MEAN SQUARE STABILITY AND CONTROL FOR INVARIANT MANIFOLDS OF NONLINEAR STOCHASTIC SYSTEMS

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Abstract: A mean square stability for the invariant manifolds of nonlinear stochastic systems is considered. The first approximation linear systems for invariant manifolds are introduced and a notion of P-stability (projective) is proposed. A criterion for P-stability is obtained. Mean square stabilization of periodic and quasiperiodic solutions of stochastically forced nonlinear systems is considered. The necessary and sufficient stabilizability conditions are presented. The methods for design of feedback stabilizing regulator for SDEs are suggested. The examples of constructive solving of stochastic control problem are demonstrated.

Keywords: Invariant manifolds, control, cycles, tori, stochastic stability

1. INTRODUCTION

Many nonlinear phenomena of mechanics observed under transition from the order to chaos are frequently connected with a chain of bifurcations: a stationary regime (equilibrium point) - periodic regime (limit cycle) - quasiperiodic regime (torus) - chaotic regime (strange attractor). Each such transition is accompanied by the loss of stability of simple attractor and new more complicated stable attractor birth. Stability analysis of appropriate invariant manifolds is key for undestanding of the complex behavior of nonlinear dynamical systems. The stability investigation and control of stochastic systems are attractive from theoretical and engineering points of view. Even weak noise can result in qualitative changes in the system's dynamics. We consider the mean square stability problem for invariant manifolds of stochastic differential equations (SDEs). One of the most important directions of stability analysis is Lyapunov function technique (LFT) (Krasovskii (1963); Kats and Krasovskii (1960); Khasminskii (1980); Kushner (1967)). LFT in research of a stationary point stochastic stability has been widely studied by many authors (see Arnold (1974); Arnold (1998); Mao (1994)).

The orbital Lyapunov functions were used in stability and sensitivity analysis via a quasipotential of stochastic forced limit cycles (Ryashko (1996); Bashkirtseva and Ryashko (2002); Bashkirtseva and Ryashko (2004)).

Deterministic LFT for stability analysis of tori and invariant manifolds is considered in (Ryashko (2001); Ryashko and Shnol (2003)).

The aim of this work is to present a common aproach to stability analysis for stochastically forced invariant manifolds (cycles, tori and etc.).

The first approximation linear systems (linear extension systems) for invariant manifolds are introduced and a notion of P-stability (projective) is proposed. A general criterion for P-stability is obtained. The stochastic stability analysis is reduced to the estimation of the spectral radius of some

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positive operator. Applications of this common theory to exponential mean square stability of stochastically forced limit cycles and invariant tori are demonstrated. For important cases of limit cycle in 2-dimensional stochastic system and 2-torus in 3-dimensional system the parametric criteria are given.

These stochastic stability criteria allow to solve relevant control problems effectively. Mean square stabilization of invariant manifolds of stochastically forced nonlinear systems is considered. The necessary and sufficient stabilizability conditions are presented. The methods for design of feedback stabilizing regulator for SDEs are suggested. The examples of constructive solving of stochastic control problem for periodic and quasiperiodic solutions are demonstrated. As shown, this approach gives the useful analytical tool for analysis and control of thin effects observed in nonlinear stochastic models.

2. STOCHASTIC STABILITY OF INVARIANT MANIFOLDS

Consider a deterministic nonlinear system

$$dx \;=\; f(x)\,dt \qquad (1)$$

where x is $n-$vector, $f(x)$ is sufficiently smooth vector-function of the appropriate dimension. It is assumed that system (1) has an smooth compact invariant manifold M (Fenichel (1971);Hirsch *et al.* (1977);Wiggins (1994)).

Consider a function $\gamma(x)$ in a neighbourhood U of the manifold M. Here $\gamma(x)$ is a point of manifold M that is nearest to x, $\Delta(x) = x - \gamma(x)$ is a vector of a deviation of a point x from manifold M. It is assumed that a neighbourhood U is invariant for systems (1). For each $x \in M$, denote by T_x the tangent subspace to M at x. Denote by N_x the orthogonal complement to T_x and by P_x the operator of orthogonal projection onto the subspace N_x.

