FREQUENCY-DOMAIN CRITERION FOR STABILITY OF OSCILLATIONS IN A CLASS OF NONLINEAR FEEDBACK SYSTEMS

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Abstract: The paper, describes the problem of stability of oscillations in nonlinear feedback systems. The concept of stability is defined in a way that makes the problem tractable using the absolute stability approach. The result is formulated in frequency domain and has the form of the Zames-Falb multiplier, which makes it amenable to geometric interpretation. Numerical examples are given to illustrate the application of the new result to cases, where the Circle Criterion is not applicable. The advantage of the new criterion is that only the period of the oscillations needs to be known, not the complete expression of the oscillatory solution.

Keywords: Absolute stability, stability analysis, nonlinear control systems, oscillators.

1. INTRODUCTION

The problem of stability of periodic motion was first formulated in the classic book by Lyapunov (1992) and has since received considerable attention from many researchers, including Lyapunov himself.

The standard approach to this problem involves the investigation of the so-called variation equation. For the local stability problem, this approach leads to the well-studied linear differential equations with periodic coefficients (Yakubovich and Starzhinskii, 1975). The stability criteria involve computation of the Floquet multipliers, and various ad hoc estimation techniques. They are applicable to both forced and autonomous oscillations.

Another approach involves the use of the fixed-point theory. It is described in some length in the monographs by Holtzman (1974) and Burton (2005). Topological methods are studied in the book by Krasnoselskii (1968). These methods are applicable only to forced oscillations.

The approach proposed in this paper differs from the previous work in several ways. First, the nonlinear variation equations are studied instead of the linearized ones, leading to global, as opposed to local, stability results. Secondly, the proposed approach uses the known absolute stability criteria and, therefore, does not require any information about the periodic solution except for its period. Finally, the resulting criteria are much easier to check than the standard ones involving Floquet multipliers found in many standard textbooks on differential equations. The results are applicable to both forced and autonomous oscillations.

2. FORMULATION OF THE PROBLEM

Our starting point is the nonlinear feedback system in the vector-matrix form:

$$\dot{x} = Ax + Bu + f(t) \tag{1}$$

$$y = Cx \tag{2}$$

$$u = \varphi(y)\,. \tag{3}$$

Here *f(t)* is a periodic vector function with the period *T*. It may be identically equal to zero, but then there is a question of the existence of periodic solutions, which is outside of the scope of this paper. The function $\varphi(y)$ represents a nonlinear feedback. Its properties will be described later in the paper, when we formulate the results. Let $y = \phi(t)$ be a periodic output with a period T corresponding to the solution *x(t)*. Let $y = \overline{\phi}(t)$ be an output of the system corresponding to another solution $\overline{x}(t)$, not necessarily periodic, and let $\sigma(t) = \overline{\phi}(t) - \phi(t)$. In order to make use of some of the previously obtained absolute stability results, we write the variation equation in the integral form:

$$\sigma(t) = \alpha(t) + \int_0^t \Omega(t-\tau)\xi(\tau)d\tau$$

$$\xi(t) = \varphi(\sigma(t) + \phi(t)) - \varphi(\phi(t)).$$

Here

$$\Omega(t) = Ce^{At}B, \ \alpha(t) = e^{At}(\overline{x}(0) - x(0)).$$

Stability of the output $\phi(t)$ will be understood in the following sense: For the output $\overline{\phi}(t)$, corresponding to any other solution

$$\sigma(\cdot) \in L_2(0,\infty)$$

and there exists a constant λ, independent of the function $\alpha(\cdot)$, such that

$$\|\sigma(\cdot)\| \le \lambda \|\alpha(\cdot)\|.$$

The bars denote the usual Euclidean L_2 norm.

For the sake of simplicity, we consider the case of a SISO system, i.e. both the function $\varphi(\sigma)$, hereafter called the nonlinearity, and its argument σ are scalar. The results can be easily extended to MIMO systems.

We now proceed to the formulation of the results.

3. STATEMENT OF THE RESULTS

The main results of this paper follow almost directly from the earlier absolute stability results for systems with time periodic nonlinearities. For this reason they will be stated without proof. Throughout the paper, we denote the transfer function of linear part of the system by *W(s)* and define it by the usual equation:

$$W(s) = C^*[sI - A]^{-1}B.$$

3.1 Analytic Criterion.

The following result is an immediate consequence of the Theorem 3.2.5 from (Altshuller, 2004) and restated for sake of completeness in the Appendix.

