# Differential Equations with Aftereffect Perturbed by Impulses

J.V. Fetisova
*Institute of Mathematics and Mechanics of Russian Academy of Sciences, Ekaterinburg, Russia*

The paper deals with formalization of notion of discontinuous solutions for nonlinear systems of differential equations with impulse functions and with aftereffect in phase coordinates. The reviews of various approaches to solve this problem for nonlinear systems of ordinary differential equations are contained in [1-2]. The main approaches to formalization problem are following. The first approach is associated with substitution of the differential equation by integral one, where the integral is treated as Lebegue-Stielties or Perron-Stielties integral. The second approach is connected with formalization of operation of discontinuous and generalized functions multiplica tion. The third is based on definition of solution using the closure of a set of smooth solutions in a space of function of bounded variation. This approach is natural from the point of view of control theory [3] where impulsive control are frequently the idealized processes with the large changes of parameters in short time intervals. The mentioned approaches can be applied to functional differential equations as well.

In [4] the latter approach was developed for systems with constant delay when the system reaction on generalized action is unique and does not depend on functions approximating this action. Here, the the case with non-unique system reaction on impulse action is considered.

Consider the following Cauchy problem

$$\begin{gathered} \dot{x}(t) = f\Big(t, x(t), x(t-\tau), \int_{-\tau}^{0} h(t,s,x(t+s))dg_s(t,s), v(t), V(t)\Big) \\ +B\Big(t, x(t), [x(t-\tau)], v(t), V(t)\Big)\ \dot{v}(t), \qquad (1) \\ x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0]. \end{gathered}$$

Here $t \in [t_0, \vartheta]$, $x(t)$ and $v(t)$ are respectively $n$- and $m$-vector functions of time, $h(t,s,x)$ – is continuous $n$-vector function, $g(t,s)$ – is measurable in $t$ and continuous in $s$ function, $f(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is an $n$-vector function, and $B(\cdot,\cdot,\cdot,\cdot,\cdot)$ is an $n\times m$-matrix function, $V(t) = \underset{[t_0,\,t]}{\mathrm{var}}\, v(\cdot)$, $v(\cdot) \in BV_m[t_0,\vartheta]$, where $BV_m[t_0,\vartheta]$ denotes Banach space of $m$-vector functions of bounded variation, $\tau > 0$ is a constant delay, $[x(t-\tau)]$ is a vector containing only continuous coordinates of vector $x(t-\tau)$, $\varphi(t)$ is an initial $n$-vector function of bounded variation with corresponding to coordinates of vector $[x(t-\tau)]$ continuous coordinates.

Assume that $f(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is measurable in $t$, continuous in rest variables and Lipschitz in $x$, $B(\cdot,\cdot,\cdot,\cdot,\cdot)$ is continuous and Lipschitz in $x$ on the set $\{t \in [t_0,\vartheta],\ \|x\| < \infty,\ \|v\| < \infty,\ V < \infty\}$, where $\|x\| = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}$, and satisfy to the following standard conditions on the same set:

$$\|f(t,x,y,z,v,V)\| \le \kappa(1+\|x\|), \quad \|B(t,x,y,v,V)\| \le \kappa(1+\|x\|),$$

where $\kappa$ is some positive constant.

Let us choose a sequence of absolutely continuous functions $v_k(t)$, $k = 1,2,\dots$ pointwise converging to $v(t) \in BV_m[t_0,\vartheta]$. Assume that a solution of Cauchy problem (1) exists for every absolutely continuous function $v_k(t)$ (the functions $v(t)$ and $v_k(t)$, $k = 1,2,\dots$ satisfy the constraint $\underset{[t_0,\vartheta]}{\operatorname{var}} v(\cdot) \le a$), and to each function $v_k(t)$, $k = 1, 2, \ldots$ corresponds the solution $x(t) = x_k(t)$ of Cauchy problem (1) with $v(t) = v_k(t)$.

*Definition 1.* A vector function of bounded variation $x(t)$ is called the *approximable solution* of Cauchy problem (1), if $x(t)$ is the pointwise limit of the sequence $x_k(t)$, $k = 1, 2, \ldots$ generated by a sequence $v_k(t)$, and $x(t)$ does not depend on the choice of $v_k(t)$.

Further, a sequence $v_k(t)$ is said to be *$V$-convergent* to $v(t) \in BV_m[t_0, \vartheta]$, if $\lim_{k\to\infty} v_k(t) = v(t)$ for all $t \in [t_0, \vartheta]$, and $\lim_{k\to\infty} \underset{[t_0,\,t]}{\operatorname{var}} v_k(\cdot) = V(t)$.

*Definition 2.* Every partial pointwise limit of the sequence $x_k(t)$, $k = 1, 2, \ldots$ generated by an arbitrary $V$-convergent sequence of absolutely continuous functions $v_k(t)$, $k = 1, 2, \ldots$ will be referred to as *$V$-solution* of the problem (1).

Let

$$z(\tau(t_i - 0)) = x^*(\tau(t_i - 0)) \ \ (z(\tau(t_i)) = x^*(\tau(t_i))) \, ,$$
$$\mu(\tau(t_i - 0)) = v^*(\tau(t_i - 0)) \ (\mu(\tau(t_i)) = v^*(\tau(t_i)))$$

be initial conditions for the system

$$\dot z(\xi) = B(t, z(\xi), [x(t_i - \tau)], \mu(\xi), V(t) + \xi - t)\eta(\xi),$$
$$\dot\mu(\xi) = \eta(\xi). \tag{2}$$

Denote by $S(t, x(t), [x(t-\tau)], \Delta v(t), V(t), \Delta V(t))$ (where $t = t_i - 0$ or $t = t_i + 0$) the set obtained by shifting the cross-section of the attainability set of (2) by the value $-x(t)$ at the instant $t + \Delta V(t)$. It is assumed in (2) that

$$\mu(\tau(t_i + \Delta V(t_i - 0)) = v(t_i) \ (\mu(\tau(t_i + \Delta V(t_i)) = v(t_i + 0)) \, ,$$

where the control $\eta(\xi)$ is subjected to the constraint $\|\eta(\xi)\| \le 1$.

