Estimating Phase Equations from Multivariate Time Series

Isao T. Tokuda , Swati Jain , Istvan Z. Kiss , and John L. Hudson

Abstract— **A novel approach is presented for extracting phase equations from multivariate time series data recorded from a network of weakly coupled limit cycle oscillators. Our aim is to estimate all the important properties of the phase equations such as natural frequencies and interaction function between the oscillators. Our approach requires the measurement of an experimental observable of the oscillators; in contrast to previous methods, it does not require measurements in isolated single, or two-oscillator setups. This non-invasive technique should be advantageous in biological systems, where extraction of few oscillators may be a difficult task. The method is most efficient when data is taken from the non-synchronized regime where the phases of the oscillators are affected by coupling but where no complete phase locking of the oscillators occurs. Applicability to experimental systems is demonstrated by using a network of electrochemical oscillators; the experimentally obtained phase model is used to predict the synchronization diagram of the system.**

I. INTRODUCTION

Synchronization is a ubiquitous phenomenon of coupled nonlinear oscillators found in many fields of natural science and engineering. Comprehensive investigations have been made from both theories and experiments on synchronization of coupled limit cycle oscillators as well as coupled chaotic oscillators [1]. One of the most important theoretical developments is the phase reduction theory of weakly coupled limit cycle oscillators [2]. This theory provides the standard methodology for analysis of various systems, ranging from electrochemical oscillators [3], neurons [4], brain activity [5], and circadian oscillations [6], [7]. However, the important problem, how to construct phase models from measurement data, remains open. It is of special interest to infer phase equations that characterize the underlying coupled system simply in terms of natural frequencies of individual oscillators and interaction functions between the oscillatory elements. Up to now, several techniques have been developed to construct phase equations from nonlinear systems. [2], [4], [8], [9], [10]. One of them is the perturbation method [2], [8], [9]. Despite its high reliability, the method has a severe limitation to measurement data, since it requires the application of external perturbations to nonlinear oscillator in an isolated condition. Other methods also have restrictions such as they are applicable only up to a system of two coupled oscillators [4], [10]. In order for its wider application

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to experimental (most notably biological) systems, which often have a network structure composed of many oscillators, much stronger technique should be established. Non-invasive approach is desired not to destroy the original configuration of the biological network.

The aim of this presentation is to demonstrate a novel approach to the development of experiment-based phase models of rhythmic, weakly interacting systems [11]. Our approach has the important practical advantages that (1) it can be applied to a network of many oscillators, (2) it is a non-invasive approach, and (3) the technique can be easily implemented into simple codes and parameter estimation has an excellent convergence property.

II. PROBLEM AND METHOD

Consider a system of N weakly coupled limit cycle oscillators:

$$\dot{x}_i = F_i(x_i) + \frac{C}{N}\sum_{j\neq i}^{N} G(x_i, x_j), \qquad (1)$$

where x_i and F_i ($i = 1, 2, \cdots, N$) represent state variables and dynamics of the i-th oscillator, C and G_i represent coupling constant and interaction function. Our assumption is that in isolated condition ($C = 0$) each oscillator F_i gives rise to a stable limit cycle with similar natural frequencies ω_i. Then the phase reduction theory [2] states that for weak global coupling C the network dynamics is reduced to phase equations of the following form:

$$\dot{\theta}_i = \omega_i + \frac{C}{N}\sum_{j=1}^{N} H(\theta_j - \theta_i). \qquad (2)$$

As a recording condition, we suppose that simultaneous measurement of all oscillators is made as $\{x_i(n\Delta t) : n = 1, \cdots, M\}_{i=1}^{N}$ (Δt: sampling time). Our goal is to infer the phase equations (2) from the measurement data under the conditions that (i) underlying dynamics (1) is unknown, (ii) coupling constant C associated with the measured data is taken from non-synchronous regime [12], and (iii) coupling type is known to be global.

Our approach to the problem can be described as follows.

1) Determine phases $\theta_i(t)$ from data $x_i(t)$. Among various definitions of phases [1], a simple formula is chosen, where phase θ is increased by 2π at every local maximum of $x(t)$ and between the local maxima the phase grows proportionally to time.

