Chaotification of a Spatiotemporal Coupled Map Lattice

Ping Li, Zhong Li, Wolfgang A. Halang and Guanrong Chen

***Abstract*— Chaotification of a spatiotemporal coupled logistic-map lattice via a state feedback control with a mod-operation is investigated. A mathematically rigorous proof shows that the controlled system satisfying certain parameter conditions is chaotic in the sense of Li-Yorke and Devanay, respectively. Moreover, the chaotification method is applicable to other coupled logisic-map lattices even with different coupling modes. Simulation results have illustrated the effects and verified the correctness of the theoretical analysis.**

I. INTRODUCTION

The coupled map lattice (CML) as a spatiotemporal chaotic system was proposed in 1983 [1]. Since it is a simple model with most essential features of spatiotemporal chaos, the CML has been extensively studied in the fields of bifurcation and chaos, pattern formation, physical biology and engineering. Recently, the CML has been applied in cryptography [2], [3], [4], [5], where spatiotemporal chaos in the CML is very desirable and plays a key role. There is a method proposed, by shifting the binary representation of the CML output the first several bits, to chaotify the CML [6], where the investigation of chaotification of the CML is numerical. Nevertheless, a mathematically rigorous and effective chaotification method for CML is desirable, which can make an originally non-chaotic dynamical system chaotic, or can enhance the existing chaos of a chaotic system. Some mathematically rigorous chaotification methods have been developed (see, e.g. [7] and some references therein) since the first mathematical chaotification method proposed by Chen and Lai [8].

Similar to the Chen-Lai method [8], a chaotification algorithm is proposed in this paper to chaotify a CML. In the controlled CML, a state feedback in each dimension is applied to guarantee the system trajectory expanding in all directions, and then a mod-operation is used to "fold" the trajectories back into a compact region whenever the expansion takes them to move out of it. Moreover, sufficient conditions for the feedback gain parameter are derived, under which the controlled CML is proved to have a snap-back repeller. According to the Marotto theorem [9], it is, thus, chaotic in the sense of Li-Yorke. Moreover, the above chaotification method with certain feedback parameter values is suitable for chaotifying other coupled logistic-map lattices with different coupling topologies. Finally, the chaotification method is applied to control some typical CMLs by choosing suitable control parameters. Simulation results show that the chaotification method makes an originally non-chaotic CML chaotic and enhances the chaos of an originally chaotic CML, and that the method is applicable to CMLs with different coupling structures.

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This paper is organized as follows. In Sec. 2, the concept of Li-Yorke chaos and the Marotto theorem are introduced. In Sec. 3, the chaotification algorithm of the CML is described, and some conditions for choosing the control parameter are obtained; moreover, the controlled CML with certain parameter values is proved to be chaotic in the sense of Li-Yorke. Simulations on controlling some typical CMLs by using the proposed chaotification method are demonstrated in Sec. 4. The last section gives some concluding remarks.

II. LI-YORKE CHAOS AND MAROTTO THEOREM

The first precise definition of discrete chaos proposed by Li and Yorke states that any one-dimensional discrete interval map having a period-three orbit exhibits chaos [10]. This is a simple criterion for chaos in one-dimensional discrete systems and was lately generalized to n-dimensional discrete systems by Marotto [9]. A correct proof of the Marotto theorem is given in [11].

Theorem 1: (Modified Marotto Theorem) Consider the n-dimensional discrete system

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k), \quad \mathbf{x}_k \in R^n, \quad k = 0, 1, 2, ..., , \qquad (1)$$

where F is a map from R^n to itself. Assume that F has a fixed point $\mathbf{x}^*$ satisfying $\mathbf{x}^* = F(\mathbf{x}^*)$.