A standard model for random forced deterministic system (1) is a system of Ito's stochastic differential equations

$$dx \;=\; f(x)dt \;+\; \sum_{r=1}^{m} \sigma_r(x)dw_r(t), \qquad (2)$$

where $w_r(t) \quad (r = 1, ..., m)$ are independent standard Wiener processes, $\sigma_r(x)$ are sufficiently smooth vector-functions of the appopriate dimension. To ensure M is an invariant of stochastic system (2) we assume that

$$\sigma_r|_M \;=\; 0 \qquad (3)$$

Definition 1. The manifold M is called exponentially stable in the mean square sense (EMS-stable) for the system (2) in U if there exist $K > 0,\ l > 0$ such that

$$E||\Delta(x(t))||^2 \;\le\; Ke^{-lt}E||\Delta(x_0)||^2$$

where $x(t)$ is a solution of system (2) with initial condition $x(0) = x_0 \in U$.

2.1 Stochastic linear extension system

An investigation on stability of a stationary point for non-linear system is connected traditionally with analysis of the appropriate first approximation linear system. The first approximation systems for stochastic nonlinear systems with limit cycles and its stability analysis are considered in Ryashko (1996). Now we extend this technique for general invariant manifold case.

Consider stochastic linear extension system for extended phase space $M \times R^n$

$$\begin{aligned} \dot{x} \;&=\; f(x), \qquad x \in M \\ \dot{z} \;&=\; F(x)\,z + \sum_{r=1}^{m} S_r(x)\,z\,\dot{w}_r, \end{aligned} \qquad (4)$$

where

$$F(x) = \frac{\partial f}{\partial x}(x)\;,\; S_r(x) = \frac{\partial \sigma_r}{\partial x}(x).$$

Solution $(x(t), 0)$ of system (4), because of presence of the family solutions $(x(t),\ f(x(t)))$ with nonvanishing component $f(x(t))$ can not be exponentially mean square stable in standard sense. Here more weak analog of exponential stability defined with the help of a projector P_x is considered.

Definition 2. The invariant manifold M of system (4) is called exponentially P-stable in the mean square sense (system (4) is P-stable for short) if there exist $K > 0,\ l > 0$ such that

$$E\|P_{x(t)}z(t)\|^2 \le Ke^{-lt}E\|P_{x_0}z_0\|^2$$

for any solution $(x(t),\ z(t))$ of system (4) with initial conditions $x(0) = x_0 \in M,\ z(0) = z_0 \in R^n$.

Consider a space Σ of symmetrical $n \times n$ matrix functions defined and sufficiently smooth on M and satisfying singularity condition

$$\forall x \in M\ \forall z \in T_x \quad V(x)z = 0.$$

Definition 3. A matrix $V(x) \in \Sigma$ is called P-positive definite if

$$\forall x \in M \;\; \forall z \qquad P_x z \neq 0 \quad \Rightarrow \quad (z, V(x)z) > 0.$$

In space Σ we shall consider a cone $\mathcal{K}$ of nonnegative definite for any $x \in M$ matrices and set

$$\mathcal{K}_P = \{V \in \Sigma | V \text{ is } P-\text{positive definite}\}.$$

Consider the matrix Lyapunov operator

$$\mathcal{L}[V] \;=\; \left(f(x), \frac{\partial V}{\partial x}\right) + \tag{5}$$

$$+F^\top(x)V + VF(x) + \sum_{r=1}^{m} S_r^\top(x)VS_r(x).$$

Theorem 1. The following statements are equivalent:
(a) Manifold M for system (2) is EMS-stable;
(b) System (4) is P-stable;
(c) $\forall W \in \mathcal{K}_P \; \exists\, v \in \mathcal{K}_P \qquad \mathcal{L}[V] \;=\; -W.$

2.2 A spectral stability criterion

Theorem 1 reduces a problem of manifold M stability to analysis of equation $\mathcal{L}[V] \;=\; -W$ decision problem in the space of $P-$positive definite matrices $\mathcal{K}_P$.

It is difficult to analyze the system stability by direct investigation of decision problem for matrix Lyapunov equation especially in cases close to critical. Here we shall consider an extension of the effective criteria (Ryashko (1979); Ryashko (1981); Ryashko (1999)) based on positive operators spectral theory (Krasnosel'skii *et al.* (1985)).

