STEADY MOTIONS OF A TETRAHEDRAL SATELLITE WITH TETHERED ELEMENTS

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Abstract

The procedure of the Routh reduction is well-known for systems of Lagrangian equations in the case, when one or few generalized coordinates are cyclic. However, if this system admits a symmetry group given by a vector field on the configuration space, but the cyclic coordinates are not given explicitly, the reduction seems difficult. Here we describe the reduction in this case.

The result is applied to the problem of motion of a system of interacting material points moving about an attracting center. In particular, the expression for the amended potencial is obtained by the proposed procedure without introduction of any special coordinates.

The obtained potential is used to analise the stationary motions of a tetrahedral satellite in a central Newtonian gravitational field. The tetrahedral structure is assumed to be regular; it is composed by rigid and tether elements. The possibility to use flexible tethers to provide a stationary tetrahedral configuration is discussed. The structure possesses spherically symmetric tensor of inertia, so we use the Routh method with the amended potential. The reactions in the links are studied using Lagrangian multipliers. The goal is to identify the stretched links so they could be replaced by massless inextensible tethers.

Key words

Routh theory, implicit symmetries, orbiting tethered systems.

1 Introduction

The recent programs of space exploration by large-scaled spatial structures presume, in particular, the launch of tetrahedral satellite formations [Guzman, Schiff, 2002], [Clemente and Atkins, 2005]. Cluster formation was successfully launched and used for Earth magnetosphere studies [Laakso et al., 2005]. In NASA program, a regular tetrahedron constellation with the characteristic dimension about ten kilometers is planned to be kept by active control tools [Bainum and Tan, 2006], [Capo-Lugo and Bainum, 2006]. However, formation maintenance by active control seems extremely expensive, so the question is whether it is possible to replace some active control elements by passive tools.

One way to maintain the formation chape passively is to connect the satellites in the formation by rigid or tethered links. Several research show the possibility to achieve a large variety of plane and three-dimensional configurations using tethered structures or an open chain of satellites connected by rigid rods [Misra and Modi, 1992], [Sarychev, 1999], [Guerman, 2003], [Guerman, 2006], [Munitsyna, 2007].

However, replacement of the active control tools by rigid rods increases the system mass and leads to several problems, such as deformability of the constructed object, and difficulties in deployment or orbital assembling of such a structure. To minimize the masses of the links, one can suggest to replace stretched rigid rods by tethers which are sufficiently light and can be delivered into the orbit in a compact form. So it is necessary to estimate the reactions in the elements of the tetrahedral structure which moves under the action of the central Newtonian gravitational force.

The case of particular interest is that of regular tetrahedron formation, i.e. the case when all satellites are identic and all links have equal lengths. Here we consider dynamics of a regular tetrahedron formed by identic satellites connected by massless rigid rods assuming that its center of mass moves in a circular Keplerian orbit. In this case the use of so called “satellite approximation” does not suffice for attitude dynamics analysis, because the known partial separation of the orbital and attitude motions is verified only if at least two moments of inertia differ from each other [Beletsky, 1965]. The inertial properties of the structure in question are described by the mass moments of the third order because all its central moments of inertia are equal to each other, and the “satellite approximation” is not applicable.

Investigation of the orbital dynamics of rigid bodies with mass distribution that possesses discrete groups of symmetry, in particular, a group of a regular tetrahedron, arises to publications [Sulikashvili, 1985], [Sulikashvili, 1987], [Sulikashvili, 1989a], [Sulikashvili, 1989b], [Burov and Sulikashvili, 1993], where the steady orbital configurations of such systems are found and the conditions of their stability are studied. It permits us to use the results obtained in [Sulikashvili, 1989a] to estimate the reactions developed in the constraints.

2 Method of study

Consider a system of the Lagrangian equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \qquad i = 1, ..., n, \qquad q = (q_1, ..., q_n) \tag{1}$$

with the Lagrange function

$$L = L(\dot{q}, q) \tag{2}$$

allowing a group of symmetry generated by a vector field

$$\frac{\partial q}{\partial \psi} = v(q) \tag{3}$$

Let

$$q = q(\psi, Q), \qquad Q = q(0, Q) \tag{4}$$

be a general solution of (3). Then by (4)

$$\frac{dq}{dt} = \frac{\partial q}{\partial \psi}\frac{d\psi}{dt} + \left(\frac{\partial q}{\partial Q}, \frac{dQ}{dt}\right) \equiv \frac{\partial q}{\partial \psi}\dot{\psi} + \frac{\partial q}{\partial Q_i}\dot{Q}_i \tag{5}$$