Moreover, assume that

(i) $F(\mathbf{x})$ is continuously differentiable in a neighborhood of $\mathbf{x}^*$, and all eigenvalues of $DF(\mathbf{x}^*)$ have absolute values large than 1, where $DF(\mathbf{x}^*)$ is the Jacobian of F evaluated at $\mathbf{x}^*$, which implies that there exist $r > 0$ and a norm $||\cdot||$ in R^n such that F is expanding in $\bar{B}_r(\mathbf{x}^*)$, the closed ball of radius r centered at $\mathbf{x}^*$ in $(R^n, ||\cdot||)$;

(ii) $\mathbf{x}^*$ is a snap-back repeller of F with $F^m(\mathbf{x}^0) = \mathbf{x}^*$, $\mathbf{x}_0 \neq \mathbf{x}^*$, for some $\mathbf{x}_0 \in B_r(\mathbf{x}^*)$ and some positive integer m, where $B_r(\mathbf{x}^*)$ denotes an open ball of radius r centered at $\mathbf{x}^*$ in $(R^n, \|\cdot\|)$. Furthermore, F is continuously differentiable in some neighborhoods of $\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{m-1}$, respectively, and $\det[DF(\mathbf{x}_j)] \neq 0$, where $\mathbf{x}_j = F(\mathbf{x}_{j-1})$ for $j = 1, 2, ..., m$.

Then, system (1) is chaotic in the sense of Li-Yorke.

III. Chaotification algorithm designed for CML

A one-way coupled logistic-map lattice, one of the most intensively investigated CML, is described by

$$x_{k+1}^j = (1-\epsilon)f(x_k^j) + \epsilon f(x_k^{j-1}), \quad (2)$$

where $f(x) = ax(1-x)$, x_k^j represents the state variable for the jth site ($j = 1, 2, ..., L$; L is the number of the sites in the CML) at time k ($k = 0, 1, 2, ...$), $\epsilon \in (0, 1)$ is the coupling strength, and $a \in (0, 4]$ is the parameter of the logistic map. The periodic boundary condition, $x_k^0 = x_k^L$ for all k, is used in the CML.

Denote $\mathbf{x}_k = [x_k^1, ..., x_k^L]^T$, and rewrite the CML (2) in a vector form as an L-dimensional map,

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k). \quad (3)$$

The controlled CML is given by

$$\mathbf{x}_{k+1} = G(\mathbf{x}_k, \mu) = F(\mathbf{x}_k) + \mu\mathbf{x}_k \pmod 1, \quad (4)$$

where μ is the control parameter.

Two lemmas from linear algebra [12] are first introduced.

Lemma 1: (Gerschgorin circle theorem) If $X^{-1}AX = D + F$, where X is a unitary matrix, $D = \text{diag}(d_1, ..., d_n)$ and $F = [f_{ij}]_{n\times n}$ has zero diagonal entries, then $\lambda(A) \subseteq \bigcup_{i=1}^n D_i$, where $D_i = \left\{ z \in \mathbf{C} : |z - d_i| \le \sum_{j=1}^n |f_{ij}| \right\}$.

Lemma 2: For a matrix $A_{N\times N}$ with eigenvalues $\lambda_1, ..., \lambda_N$, the determinant of A is equal to $\prod_{i=1}^N \lambda_i$.

In this paper, all vector inequalities are component-wise inequalities. The maximum row-sum norm for $(m \times n)$ real matrices $A = [a_{ij}]$ is defined as $\|A\|_\infty = \max\left\{ \sum_{i=1}^m |a_{ij}| : j = 1, ..., n \right\}$.

Theorem 2: The controlled CML (4) is chaotic in the sense of Li-Yorke, provided that $\mu > \max\left\{ 2\alpha, \alpha + \sqrt{1+\delta}, \frac{1}{2}\left(1 + 3\alpha + \sqrt{\alpha^2 + 6\alpha(1+\delta) + 1}\right)\right\}$, where $\alpha = a + \xi$, and δ, ξ are two small positive real values.

Proof: Let $\mathbf{x}^* = [0, ..., 0]^T = \mathbf{0}$. Then $\mathbf{x}^*$ is a fixed point of the CML (4), satisfying $\mathbf{x}^* = G(\mathbf{x}^*)$.