THEOREM 1. Suppose that:

1. *The matrix* A *is Hurwitz;*
2. *For all* t *and all* $\sigma_1 \ne \sigma_2$;

$$0 \le \frac{\varphi(\sigma_1,t) - \varphi(\sigma_2,t)}{\sigma_1 - \sigma_2} \le \mu < \infty;$$

3. *There exists a sequence* θ_n *with nonnegative terms, such that* $\sum_{n=1}^{\infty}\theta_n < 1$ *and*[1]

$$\mathrm{Re}\left\{\left[\mu^{-1} + W(i\omega)\right]Z(i\omega)\right\} >> 0 \qquad (4)$$

with

$$Z(i\omega) = 1 - \sum_{n=0}^{\infty}\theta_n e^{i\omega nT}. \qquad (5)$$

Let $y = \phi(t)$ *be the output of the system (1-3) having a period* T. *Then for the output* $y = \overline{\phi}(t)$ *corresponding to any other solution of the system (1-3)*

$$\sigma(\cdot) \in L_2(0,\infty)$$

and there exists a constant λ, *independent of the function* $\alpha(\cdot)$, *such that*

$$\|\sigma(\cdot)\| \le \lambda \|\alpha(\cdot)\|,$$

where $\sigma(t) = \overline{\phi}(t) - \phi(t)$.

The general nature of this criterion makes difficult to use since it is not clear how to find the desired sequence θ_n. However, the left-hand side of the inequality (4) has a very convenient Zames-Falb multiplier form, which makes it possible to interpret this criterion geometrically as we proceed to do in the next subsection.

3.2 Geometric Interpretation.

With a slight abuse of notation, we can rewrite the inequality (4) in the form:

$$[\mu^{-1} + \mathrm{Re}\,W(i\omega)]\,\mathrm{Re}\,Z(i\omega) - \mathrm{Im}\,W(i\omega)\,\mathrm{Im}\,Z(i\omega) > 0$$

Let us define the two functions:

$$\Phi(\omega) = \frac{\mu^{-1} + \mathrm{Re}\,W(i\omega)}{\mathrm{Im}\,W(i\omega)}, \ \Psi(\omega) = \frac{\mathrm{Im}\,Z(i\omega)}{\mathrm{Re}\,Z(i\omega)}.$$

Note that since $\mathrm{Re}\,Z(i\omega) > 0$, the function $\Psi(\omega)$ is continuous for all values of ω.

It is relatively easy to show (Lipatov, 1981) that for the type of systems under consideration the graph of the function $\Phi(\omega)$ consists of branches with asymptotes. The ends of the branches point either to $+\infty$ (Such branches are called stalactites) or

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[1] For any function $X(i\omega)$, the expression $\mathrm{Re}\,X(i\omega) >> 0$ means that there exists a constant $\delta > 0$, such that for any real number ω, $\mathrm{Re}\,X(i\omega) > \delta$.

to $-\infty$ (Such branches are called stalagmites). The inequality (4) holds if a function $\Psi(\omega)$ can be found such that its graph separates stalactites from the stalagmites.

This geometric interpretation has been used extensively for systems with stationary nonlinearities. For the time-dependent case, the best known result is the Circle Criterion, for which $\Psi(\omega) \equiv 0$.

For the expression given by the Equation (5) we have:

$$\Psi(\omega) = \frac{\sum_{n=0}^{\infty} \theta_n \sin \omega n T}{\sum_{n=0}^{\infty} \theta_n \cos \omega n T - 1}.$$

This geometric interpretation of the analytic criterion is easier to use if the infinite series are replaced with finite sums. In the next section, several numerical examples will be given to illustrate the application of the criterion.

4. NUMERICAL EXAMPLES

Consider the system with the transfer function given by:

$$W(s) = \frac{b + G(s)}{aG(s) + 1}$$

where

$$G(s) = \frac{s^2}{[(s+p)^2 + 0.81][(s+p)^2 + 1.21]}.$$

Let us choose the following numerical values: a=0, b=0.04, p=0.5 and $\mu = 20$.