We assume summation with respect to repeating Latin indices from 1 to n. Field (3) is a field of symmetry for the Lagrangian system (1), (2). Hence the function

$$\mathcal{J}_1 = \left(\frac{\partial L}{\partial \dot{q}}, v\right) = p_\psi \tag{6}$$

is the first integral of equations (1), (2). This integral is linear with respect to the impulses $\partial L/\partial \dot{q}$. Substitute (4), (5) into (6) and consider the latter expression as an equation with respect to $\dot{\psi}$. Let

$$\dot{\psi} = \dot{\psi}(\dot{Q}, Q, \psi, p_\psi) \tag{7}$$

be its solution. Moreover, we assume that the conditions of unicity of this solution are fulfilled (that is usually true for mechanical systems).

Consider a function

$$R = \left[L - \dot{\psi} p_\psi\right]_{(5),(7)} \tag{8}$$

In the general case, this function would depend on Q, $\dot{Q}$, ψ, p_ψ. One can prove the following relations.

$$\partial R/\partial \psi = 0 \tag{9}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{Q}_i} = \frac{\partial R}{\partial Q_i}, \quad i = 1, ..., n, \quad \frac{\partial R}{\partial p_\psi} = -\dot{\psi} \tag{10}$$

3 Application to the system of interacting massive points rotating about an attracting center

Consider a system of massive points $\mathcal{I}$ moving in the central Newtonian gravitational field. Assume that these points interact to each other according to the law of action and reaction. In particular, such an interaction can be realized with holonomic constraints, for example, by inextensible massless rods or tethers. We introduce an absolute frame $\mathbf{N}X_\alpha X_\beta X_\gamma$ with the origin in the attracting center, $\mathbf{X}_i = (X_\alpha, X_\beta, X_\gamma)_i^T$, $i \in \mathcal{I}$ are coordinates of the vector OX_i pointed from the origin to the i-th massive point, $\mathbf{V}_i = (V_\alpha, V_\beta, V_\gamma)_i^T$ denotes its velocity, $r_i = (\mathbf{X}_i, \mathbf{X}_i)^{1/2}$. The kinetic energy of the system and the Newtonian potential are respectively

$$T = \frac{1}{2}\sum_{i \in \mathcal{I}} m_i \mathbf{V}_i^2, \quad U_N = -fM \sum_{i \in \mathcal{I}} m_i r_i^{-1}.$$

The reaction of constraint that acts to the i-th point from the j-th one is $\mathbf{F}_{ij}$. Then the equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{V}_i} = \frac{\partial L}{\partial \mathbf{X}_i} + \sum_{j \in \mathcal{I}} \mathbf{F}_{ij}, \quad L = T - U.$$

Since the external forces are central, the vector of kinetic moment is constant in the absolute space, and the system allows the integral of motion

$$\overrightarrow{\mathcal{J}}_\psi = \sum_{i \in \mathcal{I}} m_i \mathbf{X}_i \times \mathbf{V}_i.$$

If the absolute frame is chosen so as this constant vector is colinear to OX_β, then $\overrightarrow{\mathcal{J}}_\psi = (0, p_\psi, 0)$.

Let us apply to this system the Routh - Lyapunov reduction method described above. Introduce a mobile frame $Ox_\alpha x_\beta x_\gamma$, such that $OX_\beta = Ox_\beta$, and the angle between the corresponding axes of the absolute and mobile frames is ψ. Then

$$\begin{aligned}X_\gamma &= x_\gamma \cos\psi - x_\alpha \sin\psi,\\ X_\beta &= x_\beta,\\ X_\alpha &= x_\gamma \sin\psi + x_\alpha \cos\psi.\end{aligned} \tag{11}$$

The differentiation of the expressions (11) with respect to time leads to

$$\begin{aligned}V_\gamma &= v_\gamma \cos\psi - v_\alpha \sin\psi - \dot\psi\left(x_\gamma \sin\psi + x_\alpha \cos\psi\right),\\ V_\beta &= v_\beta, \qquad\qquad (12)\\ V_\alpha &= v_\gamma \sin\psi + v_\alpha \cos\psi + \dot\psi\left(x_\gamma \cos\psi - x_\alpha \sin\psi\right),\\ \mathbf{v}_i &= \left(v_\alpha, v_\beta, v_\gamma\right)_i^T = \left(\dot x_\alpha, \dot x_\beta, \dot x_\gamma\right)^T.\end{aligned}$$

