) and with himself (internal mechanisms). Our findings show that there are transitions between different pure and mixed Nash equilibria, reported also in [3], and eventually allowing players to reach consensus. These results are also promising to solve distributed control problems [1].

2 External and internal dynamical mechanisms

Presently, in both standard replicator equation and EGN, agents dynamically change their strategy in agreement with the outcomes of their interactions with other players. Self interaction are not studied yet.

The main contribution of this work is based on the introduction of **internal** mechanisms that influence the dynamics of each player. Basically, a generic internal mechanism can be described as a game that any individual plays against itself.

Notice that this is not possible in the standard theory due to the assumption of well mixed population. On the contrary, in the EGN one can distinguish between different players, then the contributions to the payoff coming from out (external mechanism) and self (internal mechanism) games can be calculated separately.

To this aim, we allow the presence of self loops within the graph of connections. If player v plays a self game, then $a_{v,v} = 1$. In order to distinguish between the two kind of games, we indicate with B_v the payoff matrix used by player v against other individuals, and with $B_v^S = \text{diag}(\sigma_{v,1}^S, \sigma_{v,2}^S)$ the payoff matrix of his self game.

The effect of this internal mechanism is embedded in the replicator equation on network by adding the following term

$$\Delta p_v^S = a_{v,v} \left[(\sigma_{v,1}^S + \sigma_{v,2}^S) x_v - \sigma_{v,2}^S \right], \quad (6)$$

to the quantity Δp. This allows to obtain a modified version of EGN:

$$\dot{x}_v = x_v(1 - x_v)(\Delta p_v + \Delta p_v^S). \quad (7)$$

3 Consensus

It is straightforward to verify that all the 2^N points x^*, such that $x_v^* \in \{0, 1\}\ \forall v \in \mathcal{V}$, are steady states of equation (7). These points are called **pure** steady states and they represent the vertices of the hypercube $[0, 1]^N$. Recalling that x^* is a consensus steady state if $x_v^* = c\ \forall v \in \mathcal{V}$, the pure steady states such that all individuals eventually converge to the same strategy are particularly interesting because they represent a consensus. In the following, we will refer to the steady state **ALL1** when all individuals converge to the pure strategy 1, i.e. $x_v^* = 1\ \forall v \in \mathcal{V}$, and to **ALL2** in the opposite situation, where $x_v^* = 0 \; \forall v \in \mathcal{V}$. The consensus steady states that are eventually reached by the population, are also Nash equilibria.

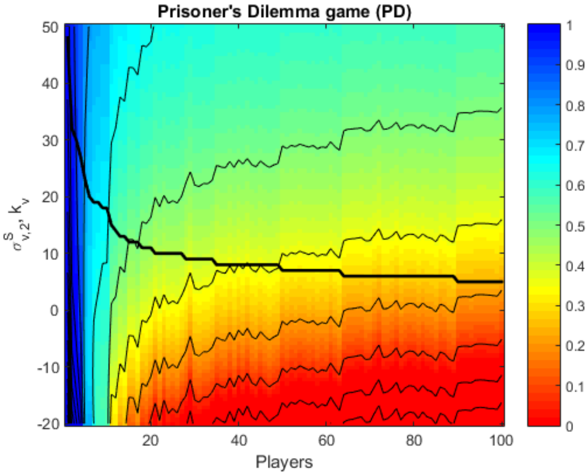

Figure 1. Value of mixed equilibrium components for $N = 100$ individuals playing a prisoner's dilemma game ($\sigma_{v,1} = -1$ and $\sigma_{v,2} = 2 \; \forall v \in \mathcal{V}$), arranged on a random network with a scale free degree distribution. All individuals play a self game, with $\sigma^S_{v,1} = 50 \; \forall v \in \mathcal{V}$. The value of mixed equilibrium component of a given player with degree k_v ($k_v \in [5, 50]$) and $\sigma^S_{v,2} \in [-20, 50]$ is represented by a color ranging from red ($x_v^* = 0$) to blue ($x_v^* = 1$). Level curves of the mixed equilibrium components are depicted with thin black lines. The degree k_v of each player is reported with a thick black line.

