CHAOTIC INSTANTONS AND GROUND QUASIENERGY LEVELS IN KICKED DOUBLE-WELL SYSTEM

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Abstract
Ground quasienergy levels in the kicked double-well system are investigated both analytically and numerically. The splitting between two lowest levels is described using resonances overlap criterion in the framework of chaotic instanton approach. Results of numerical calculations of quasienergy spectrum are in good agreement with derived phenomenological formula.

Key words
Kicked double-well, quasienergy, chaotic instanton.

Introduction

It is well known that tunneling phenomenon in double-well potential is connected with instantons (Rajaraman, 1982). Instanton contribution to tunneling amplitude determines the splitting of two ground energy levels. It was shown in the papers by another authors that dynamical tunneling probability can be controlled by external signals. Namely, in perturbed double-well potential a rate of tunneling can be many orders of magnitude greater than in undriven one (Lin and Ballentine, 1990). Furthermore, the tunneling amplitude grows as the number of frequency components of perturbation increases (Igarashi and Yamada, 2006). The decreasing of tunneling rate was found for specific parameter values of the driving force as well (Grossmann *et al.*, 1991). In this paper we concentrate on the dynamical tunneling in kicked double-well potential.

Dynamical tunneling phenomenon as well as the closely related chaos assisted tunneling play an important role in the quantum behavior of some real physical systems. For example they determine the splittings' enhancements between highly excited states of symmetric molecules (Keshavamurthy, 2003) and statistics of tunneling rates for a hydrogen atom placed in parallel, uniform, static electric, and magnetic fields, in the presence of chaotic classical dynamics (Delande and Zakrzewski, 2003). Dynamical tunneling in a sodium Bose-Einstein condensate was both experimentally and theoretically studied in the Ref. (Hensinger *et al.*, 2004).

Analytic chaotic instanton approach was proposed in order to describe enhancement of tunneling in one-dimensional spatially periodic potential (Kuvshinov *et al.*, 2002), (Kuvshinov *et al.*, 2003), (Kuvshinov and Kuzmin, 2005). Alternative approaches exploiting the mathematical apparatus of quantum field theory were suggested in (Aoki *et al.*, 1998), (Jirari *et al.*, 2001). The first approach uses nonperturbative renormalization group for analysis of tunneling in quantum mechanics (Aoki *et al.*, 1998). The later one is based on quantum instantons which are defined using an introduced notion of quantum action (Jirari *et al.*, 2001).

In this paper we adopt analytic chaotic instanton approach to double-well system and regard perturbation of the kick type for the system under investigation. We obtained phenomenological formula which describes the dependence of ground quasienergy splitting on strength and frequency of perturbation. Numerical calculations of quasienergy spectrum levels were performed to check analytical results.

1 Instantons in kicked double-well potential

Hamiltonian of the particle in the double-well potential can be written in the following form

$$H_0 = \frac{p^2}{2m} + a_0\, x^4 - a_2\, x^2, \qquad (1)$$

where m - mass of the particle, a_0, a_2 - parameters of the potential.

We consider the following perturbation

$$V_{per} = \epsilon\, x \sum_{n=-\infty}^{+\infty} \delta(t - nT), \qquad (2)$$

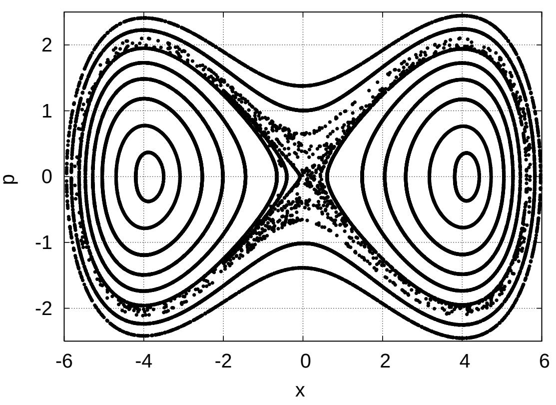

Figure 1. Stroboscopic plot for the kicked double-well potential. Closed curves correspond to regular trajectories, scattered points to chaotic ones. The model parameters are $m = 1, a_0 = 1/128$, $a_2 = 1/4, \epsilon = 0.01, \nu = 0.5$.

where ϵ - strength of the perturbation, T - period of the perturbation, t - time.