Represent the operator $\mathcal{L}$ from (5) in the form

$$\mathcal{L} \;=\; \mathcal{A} + \mathcal{S},$$

where

$$\mathcal{A}[V] = \left(f, \frac{\partial V}{\partial x}\right) + F^\top V + VF,$$

$$S[V] \;=\; \sum_{r=1}^{m} S_r^\top V S_r.$$

Consider the operator $\mathcal{P} = -\mathcal{A}^{-1}\mathcal{S}$.

Theorem 2. The manifold M is EMS-stable for stochastic system (2) if and only if it holds that
(a) The manifold M of deterministic system (1) is exponentially stable,
(b) The inequality $\rho(\mathcal{P}) < 1$ holds.

Proof is similar to the proof of the Theorem 1 in (Ryashko (1996)) and is based on the spectral theory of the positive operators (Krasnosel'skii *et al.* (1985)).

Remark 1. Spectral radius $\rho = \rho(\mathcal{P}) \neq 0$ defines bifurcation value $\varepsilon^* \;=\; \sqrt{1/\rho}$ of random noises intensity $\varepsilon \geq 0$ for a system

$$dx \;=\; f(x)dt \;+\; \varepsilon \sum_{r=1}^{m} \sigma_r(x)dw_r(t), \tag{6}$$

The manifold M for system (6) is EMS-stable for any $\varepsilon < \varepsilon^*$ and is unstable at any $\varepsilon \geq \varepsilon^*$. Case $\rho = 0$ means the system (6) is stable for any $\varepsilon \geq 0$.

Remark 2. If one can not find spectral radius ρ exactly then its estimations $\rho_1 < \rho < \rho_2$ may be useful. Actually, the inequality $\rho_2 < 1$ gives sufficient and $\rho_1 < 1$ gives necessary stability condition.

3. STABILITY OF LIMIT CYCLES

In this section we assume invariant manifold M be a limit cycle corresponding to T-periodic solution $\xi(t)$. Function $\xi(t)$ gives us the natural parametrization of the orbit M. It defines the one-to-one correspondence between cycle M points and interval $[0, T)$ time moments.

Using this parametrization, we introduce functions

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad S_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t)),$$

$$V(t) = V(\xi(t)), \quad P(t) = P(\xi(t))$$

defined on $[0, T]$. In this case Σ is the space of T-periodic symmetric $n \times n-$ matrices $V(t)$ defined and sufficiently smooth for any $t \in (-\infty, +\infty)$ such that the singularity condition $V(t)f(\xi(t)) = 0$ holds.

In space Σ we consider a cone $\mathcal{K} = \{V \in \Sigma | V(t)$ is nonnegative definite for any $t \in (-\infty, +\infty)\}$ and set $\mathcal{K}_P = \{V \in \Sigma | V$ is P-positive definite$\}$.

For the case of limit cycle the operator $\mathcal{L}$ has the following representation

$$\mathcal{L}[V] \;=\; \frac{\partial V}{\partial t} + F^\top(t)V + VF(t) + \sum_{r=1}^{m} S_r^\top(t)VS_r(t)$$

Now we can rewrite Theorem 3 in the following form

Theorem 3. Let cycle M of the system (2) be EMS-stable in U. Then for any $W \in \mathcal{K}_P$ there exists $V \in \mathcal{K}_P$ satisfying the matrix differential equation

$$\frac{\partial V}{\partial t} + F(t)^\top V + VF(t) + \sum_{r=1}^{m} S_r^\top(t)VS_r(t) = -W(t) \tag{7}$$

If for some $W \in \mathcal{K}_P$ equation (7) has a solution $V \in \mathcal{K}_P$ then limit cycle M of the system (2) is EMS-stable.