The first step is to plot the graph of the function $\Phi(\omega)$. It is shown in thin lines in Fig.1. For this example, it consists of one stalactite and one stalagmite. Clearly, the Circle Criterion is not satisfied in this case since both branches cross the horizontal axis.

Next, we choose

$$\theta_1 = 0.1,\ \theta_2 = 0.5,\ \theta_3 = 0.2$$

and plot the curve $\Psi(\omega)$ for various values of T (shown as a thick line).

Fig. 1 shows the plot for $T = 0.26\pi$. We notice that the branches of the curve $\Phi(\omega)$ are separated and conclude that the oscillations with this period are stable.

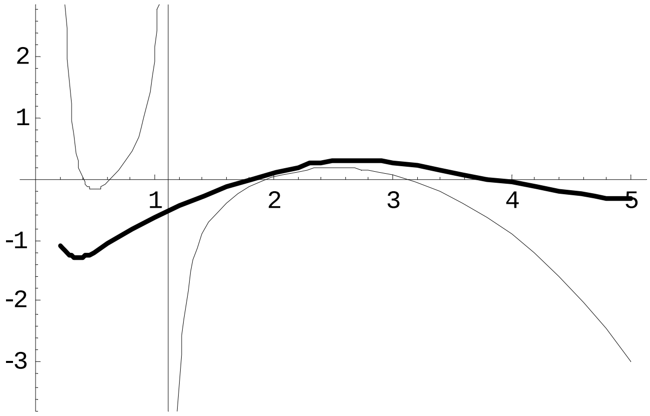

Fig. 1.

Fig. 2 shows the plot for $T = 0.3\pi$ and leads to the same conclusion, as do the plots for $0.26\pi < T < 0.3\pi$ (not shown).

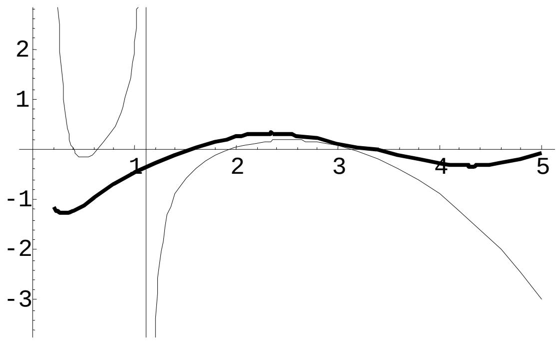

Fig. 2.

Let us now choose a different set of numerical values: a=3, b=0.167, p=0.5 and $\mu = 20$. For the function $\Psi(\omega)$ we use $\theta_1 = 0.1$, $\theta_2 = 0.4$, $\theta_3 = 0.3$.

Plotting the curves (Fig. 3), we find that the oscillations with the period $T = 0.2\pi$ are stable.

Once again, we note that the Circle Criterion is not met in this case.

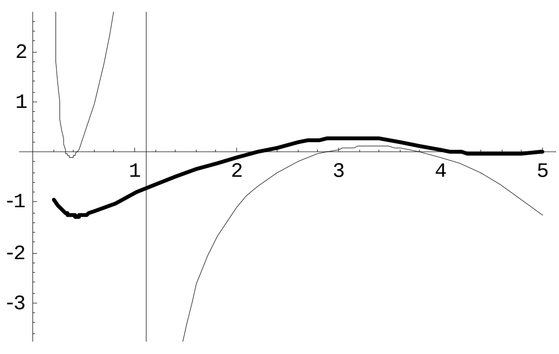

Fig. 3.

Let us now consider an example with three branches:

$$G(s) = \frac{s^2}{[(s+p)^2+0.81][(s+p)^2+1.21](s+p)^2+1.44]}$$

We use the same numerical values as in the previous example except that we set $\mu = 500$. For the function $\Psi(\omega)$ we use only one term of the series: $\theta_1 = 0.8$.

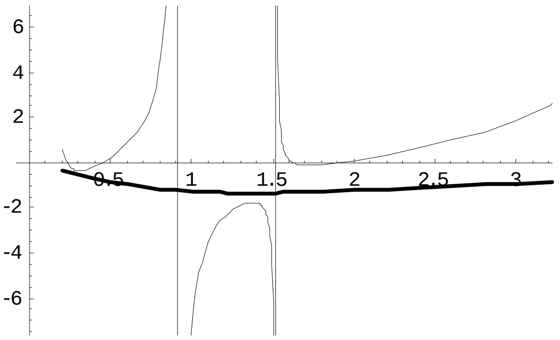

Fig. 4.