Full Hamiltonian of the system is the following:

$$H = H_0 + V_{per}. \tag{3}$$

Euclidean equations of motion of the particle in the double-well potential have a solution - instanton. In phase space of nonperturbed system instanton solution lies on the separatrix. Perturbation destroys separatrix forming stochastic layer. In this layer a number of chaotic instantons appears. Chaotic instanton can be written in the following form

$$x_{chaos} = x_{inst} + \epsilon\, \Delta x_{chaos}.$$

It is a solution of the Euclidean equation of motion. Here x_{chaos} and x_{inst} - chaotic and nonperturbed instanton solutions, respectively, Δx_{chaos} - stochastic correction.

We use the following assumptions:

1. small perturbation ($\epsilon < 0.1$),
2. uniform stochastic layer (see fig.1),
3. Euclidean chaotic instanton action is equal to nonperturbed instanton action corresponding to some nonmaximal energy. It can be approximated by the following linear form (see (Kuvshinov *et al.*, 2003))

$$S[x_{chaos}(\tau, \xi)] = S[x_{inst}(\tau, 0)] - \alpha \sqrt{\frac{m}{a_2}}\, \xi, \tag{4}$$

where $S[x_{inst}(\tau, 0)] = 2\sqrt{m}\, a_2^{3/2}/(3\, a_0)$ - nonperturbed instanton action, $\alpha = (1 + 18 \ln 2)/6$ - numerical coefficient, E - energy, E_{sep} - energy on separatrix and $\xi = E_{sep} - E$.

2 Phenomenological formula

When the separatrix is destroyed and stochastic layer is formed under the action of the perturbation a number of chaotic instantons appears. The width of the stochastic layer determines their contribution to the tunneling amplitude. The stochastic layer width estimated using resonances' overlap criterion. We calculated the parameter of the resonances' overlap for this purpose. It is equal to relation of the resonance width in the frequency scale ($\Delta\omega$) and the distance between resonances ($\delta\omega$). First one is estimated as a frequency of oscillations of the value $\Psi = \Theta - \nu\tau$, where Θ - angle variable for the particle in the system, $\nu = 2\pi/T$ is a perturbation frequency. The resonance width is the following (see chapter 5 in (Sagdeev *et al.*, 1988))

$$\Delta\,\omega \sim \sqrt{\frac{\epsilon\,\nu\,\omega^2}{\Delta H}}, \tag{5}$$

where $\Delta H = E_{sep} - E$ - distance from the separatrix, ω - frequency of the particle oscillations.

The distance between resonances is calculating using the expression for the resonance levels $\omega_n = n\,\nu$. Thus one can obtain

$$\delta\,\omega = \omega_{n+1} - \omega_n = \nu. \tag{6}$$

Using two last expressions (5) and (6) the parameter of the resonances' overlap can be written in the following form

$$\overline{K} = \frac{\Delta\omega}{\delta\omega} \sim \left[\frac{\epsilon\,\omega^2}{\nu\,\Delta H}\right]^{1/2} \gtrsim \left[\frac{\epsilon\,\nu}{\Delta H}\right]^{1/2}. \tag{7}$$

Overlap parameter is equal to unity on the boundary of the stochastic layer. Using equation (7) we can write the expression for the width of the stochastic layer in the following way

$$\Delta H_s = E_{sep} - E_{bor} \approx \tilde{k}\,\epsilon\,\nu, \tag{8}$$

where E_{bor} is the energy on the border between stochastic and regular regions, $\tilde{k}$ - some numerical parameter which can not be obtained in the framework of the criterion used.