Stability of limit cycle in 2D-space. In the case $n = 2$ one can find for spectral radius of operator $\mathcal{P}$ the following simple representation

$$\rho(\mathcal{P}) = -\frac{< \beta >}{< \alpha >}$$

Here $\alpha(t) \;=\; p(t)^\top[F^\top(t) + F(t)]p(t)$, $\beta(t) \;=\; tr(\sum S_r(t)S_r^\top(t))$, $p(t)$ is a vector orthonormal to limit cycle M at a point $\xi(t)$, brackets $< \cdot >$ mean integral with time averaging

$$< \alpha > = \frac{1}{T}\int_0^T \alpha(t)dt.$$

Inequality (famous Poincare criterion)

$$< \alpha > \; < 0$$

is necessary and sufficient condition of exponential stability of limit cycle M for the deterministic system (1). Thus, the inequality $\rho(\mathcal{P}) < 1$ written as

$$< \alpha + \beta > = < tr[2F(t) + \sum_{r=1}^{m} S_r(t)S_r^\top(t)] > < 0$$

is necessary and sufficient condition of EMS-stability of cycle M for stochastic system (2) in 2D-case (Ryashko (1996)).

4. STABILITY OF 2-TORUS

It is assumed that invariant manifold M of system (1) is an invariant two-dimensional toroidal manifold. The following parametrization of 2-torus M is considered.

Suppose some closed sufficiently smooth curve α (equator) lies on the M (see Fig.1.).

This curve is defined by function $\alpha(s)$ on the interval $0 \leq s \leq 1$ with condition $\alpha(0) = \alpha(1)$. Consider a solution $x(t,s)$ of system (1) with initial condition $x(0,s) = \alpha(s)$. It is supposed that the trajectory of $x(t,s)$ leaves the point $\alpha(s)$ of a curve α and after rotation around the torus crosses curve α again. Let $T(s) = \min\{ t > 0 \mid x(t,s) \in \alpha \}$ be the first return time of trajectory $x(t,s)$ on the curve α and $x(T(s),s)$ be the first return point. Let $\tau(s)$ be a point of the interval $[0,1)$ where $\alpha(\tau(s)) = x(T(s),s)$. Here, $\tau(s)$ is the Poincare first return function for intersections of curve α by the phase trajectories of system (1).

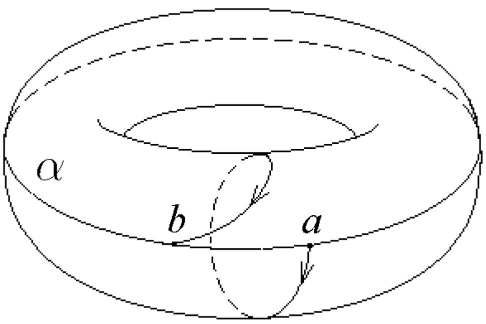

Fig. 1. α is closed curve (equator), $a = x(0,s) = \alpha(s)$ is initial point of solution $x(t,s)$, $b = x(T(s),s) = \alpha(\tau(s))$ is the first return point of solution $x(t,s)$ on the curve α

Torus M consists of phase trajectories $x(t,s)$ of system (1). Function $x(t,s)$ defines one-to-one correspondence between 2-torus M points and points of set $D = \{(t,s) \mid 0 \leq t < T(s),\ 0 \leq s < 1\}$. The vector-functions $\frac{\partial x(t,s)}{\partial t}$, $\frac{\partial x(t,s)}{\partial s}$ are linearly independent. For any point $\gamma \in M$ one can find $t = t(\gamma)$, $s = s(\gamma)$ such that $x(t,s) = \gamma$.

Using a parametrization of 2-torus M connected with a family of the solutions $x(t,s)$ one can introduce functions

$$F(t,s) = \frac{\partial f}{\partial x}(x(t,s)), \quad S_r(t,s) = \frac{\partial \sigma_r}{\partial x}(x(t,s)),$$

$$V(t,s) = V(x(t,s)), \quad P(t,s) = P(x(t,s))$$

defined on D. The equalities $x(t,s+1) = x(t,s)$, $x(T(s)+t,s) = x(t,\tau(s))$ allow to extend these functions to the whole plane $\Pi = \{(t,s) | -\infty < t < +\infty,\ -\infty < s < +\infty\}$.