Substitution of the formulae (12) into the expression for the kinetic energy results in

$$\mathcal{T}(\mathbf{x},\mathbf{v};\dot\psi) = \frac{1}{2}\left(a + 2b\dot\psi + c\dot\psi^2\right) \tag{13}$$

where

$$\begin{aligned}a &= \sum_{i\in\mathcal{I}} m_i \mathbf{v}_i^2, \quad b = \sum_{i\in\mathcal{I}} m_i \left(v_\alpha x_\gamma - v_\gamma x_\alpha\right)_i\\ c &= \sum_{i\in\mathcal{I}} m_i \left(x_\alpha^2 + x_\gamma^2\right).\end{aligned}$$

Then after some simplification the Routh function can be written as

$$R(\mathbf{x},\mathbf{v};p_\psi) = \frac{1}{2}\left[a - c^{-1}(p_\psi - b)^2\right] - U_N \tag{14}$$

The amended potential can be written as

$$U_A(\mathbf{x};p_\psi) = -R(\mathbf{x},0;p_\psi) = U_c + U_N \tag{15}$$

Its two components U_c and U_N correspond to centrifugal and Newtonian forces respectively.

4 Application to the orbiting tetrahedral body with tethered elements

According to the Routh theory [Routh, 1892] (see also, for example, [Karapetyan, 1998]) to find stationary motions of the considered system one should find the critical points of the restriction of the amended potential onto the manifold defined by the constraints. Their analysis permits one to obtain the conditions of stability for stationary motions. Moreover, the use of the Lagrangian multipliers allows to determine if the constraint in question is stretched or compressed. If the constraint turns out to be stretched, it can be implemented using a massless inextensible tether; otherwise it is necessary to examine other options for their implementation.

Assume that a regular tetrahedron with the edge ℓ composed by four massive points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{S}$ of masses $m/4$ moves in the central Newtonian gravitational field. The center of the tetrahedron is denoted as $\mathbf{O}$, the constraints are realized by massless rods or, if possible, tethers.

The central tensor of inertia of the system is spherical, so according to Steiner's theorem the moment of inertia c of the whole system with respect to the axis of rotation Nx_β does not depend on orientation of the body and depends only on distance R from the tetrahedral center $\mathbf{O}$ to this axis. If J is the principal component of the central tensor of inertia, then

$$c = mR^2 + J.$$

It means that the centrifugal term

$$U_c = \frac{p_\psi^2}{2\left(mR^2 + J\right)}$$

in the expression for the amended potential is not involved in determination of the body's orientation in stationary motions: it is involved only in determination of the "orbital radius" R.

The potential of the Newtonian attraction reads

$$U_N = -.25mG\left(r_{NA}^{-1} + r_{NB}^{-1} + r_{NC}^{-1} + r_{NS}^{-1}\right), \tag{16}$$

where G is the gravitational constant.

4.1 The Routh Function

Suppose the vectors $\overrightarrow{\mathbf{NP}}, \mathbf{P} \in \mathcal{I} = \{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{S}\} \cup \mathbf{O}$ are given by their components in the rotating frame. Since $\mathbf{O}$ is the system's center of mass, one can write

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{S} = \mathbf{O}. \tag{17}$$

Six constraints that fix the distances between tetrahedron's vertices can be written as

$$f_{\mathbf{PQ}} = (\mathbf{P} - \mathbf{Q})^2 - \ell^2 = 0, \quad \mathbf{P} \neq \mathbf{Q}, \quad P, Q \in \mathcal{I}. \tag{18}$$

The expression for the Routh function can be written as

$$W_\lambda = U_A + \frac{1}{2}\sum\nolimits_{I\neq J\in\mathcal{I}} \lambda_{IJ} f_{IJ}. \tag{19}$$

The equations of relative equilibria

$$\frac{\partial W_\lambda}{\partial x_{\alpha i}} = 0, \quad \frac{\partial W_\lambda}{\partial x_{\beta_i}} = 0, \quad \frac{\partial W_\lambda}{\partial x_{\gamma_i}} = 0. \tag{20}$$

are cumbersome, so we don't represented them here. These equations should be completed by six equations of constraints (18). The resulting system is non-linear and too complex to be solved directly. However, their solutions, at least their part related to the structure of the equilibria configurations, are known.