Furthermore, system (7) presents at most one isolated **mixed** steady state x^*, hereafter called **ALLM**, characterized by $x_v^* \in (0, 1) \; \forall v \in \mathcal{V}$. According to [13; 7], **ALLM** steady state $[x_1^*, x_2^*, \ldots, x_N^*]$ is obtained by solving the following system of equations:

$$\Delta p_v + \Delta p_v^S = 0 \; \forall v \in \mathcal{V}. \qquad (8)$$

Unlike pure steady states, **ALLM** is not feasible when $\exists v \in \mathcal{V} : x_v^* \notin (0, 1)$. Moreover, since its components are in general different, **ALLM** is not guaranteed to be a consensus steady state. Notice that the **ALLM** steady state is always a Nash equilibria, whenever it is feasible.

This fact is highlighted in Figures 1, 2 and 3, where the population is arranged on a random network with a scale free degree distribution. Here, $\sigma_{v,1}$ and $\sigma_{v,2}$ are fixed for all players according to 3 different prototypical games: Prisoner's Dilemma (PD) in Figure 1, Stag Hunt (SH) in Figure 2 and Chicken game (CH) in Figure 3. $\sigma^S_{v,1}$ has been also set to 50 for all games and all players. We report the value of **ALLM** components of any player characterized by a given degree k_v and

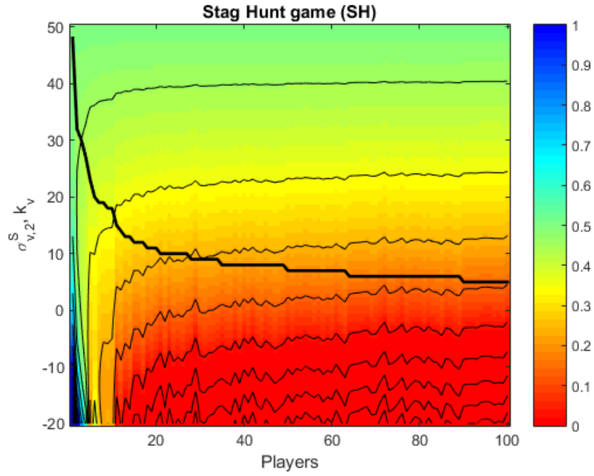

Figure 2. Value of mixed equilibrium components for $N = 100$ individuals playing a Stag Hunt game ($\sigma_{v,1} = 1$ and $\sigma_{v,2} = 1 \; \forall v \in \mathcal{V}$), arranged on a random network with a scale free degree distribution. All individuals play a self game, with $\sigma^S_{v,1} = 50 \; \forall v \in \mathcal{V}$. The value of mixed equilibrium component of a given player with degree k_v ($k_v \in [5, 50]$) and $\sigma^S_{v,2} \in [-20, 50]$ is represented by a color ranging from red ($x_v^* = 0$) to blue ($x_v^* = 1$). Level curves of the mixed equilibrium components are depicted with thin black lines. The degree k_v of each player is reported with a thick black line.

for a given $\sigma^S_{v,2}$ by using colors ranging from red ($x_v = 0$) to blue ($x_v = 1$). Moreover, we report with a thick black line the degree of each player and with thin black lines the level curves of mixed equilibrium, along which all players components are the same, thus **ALLM** is a consensus. We notice a strong similarity between the curve of the degree of each player and the level curves of the mixed equilibrium; in particular, $\sigma^S_{v,2}$ is proportional to $-k_v$ in Figures 1 and 2 while it is proportional to k_v in Figure 3. The same results hold for variable $\sigma^S_{v,1}$ when $\sigma^S_{v,2}$ is fixed. This fact suggests that linear relationships between $\sigma^S_{v,1}$, $\sigma^S_{v,2}$ and k_v ensure **ALLM** to have all equal components.