Define the jth eigenvalues of $DG(\mathbf{x})$ as λ_j ($j = 1, 2, ..., L$). Since the differentiation and the mod-operation are interchangable, one has $DG(\mathbf{x}) = DF(\mathbf{x}) + \mu I$. By Lemma 1, one has

$$|\lambda_j - \mu| < \|DF(\mathbf{x})\|_\infty, \quad \forall j = 1, ..., L. \quad (5)$$

Due to (6), where $\mathbf{x} = [x^1, x^2, ..., x^L]^T$, $f'(x) = a(1-2x)$, one has $\|DF(\mathbf{x})\|_\infty = \max\{|a(1-\epsilon)(1-2x^j)| + |a\epsilon(1-2x^{j-1})| : j = 1, 2, ..., L\} \le a$, $\forall$ $\mathbf{x} \in [\mathbf{0}, \mathbf{1}]$, where $\mathbf{1} = [1, ..., 1]^T$, so that (5) is implied by $|\lambda_j| > \mu - a, (\forall j = 1, ..., L)$. Under the theorem condition, $\mu > \alpha + \sqrt{1+\delta}$, one has

$$|\lambda_j| > 1, \forall j = 1, ..., L. \quad (7)$$

Therefore, all eigenvalues of $DG(\mathbf{x}^*)$ are bigger than 1 in absolute value. Since $DG(\mathbf{x}) = DF(\mathbf{x}) + \mu I$, one has $\|DG(\mathbf{x})\|_\infty = \max\left\{|a(1-\epsilon)(1-2x^j) + \mu| + |a\epsilon(1-2x^{j-1})| : j = 1, ..., L\right\}$, $\forall \mathbf{x} \in [0, 1]$.

By taking $r = \frac{1+\delta}{(\mu-\alpha)^2}$, one has $\min\left\{\|DG(\mathbf{x})\|_\infty : \mathbf{x} \in \bar{B}_r(\mathbf{x}^*)\right\} = a\left(1 - \frac{2(1+\delta)}{(\mu-\alpha)^2}\right) + \mu$. Under the theorem condition, $\mu > \alpha + \sqrt{1+\delta}$, one has $\|DG(\mathbf{x})\|_\infty \ge a\left(1 - \frac{2(1+\delta)}{(\mu-\alpha)^2}\right) + \mu > 1$, hence, $G(\mathbf{x})$ is expanding for all $\mathbf{x} \in \bar{B}_r(\mathbf{x}^*)$.

Next, similar to the proof of Theorem 3 in [13], [14], define $\bar{G}(\mathbf{x}) = \bar{F}(\mathbf{x}) + \mu\mathbf{x} - \mathbf{1}$, and $\upsilon(\mathbf{x}) = \max\{0, \mathbf{x} - \bar{G}(\mathbf{x})\} \equiv [\max\{0, x_1 - \bar{G}(x_1)\}, ..., \max\{0, x_n - \bar{G}(x_n)\}]^T$. Since $F(\mathbf{0}) = \mathbf{0}$ and F is differentiable for $\mathbf{x} \in [\mathbf{0}, \mathbf{1}]$, F is bounded for any bounded $\mathbf{x}$. Hence, there exists a number $1 \ge \tau \ge \varrho > 0$, such that for any $\mathbf{x} \in \Omega_\varrho \equiv \{\mathbf{x} \in R^n | \mathbf{x} \in [\mathbf{0}, \varrho\mathbf{1}]\}$, one has $\upsilon(\mathbf{x}) \in \Omega_\tau \equiv \{\mathbf{x} \in R^n | \mathbf{x} \in [\mathbf{0}, \tau\mathbf{1}]\}$. Furthermore, define a scalar function,

$$\omega(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_\varrho, \\ \min\limits_{x_i > \varrho} \left(\frac{\varrho}{x_i}\right), & \mathbf{x} \in \Omega_\tau - \Omega_\varrho, \end{cases}$$

and let $\phi(\mathbf{x}) = \omega(\mathbf{x})\mathbf{x}$, $\quad \varphi(\mathbf{x}) \equiv \upsilon(\phi(\mathbf{x}))$. It can be easily verified that $\phi(\mathbf{x}) \in \Omega_\varrho$, $\quad \varphi(\mathbf{x}) \in \Omega_\tau$, $\quad \forall \mathbf{x} \in \Omega_\tau$. Note also that $\varphi(\mathbf{x})$ is continuous on Ω_τ. It follows from the Brouwer fixed point theorem in functional analysis [15] that there exists $\mathbf{x}_1 \in \Omega_\tau$, such that $\mathbf{x}_1 = \varphi(\mathbf{x}_1)$, or equivalently,

$$\mathbf{x}_1 = \upsilon(\phi(\mathbf{x}_1)) = \max\{\mathbf{0}, \phi(\mathbf{x}_1) - \bar{G}(\phi(\mathbf{x}_1))\}. \quad (8)$$