2) Fit the phases $\{\theta_i(t)\}$ to the phase equations:

$$\dot{\theta}_i = \omega_i + \frac{C}{N}\sum_{j=1}^{N}\tilde{H}(\theta_j - \theta_i), \qquad (3)$$

$$\tilde{H}(\Delta\theta) = \sum_{j=1}^{D} a_j\sin j\Delta\theta + b_j(\cos j\Delta\theta - 1), (4)$$

by estimating the unknown parameters $p = \{\omega_i, a_j, b_j\}$ via multiple-shooting method [13]. Note that the interaction function H, which is in general nonlinear and periodic with respect to 2π, is approximated by the Fourier expansion up to the order of D. [Assuming a difference coupling, the interaction function is set to be zero for zero phase difference, *i.e.,* $\tilde{H}(0) = 0$].
In the multiple-shooting, we denote time evolution of the phase equations (3),(4) with respect to initial condition $\theta(0)$ by $\theta(t) = \phi^t(\theta(0), p)$. Then, at each sampling time $t = i\Delta t$, the phase equation must satisfy the following boundary conditions

$$\theta((n+1)\Delta t) = \phi^{\Delta t}(\theta(n\Delta t), \mathbf{p}). \qquad (5)$$

With respect to the unknown parameters p, we solve the above nonlinear equations by the generalized Newton method. In practice, the evolution function ϕ^t is integrated numerically. For the computation of the gradients $\partial\phi/\partial(p)$ which are needed for the Newton method, variational equations of the phase equations (3),(4) are also solved numerically.

3) To avoid the over-fitting problem, cross-validation technique is utilized to determine the optimum number of higher harmonics in the interaction function, D [14]. We divide the multivariate data into two parts. For the first half data, the parameter values p are estimated. Then we apply the estimated parameters to the latter half data and measure the error. The order number D giving rise to the minimum error for the latter data is considered to be the optimum.

III. Application to Electrochemical Oscillators

Let us apply our technique to experimental data from electrochemical oscillatory system. The experiments were carried out in a standard electrochemical cell containing 3 mol/dm^3 sulfuric acid kept at a temperature of 11 ^oC with an 8×8 array of 1 mm diameter nickel, a $Hg/Hg_2SO_4/K_2SO_4$ reference, and a Pt counter electrode. (For the experiments with 32 electrodes only half of the electrodes were connected in the array.) The currents, proportional to the rate of electro-dissolution, were measured at a frequency of 100 Hz using zero resistance ammeters. The electrodes, held at a potential of $V = 1.105$V, were connected to a potentiostat through one series (collective) resistor, R_s, and through N parallel resistors (N is the number electrodes). The dimensionless interaction strength $K = NR_s/R_p$ was controlled through the external resistors by keeping the equivalent resistance $R_{eq} = R_p + NR_s$ constant[3]. (The physical coupling

strength is obtained as $C = K/(R_{eq}C_d)$, $C_d = 1.3$mS as measured by impedance spectroscopy.)

First, we consider a network of 32 electrochemical oscillators. Multivariate data $\{y_i(t)\}_{i=1}^{32}$ were measured from all $N = 32$ oscillators with a coupling strength of $C = 0.018$ 1/s, which gives rise to non-synchronized dynamics. The recording interval of 5 sec was utilized for the parameter estimation. Figures 1 and 2 show the estimation results based on the optimum $D = 1$. Natural frequencies are well estimated with slightly higher values than those obtained with $C = 0$. Moreover, the shape of the estimated interaction function $\tilde{H}(\Delta\theta)$ is in a good agreement with that estimated by applying the perturbation method to a single isolated electrochemical oscillator [9]. The interaction function consists of strong $\sin(\Delta\phi)$ and $1 - \cos(\Delta\phi)$ terms and higher harmonics are very weak; this functional form is consistent with theoretical prediction of Stuart-Landau oscillators close to a Hopf-bifurcation [2].
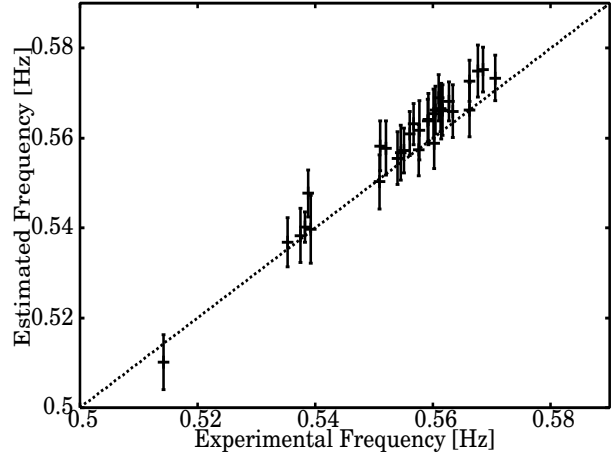