In order to prove the validity of these findings, let suppose that all players use the same payoff matrix $B_v = B = \mathrm{diag}(\sigma_1, \sigma_2)$ and that each player is influenced by an internal mechanism ($a_{v,v} = 1 \; \forall v \in \mathcal{V}$) represented by a self game. Assuming that **ALLM** is a consensus equilibrium, we substitute x_v^* with x^M and we obtain:

$$\Delta p_v = \sum_{w=1}^{N} a_{v,w} \left[(\sigma_{v,1} + \sigma_{v,2}) x^M - \sigma_{v,2} \right] = \\ = k_v \left[(\sigma_1 + \sigma_2) x^M - \sigma_2 \right], \qquad (9)$$

and

$$\Delta p_v^S = (\sigma^S_{v,1} + \sigma^S_{v,2}) x_M - \sigma^S_{v,2}. \qquad (10)$$

Plugging the results (9) and (10) into equation (8), we get that:

$$x^M = \frac{k_v \sigma_2 + \sigma_{v,2}^S}{k_v(\sigma_1 + \sigma_2) + \sigma_{v,1}^S + \sigma_{v,2}^S} \; \forall v \in \mathcal{V}. \quad (11)$$

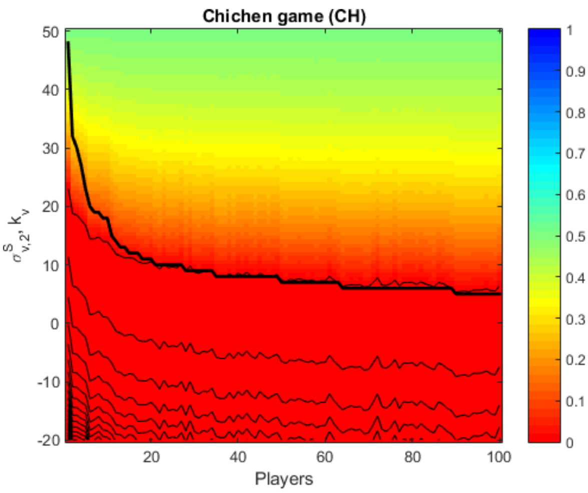

Figure 3. Value of mixed equilibrium components for $N = 100$ individuals playing a Chicken game ($\sigma_{v,1} = -1$ and $\sigma_{v,2} = -1 \; \forall v \in \mathcal{V}$), arranged on a random network with a scale free degree distribution. All individuals play a self game, with $\sigma_{v,1}^S = 50 \; \forall v \in \mathcal{V}$. The value of mixed equilibrium components of a given player with degree k_v ($k_v \in [5, 50]$) and $\sigma_{v,2}^S \in [-20, 50]$ is represented by a color ranging from red ($x_v^* = 0$) to blue ($x_v^* = 1$). Level curves of the mixed equilibrium components are depicted with thin black lines. The degree k_v of each player is reported with a thick black line.

If we assume that the internal mechanisms are characterized by the parameters $\sigma_{v,1}^S = \mu_1 k_v$ and $\sigma_{v,2}^S = \mu_2 k_v$, with μ_1 and μ_2 constant parameters, then **ALLM** really is a consensus steady state. Indeed, solution

$$x_v^* = x^M = \frac{\sigma_2 + \mu_2}{\sigma_1 + \sigma_2 + \mu_1 + \mu_2} \quad (12)$$

satisfies equation (11) $\forall v \in \mathcal{V}$ since its components do not depend on $\sigma_{v,1}^S$, $\sigma_{v,2}^S$ and k_v anymore, thus confirming the findings obtained from Figures 1, 2 and 3. Of course, this equilibrium is feasible if $x^M \in (0, 1)$. Notice that for $\mu_1 = \mu_2 = 0$, x^M reduces to $\frac{\sigma_2}{\sigma_1+\sigma_2}$, which corresponds to the mixed equilibrium of the standard replicator equation (1), where self games are not present.