The tunneling amplitude for the perturbed system is a sum of the amplitude in the nonperturbed case and the amplitude of tunneling via chaotic instantons. The later can be evaluated by integration over action of the tunneling amplitude in nonperturbed system. Using expression (4) this integral could be transformed to the integral over the energy from zero up to the width of the stochastic layer (8):

$$A_{chaos} = \alpha\sqrt{\frac{m}{a_2}}\,\tilde{N} \int_0^{\Delta H_s} d\,\xi \int_{-\infty}^{+\infty} d\,c_0 \times \\ \times \sqrt{S[x_{chaos}(\tau, \xi)]}\, exp\,(-S[x_{chaos}(\tau, \xi)]), \tag{9}$$

where $\tilde{N}$ is a normalize factor. To calculate contribution of chaotic instantons we use approximate expression for the chaotic instanton action (4) and assumptions from the previous section. Integration over c_0 gives the contribution of zero modes (Vainshtein, A.I. *et al.*, 1982). As the result we get the following expression for the amplitude:

$$A = A_{inst} + A_{chaos} \approx \approx \tilde{N}\,\sqrt{S^{inst}}\,e^{-S^{inst}}\,\Gamma\,exp\left(\alpha\sqrt{\frac{m}{a_2}}\,\Delta H_s\right), \quad (10)$$

where A_{inst} is tunneling amplitude in nonperturbed system, Γ - a time of the tunneling which is put to infinity at the end. The last exponential factor in the expression (10) is responsible for the tunneling enhancement in the perturbed system. In the nonperturbed case the width of the stochastic layer is equal to zero and the expression (10) coincides with the known expression describing the ordinary tunneling.

We can write phenomenological formula for quasienergy splitting ($\Delta\eta$) using the expression (10):

$$\Delta\eta(\epsilon,\nu) = 2\sqrt{\frac{6}{\pi}}\,\sqrt{S^{inst}}\,e^{-S^{inst}}(1 + k\,\epsilon\,\nu), \quad (11)$$

where

$$k = \alpha\sqrt{\frac{m}{a_2}}\,\tilde{k}.$$

We fix this phenomenological parameter value using the results of numerical simulations. For this purpose we perform the linear fitting of the numerical data for the dependencies on the perturbation strength and take average value of the parameter over these dependencies. As the result we have single parameter k for our phenomenological formula explaining all these dependencies.

3 Numerical calculations

For the computational purposes it is convenient to choose as basis vectors the eigenvectors of harmonic oscillator. In this representation matrix elements of Hamiltonian (1) and the perturbation (2) are real and symmetric. They have the following forms ($n \geq m$):

$$H^0_{m\,n} = \delta_{m\,n}\left[\hbar\omega\left(n + \frac{1}{2}\right) + + \frac{g}{2}\left(\frac{3}{2}\,g\,a_0\,(2m^2 + 2m + 1) - a_2'(2m+1)\right)\right] + \delta_{m+2\,n}\,\frac{g}{2}\,(g\,a_0(2m+3) - a_2')\,\sqrt{(m+1)(m+2)} + + \delta_{m+4\,n}\,\frac{a_0 g^2}{4}\sqrt{(m+1)(m+2)(m+3)(m+4)},$$

$$V_{m\,n} = \epsilon\,\delta_{m+1\,n}\,\sqrt{\frac{g}{2}}\,\sqrt{m+1}, \quad (12)$$

where $g = \hbar/m\omega$ and $a_2' = a_2 + m\,\omega^2/2$, $\hbar$ - Planck constant which we put equal to 1, ω - frequency of harmonic oscillator which is arbitrary, and so may be adjusted to optimize the computation. We used the value $\omega = 0.2$ with parameters $m = 1$, $a_0 = 1/128$, $a_2 = 1/4$. In numerical simulations size of matrices was chosen to be equal to 200×200. Simulations with larger matrices give the same results.

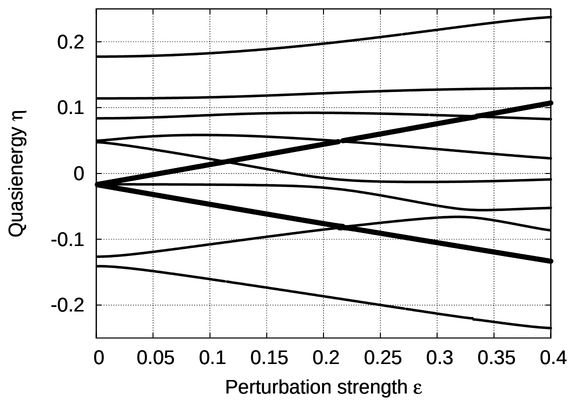

Figure 2. Quasienergy spectrum for the ten levels with the lowest average energy. The model parameters are $m = 1$, $a_0 = 1/128$, $a_2 = 1/4$ and $\nu = 0.5$. Thick lines - doublet with the minimal energy.