**Theorem 1.** Every partial pointwise limit of the sequence $x_k(t)$ generated by a sequence of absolutely continuous functions $v_k(t)$, $k = 1, 2, \ldots$, where $v_k$ is $V$-convergent to $v(t) \in BV_m[t_0, \vartheta]$ is the solution of the integral inclusion

$$x(t) \in \varphi(t_0) + \int\limits_{t_0}^{t} f\Big(\xi, x(\xi), x(\xi - \tau), \int_{-\tau}^{0} h(\xi, s, x(\xi + s)) dg_s(\xi, s), v(\xi), V(\xi)\Big)\, d\xi$$
$$+ \int\limits_{t_0}^{t} B\Big(\xi, x(\xi), [x(\xi - \tau)], v(\xi), V(\xi)\Big)\, dv^c(\xi)$$
$$+ \sum_{t_i \le t,\, t_i \in \Omega_-} S\Big(t_i, x(t_i - 0), [x(t_i - \tau)], \Delta v(t_i - 0), V(t_i - 0), \Delta V(t_i - 0)\Big)$$
$$+ \sum_{t_i < t,\, t_i \in \Omega_+} S\Big(t_i, x(t_i), [x(t_i - \tau)], \Delta v(t_i + 0), V(t_i), \Delta V(t_i + 0)\Big), \tag{3}$$
$$x(t) = \varphi(t), \ t \in [t_0 - \tau, t_0].$$

where $v^c(\xi)$ is the continuous part of the function of bounded variation $v(\xi)$, $\Omega_-(\Omega_+)$ is the set of points at which the function $V(t)$ is discontinuous from the left (from the right).

For every solution $x(t)$ of the inclusion (3) generated by the pair $(v(t), V(t))$ there exists a sequence of absolutely continuous functions $v_k(t)$, $k = 1, 2, \ldots$ pointwise converging to $v(t)$ and the corresponding sequence $x_k(t)$ of solutions of (1) pointwise converging to $x(t)$.

Consider now the following problem

$$\dot{x}(t) = f\Big(t, x(t), x(t-\tau), \int_{-\tau}^{0} h(t,s,x(t+s))dg_s(t,s)\Big) + B\Big(t, x(t), [x(t-\tau)]\Big)\ \dot{v}(t), \tag{4}$$

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0].$$

**Theorem 2.** Let all the conditions given above are satisfied. Moreover we assume there exist the partial derivatives $\partial b_{ij}/\partial x_\nu$ of elements of the matrix function $B(\cdot,\cdot,\cdot)$ which satisfy the following equalities

$$\sum_{\nu=1}^{n} \frac{\partial b_{ij}}{\partial x_\nu} b_{\nu l} = \sum_{\nu=1}^{n} \frac{\partial b_{il}}{\partial x_\nu} b_{\nu j}$$

$i = 1, 2, ..., n;\ j, l = 1, 2, ..., m.$

Then for any vector function of bounded variation $v(t)$ there exists the approximable solution $x(t)$ of (4), which satisfies to the integral equation

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} f\Big(\xi, x(\xi), x(\xi-\tau), \int_{-\tau}^{0} h(\xi,s,x(\xi+s))dg_s(\xi,s)\Big)\, d\xi$$

$$+ \int_{t_0}^{t} B\Big(\xi, x(\xi), [x(\xi-\tau)]\Big)\, dv^c(\xi)$$

$$+ \sum_{t_i \le t,\, t_i \in \Omega_-} S\Big(t_i, x(t_i - 0), [x(t_i-\tau)], \Delta v(t_i - 0)\Big) + \sum_{t_i < t,\, t_i \in \Omega_+} S\Big(t_i, x(t_i), [x(t_i-\tau)], \Delta v(t_i + 0)\Big),$$

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0].$$

Here $v^c(\xi)$ is the continuous part of the function of bounded variation $v(\xi)$,

$$S\Big(t, x(t), [x(t-\tau)], \Delta v\Big) = z(1) - x,$$

$$\dot{z}(\xi) = B\Big(t, z(\xi), [x(t-\tau)]\Big)\Delta v(t), \qquad z(0) = x,$$

$\Omega_-(\Omega_+)$ is the set of points at which the function $v(t)$ is discontinuous from the left (from the right),

$$\Delta v(t-0) = v(t) - v(t-0),\ \ \Delta v(t+0) = v(t+0) - v(t).$$

*References:*


1. A.F. Filippov. *Differential equations with a Discontinuous Right-Hand Side,* Moskva: Nauka, 1985.
2. Zavalishchin, S.T., Sesekin, A.N., *Dynamic Impulse Systems: Theory and Applications,* Dordrecht, Netherlands: Kluwer Academic Publishers, 1997.
3. N.N. Krasovskii. *Motion control theory. Linear systems.* Moskva: Nauka, 1968.
4. J.V. Fetisova, A.N. Sesekin. Discontinuous solutions of differential equations with time delay. Wseas transactions on systems. Issue 5, Volume 4, 2005. P. 487-492.