Stability of 2-torus in 3D-space. In the case $n = 3$ one can find for spectral radius of operator $\mathcal{P}$ an explicit expression

$$\rho(\mathcal{P}) = \max_s \left\{ -\frac{< \beta >}{< \alpha >} \right\}$$

Here $\alpha = p^\top[F^\top + F]p$, $\beta = tr(\sum S_r S_r^\top)$, brackets $< \cdot >$ mean time averaging

$$< \alpha > = \lim_{T\to\infty} \frac{1}{T}\int_0^T \alpha(t)dt$$

Inequality

$$\max_s < \alpha > < 0$$

is necessary and sufficient condition for exponential stability of 2-torus M for the deterministic system (1). Thus, the inequality $\rho(\mathcal{P}) < 1$ written as

$$\max_s < \alpha + \beta > =$$

$$\max_s < tr[2F(t,s) + \sum_{r=1}^{m} S_r(t,s)S_r^\top(t,s)] > < 0$$

is necessary and sufficient condition of EMS-stability of 2-torus M for stochastic system (2) in three-dimensional case.

Remark 3. The function $\gamma(s) = < \alpha + \beta >$ for quasiperiodic case is a constant

$$\gamma(s) \equiv \frac{1}{S}\int_M \left[2\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{r=1}^{m}\sum_{i,j=1}^{n} \left(\frac{\partial \sigma_{ri}}{\partial x_j}\right)^2 \right] dx,$$

where S is area of torus M.

If $x(t,s)$ is a cycle with period $T(s)$: $x(t + T(s),s) = x(t,s)$ then

$$\gamma(s) = \frac{1}{T(s)} \int\limits_0^{T(s)} (\alpha(t,s) + \beta(t,s))dt.$$

If torus consists of cycles then values of function $\gamma(s)$ can be various for different cycles. If solution $x(t,s)$ converges to a limit cycle $x(t,s^*)$ as $t \to \infty$ then $\gamma(s) = \gamma(s^*)$. If a torus consists of alternating stable and unstable cycles then $\gamma(x)$ is step function.

5. STABILIZATION

Consider a stochastic system with a control of the form

$$dx = f(x,u)dt + \sum_{r=1}^{m} \sigma_r(x) dw_r(t), \quad (8)$$

where x is $n-$dimensional state variable, u is l-dimensional vector of control functions, $f(x,u)$, $\sigma(x)$ are vector functions of the appropriate dimension, $w_r(t) \quad (r = 1,...,m)$ are independent standard Wiener processes. It is supposed that for $u = 0$ the system (8) has invariant manifold M.

The stabilizing regulator we shall select from the class $\mathcal{R}$ of admissible feedbacks $u = u(x)$ satisfying conditions:

(a) $u(x)$ is sufficiently smooth and $u|_M = 0$;

(b) for the deterministic system

$$dx = f(x,u)dt$$

the manifold M is exponentially stable in the neighbourhood U of M.

The analysis of stabilization problem of manifold M for system (8) with control function $u = u(x)$ is connected with investigation of corresponding linear extension system

$$\begin{aligned} \dot{x} &= f(x,0), \qquad x \in M \\ \dot{z} &= F_u(x)\, z + \sum_{r=1}^{m} S_r(x)\, z\, \dot{w}_r \end{aligned} \quad (9)$$

where

$$F_u(x) = \frac{\partial f(x,0)}{\partial x} + \frac{\partial f(x,0)}{\partial u}\frac{\partial u}{\partial x}$$

$$S_r(x) = \frac{\partial \sigma_r(x)}{\partial x}.$$

Consider Taylor's expansion of control function $u(x)$ at a point γ

$$u(x) = u(\gamma) + \frac{\partial u}{\partial x}(\gamma)(x-\gamma) + O(\|x-\gamma\|^3).$$

For $\gamma = \gamma(x) \in M$ we get

$$u(x) = \frac{\partial u}{\partial x}(\gamma(x))\Delta(x) + O(\|\Delta(x)\|^3).$$

As we see, a first approximation control function near M for small deviations $\Delta(x) = x - \gamma(x)$ is the feedback

$$u_1(x) = \frac{\partial u}{\partial x}(\gamma(x))\Delta(x). \quad (10)$$

As it follows from (9), the stabilization capabilities of control u are completely determined by first approximation $u_1(x)$ of a function $u(x)$ and are independent on higher order terms. It allows to restrict our consideration by simpler regulators in the following form

$$u(x) = K(\gamma(x))\Delta(x). \quad (11)$$

Here $K(x)$ is the feedback matrix coefficient.