According to [Sulikashvili, 1989a], there exist three dynamically different classes of relative equilibria:

I. Stable relative equilibria, when one of the tetrahedron vertices points to the Earth;
II. Unstable relative equilibria with the instability degree $\chi = 1$, when the middle points of a couple of tetrahedron's skew edges are located at the local vertical;
III. Unstable relative equilibria with the instability degree $\chi = 2$, when the center of a tetrahedron face points to the Earth.

For all of them the center $\mathbf{O}$ moves in the plane $\mathbf{N}X_\gamma X_\alpha$. It was shown in [Sulikashvili, 1989a] that each of the above classes is given by one-parametric family of solutions that can be obtained from each other by rotation about the local vertical. It holds true because for the system in question all central moments of inertia are equal. Moreover, it was proved that there is no more classes of relative equilibria.

4.2 Reactions in the links

To determine the reactions in constraints consider the above families of solutions. The analytical expressions for the obtained reactions are also cumbersome. However, one can assume that the size of the tetrahedron is considerably smaller than its distance from the attracting center. Using the respective small parameter to develop the expressions for reactions, one obtains the following results.

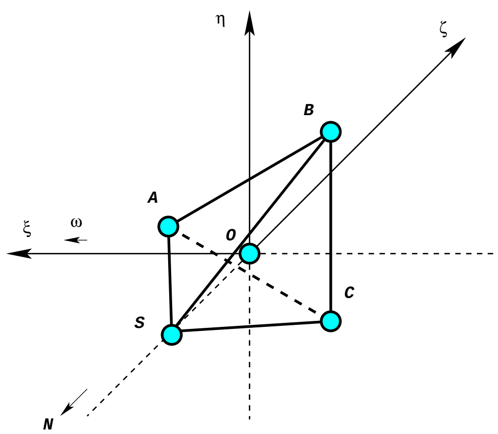

Figure 1. Equilibrium configuration of Class I.

EQUILIBRIUM OF CLASS I.

Assume that the satellite is pointed to the earth by its vertex $\mathbf{S}$. The calculation shows that for this relative equilibrium the main terms of Lagrangian multipliers read:

$$\lambda_{AS} = \lambda_{BS} = \lambda_{CS} = \frac{(2\sqrt{6}+3\ell)\sqrt{6}}{16} \quad (21)$$
$$\lambda_{AB} = \lambda_{AC} = -\frac{1}{4} + \frac{\sqrt{6}\ell}{48}, \quad \lambda_{BC} = -\frac{3}{4} + \frac{\sqrt{6}\ell}{48}$$

The edge length ℓ is small compared to 1, so according to (21) λ_{AS}, λ_{BS}, and λ_{CS} are positive. The respective constraints are stretched, and the point $\mathbf{S}$ can be attached to other vertices by massless inextensible tethers. At the same time, the other three multipliers are negative, so the respective constraints are stressed. Their implementation should include either massless rigid rods or some active control tools.

Since the stability of relative equilibria of Class I, they are the most important for applications.

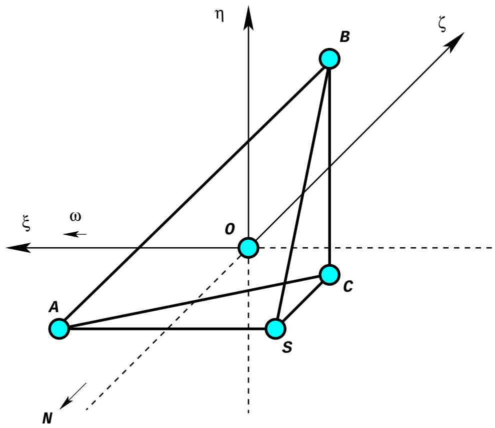

Figure 2. Equilibrium configuration of Class II.

EQUILIBRIUM OF CLASS II.

Consider the relative equilibrium of Class II with the following coordinates of the tetrahedron vertices For this equilibrium, the middle points of tetrahedron's skew edges $\mathbf{AS}$ and $\mathbf{BC}$ lie on the local vertical $\mathbf{O}\zeta$. The edge $\mathbf{AS}$ lies in the plane of the orbit below the center of mass $\mathbf{O}$, while the edge $\mathbf{BC}$ is located above the point $\mathbf{O}$, and the points $\mathbf{B}$ and $\mathbf{C}$ are symmetric to each other with respect to the orbit plane.