Fig. 1. Estimated natural frequencies (vertical axis) $\{\omega_i\}_{i=1}^{32}$ of 32 electrochemical oscillators vs. the measured natural frequencies (horizontal axis).
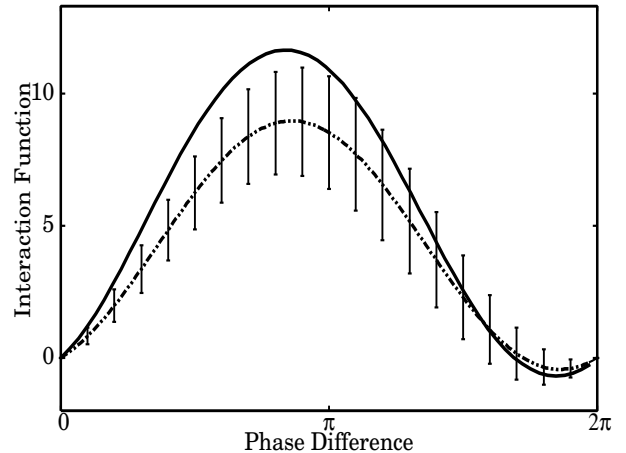

Fig. 2. Interaction function $H(\Delta\theta)$ estimated by the present method from multivariate data of a network of 32 electrochemical oscillators (dotted line) is compared with that estimated by applying the perturbation to a single isolated electrochemical oscillator (solid line) [9].

Finally, we demonstrate capability of our approach to predicting the synchronization structure. For another network of 64 globally coupled electrochemical oscillators a synchronization diagram (dependence of the order parameter Φ on the coupling strength in the range of $C \in [0, 0.17]$) has been previously obtained [15]. We examine the power of the extracted experiment-based phase model in recovering this diagram. The estimated model is simulated to produce the synchronization diagram. Fig. 3a shows the dependence of the order parameter Φ on the coupling strength when only a single data set with $K = 0.033$ close to the Kuramoto transition point was used to obtain the phase model. (The order parameter can be computed according to $Re^{i\Phi} = \frac{1}{N}\sum_{j=1}^{N}e^{i\phi_j}$ [2], [3].) In spite of the drift, relative shortness of the time series, and the larger (64) population size, the phase model gives an excellent prediction of the order *vs.* coupling strength curve.
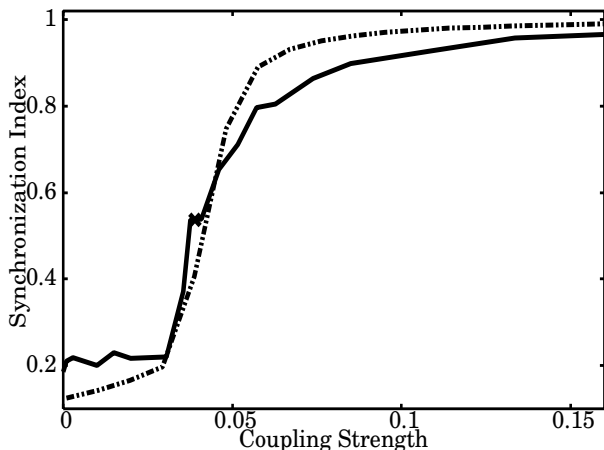

Fig. 3. Synchronization diagram of a network of 64 electrochemical oscillators. The order parameter Φ is plotted against the coupling strength K. The model prediction (dotted line) is compared with the experimental curve (solid line with crosses). Location of the measured data at $K = 0.039$ is marked by a box.

IV. CONCLUSIONS

An approach has been presented for inferring phase equations from multivariate time series. Analysis of measurement data from electrochemical oscillators demonstrated practical applicability to experimental systems. The present approach is rather general and can be extended to more complex situations such as the case of nonuniform coupling $C_{i,j}$. Such extended cases will be investigated in forthcoming studies. One of the future challenges is to apply the present approach to a network of oscillators in biological systems. Recent technological advances made simultaneous (optical or electrical) measurements of the rhythmic cells possible in networks; examples include circadian gene expressions of suprachiasmatic nucleus (SCN) [6] or the synchronous electrical activity in neuronal pacemakers [16]. Such recording condition would facilitate the construction of phase models from experimental data that could be used for the simple yet accurate description of a large number of rhythmic, interacting cells.

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