Fig. 4 shows the plot for $T = 0.1\pi$. The plot for $T = 0.5\pi$ is shown in Fig.5.

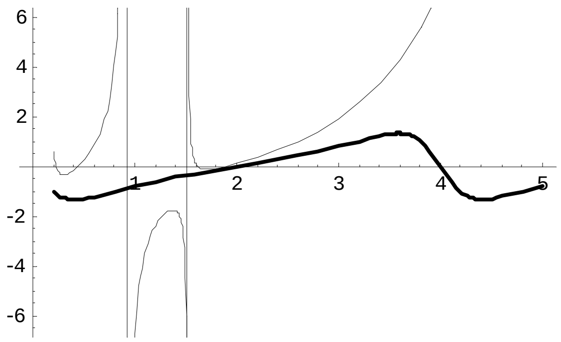

Fig. 5.

We observe that the stalactites are separated from the stalagmites and conclude that the oscillations are stable. The same conclusion is reached for $0.1\pi < T < 0.5\pi$. Therefore, in this example, it turns out to be easier to guess the desired function $\Psi(\omega)$, and the result is applicable to a wider range of the periods.

5. DISCUSSION

We have developed an analytic criterion for stability of oscillations in nonlinear feedback systems and illustrated numerically how this criterion can be applied, via geometric interpretation.

It is important to notice that we only needed to know the period of oscillations, not the expressions for the periodic solutions of the system under consideration. In addition, as in the absolute stability problem, we did not require the expressions for the nonlinearity. The only information needed was it satisfied the sector condition. This makes the result applicable to a wide class of systems.

Future research may proceed in the following two directions. First, investigation of the numerical examples required a considerable amount of "guesswork" to find the desired function $\Psi(\omega)$. It will be advantageous to find a more systematic method or algorithm for finding this function. Secondly, the range of oscillation periods for which each choice is applicable is rather narrow. Therefore, it will be worthwhile to determine which function gives the best range for a given example.

6. APPENDIX

Here we restate for the sake of completeness an earlier absolute stability criterion for systems with time-periodic feedback. The system under consideration has a linear block described by the Volterra integral equation:

$$\sigma(t) = \alpha(t) + \int_0^t \Omega(t-\tau)\xi(\tau)d\tau \; .$$

The nonlinear feedback block is given by:

$$\xi(t) = \varphi(\sigma(t), t) \; .$$

The transfer function (frequency characteristic) in this case is defined as the Fourier transform of the kernel:

$$W(i\omega) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Omega(t) e^{-i\omega t} dt$$

The absolute stability criterion is stated as follows.

THEOREM 2. Suppose that:

1. $\alpha(\cdot) \in L_2(0,\infty)$; $|\Omega(t)| < \eta e^{-\varepsilon t}$, $\varepsilon > 0$;
2. *For all* t *and all* $\sigma_1 \neq \sigma_2$;
$$0 \le \frac{\varphi(\sigma_1, t) - \varphi(\sigma_2, t)}{\sigma_1 - \sigma_2} \le \mu < \infty \; ;$$
3. $\varphi(\sigma, t+T) = \varphi(\sigma, t)$;
4. *There exists a sequence* θ_n *with nonnegative terms, such that* $\sum_{n=1}^{\infty} \theta_n < 1$ *and*
$$\mathrm{Re}\left\{\left[\mu^{-1} + W(i\omega)\right] Z(i\omega)\right\} >> 0$$
with
$$Z(i\omega) = 1 - \sum_{n=0}^{\infty} \theta_n e^{i\omega nT}$$

Then for any solution $\sigma(t)$, $\sigma(\cdot) \in L_2(0,\infty)$ *and there exists a constant* λ , *independent of the function* $\alpha(\cdot)$, *such that* $\|\sigma(\cdot)\| \le \lambda \|\alpha(\cdot)\|$.

Clearly, if the conditions of the Theorem 1 are met, so are the conditions of the Theorem 2. Therefore, the former is the immediate consequence of the latter.

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