The Lagrangian multipliers in this case are:

$$\lambda_{AB} = \lambda_{AC} = \lambda_{CS} = \lambda_{BS} = \frac{3}{4} \quad (22)$$

$$\lambda_{AS} = -\frac{3}{4} - \frac{3\sqrt{2}\ell}{8}, \quad \lambda_{BC} = -\frac{5}{4} - \frac{3\sqrt{2}\ell}{8} \quad (23)$$

One can see that λ_{AB}, λ_{AC}, λ_{BS}, and λ_{CS} are positive, so the respective constraints are stretched, and the points $\mathbf{A}$ and $\mathbf{S}$ can be attached to the segment $\mathbf{BC}$ with massless inextensible tethers. Since $\ell \ll 1$, (23) imply that the multipliers λ_{AS} and λ_{BC} are negative, the respective links are compressed, and the constraints $\mathbf{AS}$ and $\mathbf{BC}$ should be implemented using either massless rigid rods or some active control tools.

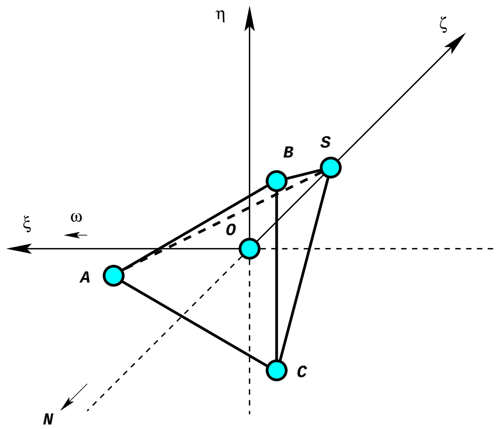

Figure 3. Equilibrium configuration of Class III.

EQUILIBRIUM OF CLASS III.

Consider the following relative equilibrium of Class III In this configuration, the center of the face $\mathbf{ABC}$ points to the Earth, while the point $\mathbf{A}$ lies in the orbit plane, and the points $\mathbf{B}$ and $\mathbf{C}$ are symmetric to each other with respect to the orbit plane.

For this relative equilibrium the Lagrangian multipliers are

$$\lambda_{AS} = \lambda_{BS} = \lambda_{CS} = -\frac{(-2\sqrt{6} + 3\ell)\sqrt{6}}{16} \qquad (24)$$

$$\lambda_{AB} = \lambda_{AC} = -\frac{1}{4} - \frac{\sqrt{6}\ell}{48}, \quad \lambda_{BC} = -\frac{3}{4} - \frac{\sqrt{6}\ell}{48} \qquad (25)$$

Since $\ell \ll 1$, relations (24) mean that λ_{AS}, λ_{BS}, and λ_{CS} are positive, the respective constraints are stretched, and the point $\mathbf{S}$ can be attached to other vertices by massless inextensible tethers. Meanwhile, equalities (25) imply $\lambda_{AB} < 0$, $\lambda_{AC} < 0$, and $\lambda_{BC} < 0$, so the links $\mathbf{AB}$, $\mathbf{AC}$, and $\mathbf{BC}$ are stressed. Thus the respective constraints should be kept either using massless rigid rods or by some active control tools.

5 Conclusion

We described the Routh reduction for symmetries given implicitly. The proposed approach was used to write down the amended potential for the system of interacting massive points moving under the action of the central Newtonian attraction. For special orbital system composed with four identical satellites in vertices of the regular tetrahedron realized by six constraints, expressing the constancy and equality of distances between each pair of satellites the orbital dynamics is considered. Since the configuration possesses the symmetry group of regular tetrahedra, the usually used "satellite approximation" for the Newtonian potential proves insufficient, and we applied the terms up to the third order ("post-satellite approximation").

When system's center of mass describes a circular orbit, there exist three group of equilibrium orientations. For these configurations, we have found the reactions in the links.

In a stable equilibrium configuration, three links that are parallel to the local horizontal plane are compressed, and so they should either be rigid or be kept stretched by an active control effort. The other three links are stretched, so one can use tethers in these elements of tetrahedron.

The other two groups of equilibria are unstable.

For the configuration with instability degree one, the two links orthogonal to the local vertical should be rigid, while the other four constraints can be implemented using tethers.

For the configuration with instability degree two, three links that lie in the local horizontal plane are compressed and so should be rigid, while the other three links can be made by tethers.

This study can be extended to the cases of other regular polyhedra.

The obtained results can be applied for formation flying in order to minimize the control efforts necessary to maintain the formation shape and orientation.

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