Suppose $\mathbf{x}_1 \notin \Omega_\varrho$. Then, there exists an index i satisfying $1 \le i \le L$, such that $\varrho < x_1^i = \|\mathbf{x}_1\|_\infty \le \tau$, $\quad \phi_i(\mathbf{x}_1) = \omega(\mathbf{x}_1)x_1^i = \varrho$. Since $\phi(\mathbf{x}_1) \in \Omega_\varrho$, for a $\xi \in \Omega_\varrho$, one has $|F_i(\phi(\mathbf{x}_1))| \le \|F(\phi(\mathbf{x}_1))\|_\infty \le \|F'(\xi)\|_\infty \|\phi(\mathbf{x}_1)\| < \alpha\varrho$; that is, $-\alpha\varrho < F_i(\phi(\mathbf{x}_1)) < \alpha\varrho$. Therefore, $\bar{G}_i(\phi(\mathbf{x}_1)) = F_i(\phi(\mathbf{x}_1)) + \mu\varrho - 1 > (\mu - \alpha)\varrho - 1 = 0$. Consequently, $x_1^i > \max\{0, \phi_i(\mathbf{x}_1) - \bar{G}_i(\phi(\mathbf{x}_1))\}$, which contradicts (8). Hence, one has $\mathbf{x}_1 \in \Omega_\varrho$, so that $\omega(\mathbf{x}_1) = 1$. Consequently, $\phi(\mathbf{x}_1) = \mathbf{x}_1$ and Eq. (8) becomes

$$\mathbf{x}_1 = \max\{\mathbf{0}, \mathbf{x}_1 - \bar{G}(\mathbf{x}_1)\}. \quad (9)$$

Suppose $\bar{G}(\phi(\mathbf{x}_1)) < \mathbf{0}$. Then, $\mathbf{x}_1 < \mathbf{x}_1 - \bar{G}(\phi(\mathbf{x}_1))$, which contradicts (9). Therefore,

$$\bar{G}(\mathbf{x}_1) \ge \mathbf{0}, \quad (\mathbf{x}_1)^T\bar{G}(\mathbf{x}_1) = \mathbf{0}. \quad (10)$$

Since $\|\mathbf{x}_1\|_\infty \le \varrho$, one has $\|F(\mathbf{x}_1)\|_\infty \le \alpha\|\mathbf{x}_1\|_\infty < \alpha\varrho$; that is, $-\alpha\varrho < F(\mathbf{x}_1) < \alpha\varrho$. Suppose that there exists an index j, satisfying $1 \le j \le L$, such that $x_1^j \le \rho$. Then $\bar{G}_i(\mathbf{x}_1) = F_i(\mathbf{x}_1) + \mu x_1^j - 1 < \alpha\varrho + \mu\rho - 1 = 0$, which contradicts inequality (10). Therefore, $\mathbf{x}_1 > \rho\mathbf{1}$ and $\bar{G}(\mathbf{x}_1) = \mathbf{0}$. Hence, there exists $\mathbf{x}_1$ satisfying $\rho\mathbf{1} < \mathbf{x}_1 \le \varrho\mathbf{1}$ such that $\bar{G}(\mathbf{x}_1) = F(\mathbf{x}_1) + \mu\mathbf{x}_1 - \mathbf{1} = \mathbf{0}$; that is

$$G(\mathbf{x}_1) = \mathbf{0}. \quad (11)$$

Under the theorem condition, $\mu > \frac{1}{2}\left(3\alpha + 1 + \sqrt{\alpha^2 + 6\alpha(1+\delta) + 1}\right)$, for $\rho > r$, one has $\mathbf{x}_1 \notin B_r(\mathbf{x}^*)$.