The formulas $\sigma_{v,1}^S = \mu_1 k_v$ and $\sigma_{v,2}^S = \mu_2 k_v$ are not the unique way to ensure the existence of a consensus equilibrium, although they provide a sufficient condition. Anyway, they are appealing for a particular reason: the payoff Δp_v is the results of all the k_v interactions of individual v with its neighbors. Hence, Δp_v^S must be strong enough compared to Δp_v in order to have a balance between the internal and the external mechanisms, mathematically nor Δp_v neither Δp_v^S must be neglectable in equation 8. This fact has very significant implications for the applications to social context; in fact, the self confidence of players' on his internal decision must be as strong as the connectivity level (social role) of the player himself in the society.

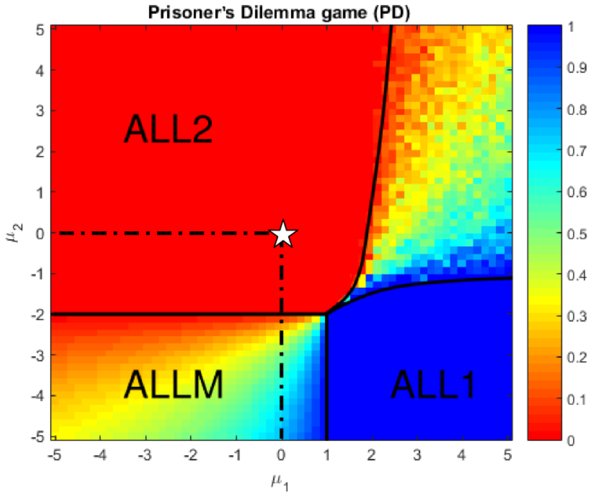

Figure 4. Average value of $N = 100$ agents asymptotic strategy playing a prisoner's dilemma game ($\sigma_{v,1} = -1$ and $\sigma_{v,2} = 2 \; \forall v \in \mathcal{V}$) on a generic network. All players are subject to internal mechanisms (self games) with $\sigma_{v,1}^S = \mu_1 k_v$ and $\sigma_{v,2} = \mu_2 k_v \; \forall v \in \mathcal{V}$. For each value of μ_1 and μ_2, the value of average asymptotic strategy is represented with a color ranging from red ($x_v^* = 0$) to blue ($x_v^* = 1$). Black lines delimit the regions where the system reaches a consensus equilibrium; **ALL2** in the upper left corner, **ALL1** in the lower right corner and **ALLM** in the lower left corner. In the remaining region, the system does not reach a consensus steady state. The white star indicates the simulation result when self games are not used (i.e. $\mu_1 = \mu_2 = 0$).

3.1 Stability of consensus steady states

The results of the previous section clearly state that the presence of the equilibrium **ALLM** strongly depends on the internal mechanisms, described by parameters $\sigma_{v,1}^S$ and $\sigma_{v,2}^S$. Furthermore, our findings show that the internal mechanisms can change the stability properties of the steady states, driving the dynamics of the whole population towards unexpected situations. For example, consider a population where all individuals play a Prisoner's Dilemma game (PD). More formally, suppose that $B_v = B = \text{diag}(\sigma_1, \sigma_2)$, with $\sigma_1 < 0$ and $\sigma_2 > 0$. In this case, it is well known that the whole population will prefer strategy 2 (defection) instead of strategy 1 (cooperation) [8]; in detail, **ALL2** is asymptotically stable, **ALL1** is unstable and **ALLM** is not feasible.

Figure 4 shows the results of simulation experiments of a population of $N = 100$ individuals all playing the same PD game on a arbitrary graph. Each individual plays an internal game described by different values of $\sigma^S_{v,1}$ and $\sigma^S_{v,2}$ for each player. In particular, we set $\mu_1 = \frac{\sigma^S_1}{k}$ and $\mu_2 = \frac{\sigma^S_2}{k}$. This guarantees that **ALLM** is a consensus steady state as described by equation (12). For each value of μ_1 and μ_2, we have performed several simulations of system (7) starting from random initial conditions ($x_v(0) \in (0, 1)\ \forall v \in \mathcal{V}$). Then, we have reported the average of the components of the stable steady stated to which the agents have converged asymptotically.