We calculate eigenvalues of the one period evolution operator $e^{-iHT}e^{-iV}$ and obtain quasienergy levels which are related with the evolution operator eigenvalues through the expression $\eta_k = i\,\ln\lambda_k/T$. Then we get the ten levels with the lowest average energy which is calculated using the formula $\langle v_i|\,H_0 + V/T\,|v_i\rangle$ ($|v_i\rangle$ are the eigenvectors of the one period evolution operator). The dependence of quasienergies of this ten levels on the strength of the perturbation is shown in the figure 2. Two levels with the minimal average energy (thick lines in the figure 2) has a linear dependence on the strength of the perturbation in the considered region. They are strongly influenced by the perturbation while some of the quasienergy states are not.

Performed numerical calculations give the dependence of the quasienergy splitting on the strength (fig.3(a)) and the frequency (fig.3(b)) of the perturbation. Numerical results are in good agreement with the phenomenological formula (11) in the regions of perturbation strength and frequency which are shown in the figures 3(a) and 3(b).

Analytical dependence of the quasienergy splitting on the strength and the frequency of the perturbation (11) is linear. Numerical points lie close to the analytical results and have linear dependence as well (fig.3). The agreement between numerical simulations and analytical expression is good in the parametric region considered.

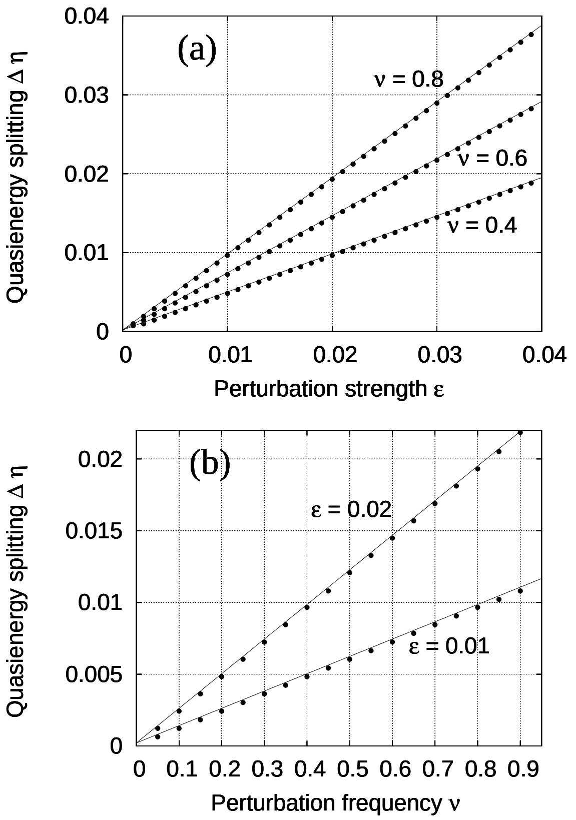

Figure 3. Quasienergy splitting as a function of the strength (a) and frequency (b) of the perturbation. Lines - phenomenological formula, points - numerical results. The model parameters are $m = 1$, $a_0 = 1/128, a_2 = 1/4$.

In addition to the results for the quasienergy splitting it should be mentioned the connection of this splitting to the tunneling phenomenon. As shown by formulas (8), (10) and (11) the growth of splitting (when the strength or frequency of perturbation rises) increases tunneling amplitude as well. But this enhancement of tunneling results in the strengthening of the small tunneling oscillations rate, whereas the initial wave packet as a whole is frozen.

Conclusions

Double-well system is investigated in presence of external kick perturbation. Analytic chaotic instanton approach is applied for this system in order to obtain the phenomenological formula for the ground quasienergy splitting. The formula has the single numerical parameter which is fixed from the numerical results. This formula describes ground quasienergy splitting as a function of strength and frequency of perturbation. It predicts linear dependence of the ground quasienergy splitting on these parameters. Numerical results for the quasienergy splitting as a function of the perturbation frequency and strength demonstrate linear dependence as well. They are in a good agreement with the analytical formula. The only restriction is that the perturbation strength has to be sufficiently small (in the case considered $\epsilon \lesssim 0.1$) in order the chaotic instanton approach used to be valid.

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