Define $\hat{G}(\mathbf{x}) = F(\mathbf{x}) + \mu\mathbf{x} - \mathbf{x}_1$, similar to the above process, it can be proved that there exists $\mathbf{x}_0$ satisfying $\mathbf{x}_0 \le \sigma\mathbf{1}$,

$$DF(\mathbf{x}) = \begin{bmatrix} (1-\epsilon)f'(x^1) & 0 & \cdots & 0 & \epsilon f'(x^L) \\ \epsilon f'(x^1) & (1-\epsilon)f'(x^2) & 0 & \cdots & 0 \\ 0 & \epsilon f'(x^2) & (1-\epsilon)f'(x^3) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \epsilon f'(x^{L-2}) & (1-\epsilon)f'(x^{L-1}) & 0 \\ 0 & 0 & \cdots & \epsilon f'(x^{L-1}) & (1-\epsilon)f'(x^L) \end{bmatrix}, \quad (6)$$

where $\sigma = \frac{(\mathbf{x}_1)_{\min}}{\mu-\alpha}$, $(\mathbf{x}_1)_{\min} = \min\{(x_1)_j : j = 1, ..., L\}$ such that

$$\hat{G}(\mathbf{x}_0) \geq \mathbf{0}, \quad (\mathbf{x}_0)^T \hat{G}(\mathbf{x}_0) = \mathbf{0}. \quad (12)$$

Since $\|\mathbf{x}_0\|_\infty \leq \sigma$ and $\mu > 2\alpha$, one has $\|F(\mathbf{x}_0)\|_\infty \leq \alpha\|\mathbf{x}_0\|_\infty < \alpha\sigma < (\mathbf{x}_1)_{\min}$. Suppose that there exists an index j, $1 \leq j \leq L$, such that $x_0^j = 0$. Then $\hat{G}_j(\mathbf{x}_0) = F_j(\mathbf{x}_0) - x_1^j < 0$, which contradicts inequality (12). Therefore, $\mathbf{x}_0 > \mathbf{0}$ and $\hat{G}(\mathbf{x}_0) = \mathbf{0}$; that is,

$$G(\mathbf{x}_0) = \mathbf{x}_1. \quad (13)$$

Since $\mathbf{x}_0 \leq \frac{(\mathbf{x}_1)_{\min}}{\mu-\alpha}\mathbf{1} < r\mathbf{1}$, $\mathbf{x}_0 \in B_r(\mathbf{x}^*)$. According to Eqs. (11) and (13), $G^m(\mathbf{x}_0) = \mathbf{0}$ with $m = 2$.

Define the eigenvalues of $DG(\mathbf{x}_1)$ as $\lambda_{1j} = |\lambda_{1j}|e^{i\theta_{1j}}$, where $|\lambda_{1j}|$ and θ_{1j} are the module and the phase of λ_{1j}, $j = 1, 2, ..., L$, respectively. By Lemma 2, one has $\det[DG(\mathbf{x}_1)] = \prod_{j=1}^L \lambda_{1j} = \left(\prod_{j=1}^L |\lambda_{1j}|\right) e^{(\sum_{j=1}^L \theta_j)i}$. According to Eq. (7), $|\lambda_{1j}| > 1$ $(j = 1, 2, ..., L)$, one has $\det[DF(\mathbf{x}_1)] \neq 0$. Similarly, $\det[DF(\mathbf{x}_0)] \neq 0$. Therefore, $\mathbf{x}^* = \mathbf{0}$ is a snap-back repeller of $G(\mathbf{x})$. It follows from Theorem 1 that the controlled CML (4) is chaotic in the sense of Li-Yorke.

IV. Simulations

To verify the above theoretical remarks, some typical CMLs are used to simulate the chaotification effect in this section. In the simulation, the initial conditions are chosen randomly and the Lyapunov exponents are computed by deleting the first 10^3 state values and using the successive 10^4 state values.

Firstly, the CML (2) with $\epsilon = 0.6$, $L = 32$ and $a = 1$ is investigated. Its Lyapunov exponents are shown in Fig. 1(a) in descending order. It is seen that all Lyapunov exponents are negative. By using the chaotification method described in Sec. III with $\mu = 4$ and $\delta = 0.1$ and $\xi = 0.1$ determined by Theorem 2, the Lyapunov exponents of the controlled CML are plotted in Fig. 1(b). It can be seen that all Lyapunov exponents are positive, thus the controlled CML is chaotic.