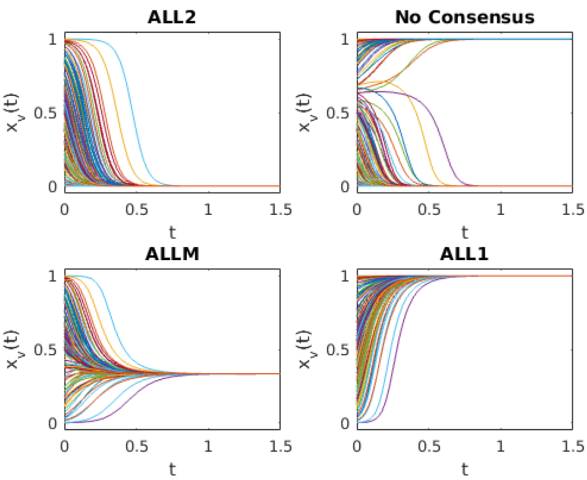

Figure 5. Dynamical behavior of a population of $N = 100$ individuals playing a PD game ($\sigma_{v,1} = -1$ and $\sigma_{v,2} = 2\ \forall v \in \mathcal{V}$), arranged on arbitrary network in four different regimes: the consensus **ALL2** is reached with $\mu_1 = 0$ and $\mu_2 = 0$ (upper left panel); no consensus is reached for $\mu_1 = 3$ and $\mu_2 = -1$ (upper right panel); the consensus **ALL1** is reached for $\mu_1 = 2$ and $\mu_2 = -4$ (lower right panel); the consensus **ALLM** is reached for $\mu_1 = -1$ and $\mu_2 = -3$ (lower left panel).

In particular, in Figure 4 we can identify four regions:

1. the upper left corner is the region where the consensus is reached and corresponds to the globally attracting steady state **ALL2**;
2. the lower right corner is the region where the consensus is reached and corresponds to the globally attracting steady state **ALL1**;
3. the lower left corner is the region where the consensus is reached and corresponds to the globally attracting steady state **ALLM**;
4. the upper right corner is the region where **ALL1**, **ALL2** and **ALLM** (when feasible) are not globally stable, thus the consensus is never reached.

Details of the dynamics are depicted in Figure 5; here, we run the systemfor different values of μ_1 and μ_2, chosen on the basis of Figure 4. In particular, the left upper panel shows the **ALL2** consensus, the lower left panel shows the **ALLM** consensus, the upper right panel shows the situation with no consensus and finally the lower right panel shows the **ALL1** consensus. Notice that, except for consensus on the state **ALL2**, all the other cases are unexpected when compared to the well known asymptotic behavior in the PD game, where strategy 2 (defection) always dominate all the others. Indeed, when self loops are not present (i.e. $\mu_1 = \mu_2 = 0$), the system converges to **ALL2** as indicated by the white star in Figure 4.

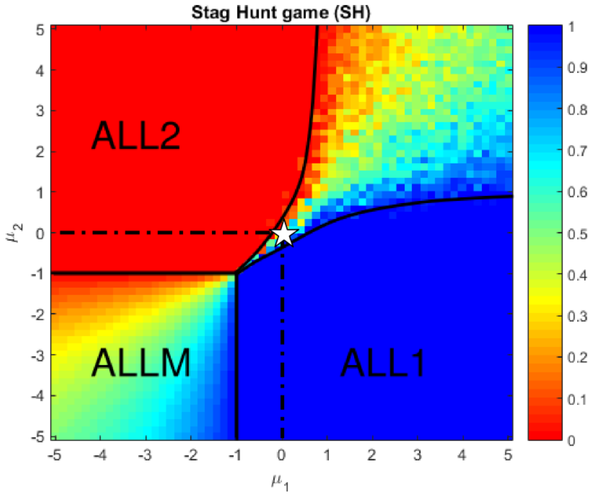

Figure 6. Average value of $N = 100$ agents asymptotic strategy playing a Stag Hunt game ($\sigma_{v,1} = 1$ and $\sigma_{v,2} = 1\ \forall v \in \mathcal{V}$) on a generic network. All players are subject to internal mechanisms (self games) with $\sigma^S_{v,1} = \mu_1 k_v$ and $\sigma_{v,2} = \mu_2 k_v\ \forall v \in \mathcal{V}$. For each value of μ_1 and μ_2, the value of average asymptotic strategy is represented with a color ranging from red ($x^*_v = 0$) to blue ($x^*_v = 1$). Black lines delimit the regions where the system reaches a consensus equilibrium; **ALL2** in the upper left corner, **ALL1** in the lower right corner and **ALLM** in the lower left corner. In the remaining region, the system does not reach a consensus steady state. The white star indicates the simulation result when self games are not used (i.e. $\mu_1 = \mu_2 = 0$).

Figures 6 and 7 show the same simulation experiment for Stag Hunt and Chicken Games. Similar results hold, altough the geometry of the consensus regions are different. Notably, also in this case the consensus is reachable, including the case **ALLM**. This is particularly important because the self games has the role of stabilizing a steady state, e.g. the mixed steady state of the Stag Hunt game, that is unstable in the standard replicator and in the EGN equations [7]. When self loops are not present (i.e. $\mu_1 = \mu_2 = 0$), the system converges to a non-consensus steady state as indicated by the white stars in Figures 6 and 7.

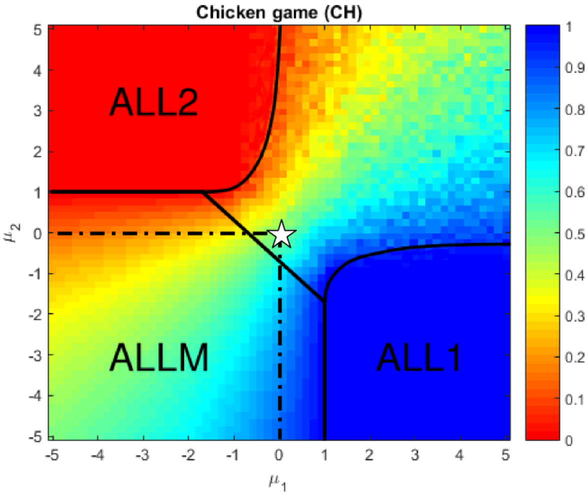

Figure 7. Average value of $N = 100$ agents asymptotic strategy playing a Chicken game ($\sigma_{v,1} = -1$ and $\sigma_{v,2} = -1\ \forall v \in \mathcal{V}$) on a generic network. All players are subject to internal mechanisms (self games) with $\sigma^S_{v,1} = \mu_1 k_v$ and $\sigma_{v,2} = \mu_2 k_v\ \forall v \in \mathcal{V}$. For each value of μ_1 and μ_2, the value of average asymptotic strategy is represented with a color ranging from red ($x^*_v = 0$) to blue ($x^*_v = 1$). Black lines delimit the regions where the system reaches a consensus equilibrium; $\mathbf{ALL2}$ in the upper left corner, $\mathbf{ALL1}$ in the lower right corner and $\mathbf{ALLM}$ in the lower left corner. In the remaining region, the system does not reach a consensus steady state. The white star indicates the simulation result when self games are not used (i.e. $\mu_1 = \mu_2 = 0$).

4 Conclusion

In this paper, the concept of internal mechanisms of players in evolutionary game on networks has been introduced. These are described as self games played by individuals alongside with the games played with their neighbors (external mechanisms). Simulation results showed that the presence of internal mechanisms may strongly influence the whole system dynamics by introducing steady states that are not feasible in the standard evolutionary game theory. In particular, our findings show that the parameters characterizing the internal mechanisms can be chosen in order to obtain a mixed Nash equilibrium, which is also a consensus steady state. This fact is very significant in the prisoner's dilemma game, where only dominant pure strategies are allowed in the standard theory. Finally, the results of our study can be used to control networked systems to drive their dynamics towards a desired consensus steady state by choosing suitable parameters that characterize the self games of each player.

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