Secondly, considering the CML (2) with $\epsilon = 0.95$, $L = 32$ and $a = 3.8$. Its Lyapunov exponents are plotted in Figs. 2(a). It can be seen that all Lyapunov exponents are positive, thus the CML is chaotic. Controlling this CML with $\mu = 10$ by letting $\delta = 0.1$ and $\xi = 0.1$, the Lyapunov exponents are plotted in Figs. 2(b). It is clear that all Lyapunov exponents are increased, thus the controlled CML has stronger chaos. Therefore, the chaotifiction method can also enhance the chaoticity of an originally chaotic CML.

Finally, consider the globally coupled logistic-map lattice described by

$$x_{k+1}^j = (1-\epsilon)f(x_k^j) + \sum_{i=1, i\neq j}^{L} \frac{\epsilon}{L-1} f(x_k^i), \quad (14)$$

where $f(x) = x(1-x)$, $L = 32$ and $\epsilon = 0.6$. Its Lyapunov exponents are plotted in Figs. 3(a). It is shown that all Lyapunov exponents are negative, thus the CML is nonchaotic. Under the control of this CML with $\mu = 4$ by letting $\delta = 0.1$ and $\xi = 0.1$, the Lyapunov exponents become positive, which is plotted in Figs. 3(b). Thus the controlled CML is chaotic. It is noted that Theorem 2 is also applicable to the CMLs where the sum of all coupling strengths of all lattices is equal to 1, which is consistent with the above Remark 1.

V. Conclusions

A spatiotemporal coupled logistic-map lattice is controlled by a chaotification algorithm, which is based on state feedback control with a mod-operation. Some conditions for choosing the control parameter have been derived. The controlled system with certain parameters satisfying these conditions is proved rigorously to be chaotic in the sense of Li-Yorke. The results are also applicable to other types of coupled logistic-map lattices with different coupling topologies. Simulation results illustrate and verify the theoretical analysis.

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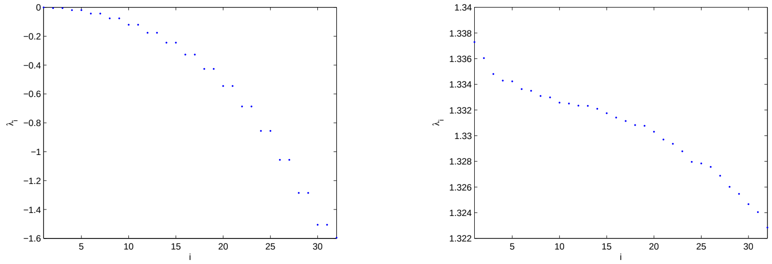
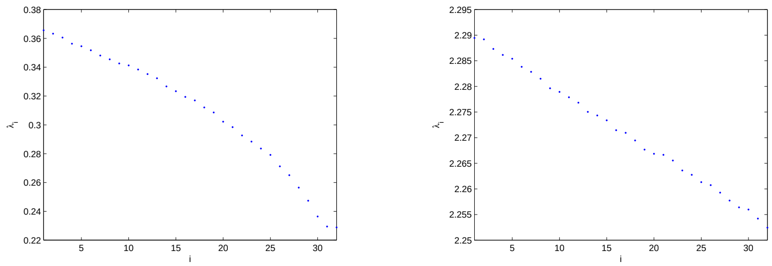


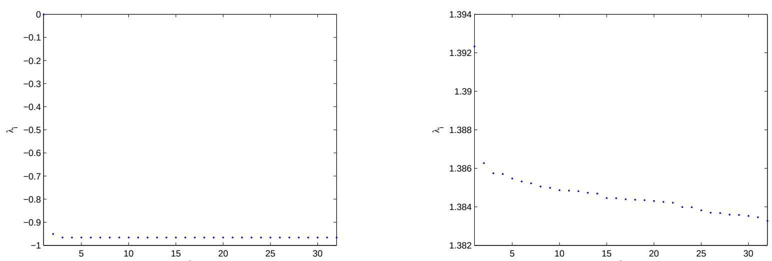
(a) (b)

Fig. 1. chaotification of an originally non-chaotic one-way coupled logistic-map lattice



(a) (b)

Fig. 2. Chaotification of an originally chaotic one-way coupled logistic-map lattice



(a) (b)

Fig. 3. Chaotification of a globally coupled logistic-map lattice

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