Under these restrictions linear extension system is following

$$\begin{aligned} \dot{x} &= f(x,0), \qquad x \in M \\ \dot{z} &= F_K(x)\, z + \sum_{r=1}^{m} S_r(x)\, z\, \dot{w}_r \end{aligned} \quad (12)$$

where

$$F_K(x) = F(x) + B(x)K(x)P(x),$$

$$F(x) = \frac{\partial f(x,0)}{\partial x},\ B = \frac{\partial f(x,0)}{\partial u}.$$

Consider set of feedback matrices

$$\mathcal{K} = \{K(x)|\ \text{system (12) for } \mathrm{S_r} = 0 \text{ is } \mathrm{P-stable}\}$$

and operators

$$\mathcal{A}_K[V] = \left(f, \frac{\partial V}{\partial x}\right) + F_K^\top V + VF_K,$$

$$S[V] = \sum_{r=1}^{m} S_r^\top V S_r,$$

$$\mathcal{P}_K = -\mathcal{A}_K^{-1}\mathcal{S}$$

Theorem 4. The manifold M is EMS-stabilizable for stochastic system (8) with feedback (11) if and only if it holds that

(a) $\mathcal{K} \neq \emptyset$,

(b) The inequality $\inf_{K\in\mathcal{K}} \rho(\mathcal{P}_K) < 1$ holds.

The feedback (11) with $K \in \mathcal{K}$ EMS-stabilizes stochastic system (8) if inequality $\rho(\mathcal{P}_K) < 1$ holds.

This Theorem reduces stabilization problem to minimization of operator $\mathcal{P}_K$ spectral radius.

5.1 Examples

Stabilization of cycle. For the case of cycle on a plane ($n = 2$) one can find for spectral radius of corresponding operator $\mathcal{P}_K$ the following simple representation

$$\rho(\mathcal{P}_K) = -\frac{<\beta>}{<\alpha> + 2 < b^\top K p>}$$

Here $\alpha = p^\top[F^\top + F]p,\ \beta(t) = tr(\sum S_r S_r^\top),\ b = B^\top p.$

Under restriction $b(t) \neq 0$ at some point $t \in [0, T]$ the spectral radius ρ_K is completely controllable by choice of K. For any $0 < \rho_K < 1$ one can find matrix K from equation

$$< b^\top K p > = -\frac{<\beta>}{2\rho_K} - \frac{<\alpha>}{2}$$

The regulator (11) with found here matrix K will stabilize cycle M for stochastic system (8).

Stabilization of torus. For the the case of 2-torus in 3D-space a spectral radius of operator $\mathcal{P}_K$ can be written as

$$\rho(\mathcal{P}_K) = \max_s \left\{ -\frac{<\beta>}{<\alpha> + 2 < b^\top K p>} \right\}$$

The necessary and sufficient conditions of stability $\rho(\mathcal{P}_K) < 1$ look like

$$\max_s < \alpha + \beta + 2b^\top K p > \ < 0.$$

This inequality allows to find matrix K of stabilizing regulator (11) for stochastic system (8) in case of 2-torus constructively.

6. CONCLUSION

Mean square stability analysis of the invariant manifolds of nonlinear stochastic systems was developed. Obtained criterion of P-stability allows (see Theorems 1) to investigate nonlinear systems stability using the first approximation linear systems. A spectral variant of P-stability criterion (see Theorem 2) is useful tool for constructive analysis and stabilization of limit cycles and tori.

REFERENCES

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications.* New York: Wiley.

Arnold, L. (1998). *Random Dynamical Systems.* Berlin: Springer.

Bashkirtseva, I. A. and L. B. Ryashko (2002). Sensitivity analysis of stochastically forced Lorenz model cycles under period doubling bifurcations. *Dynamic Systems and Applications*, 11, 293–310.

Bashkirtseva, I. A. and L. B. Ryashko (2004). Stochastic sensitivity of 3D-cycles. *Mathematics and Computers in Simulation*, 66, 55–67.

Fenichel, N. (1971). Persistence and smoothness of invariant manifolds for flows.*Indiana University Mathematics Journal*, 2, 193–226.

Hirsch, M. W., C. C. Pugh and M. Shub (1977). *Invariant Manifolds.* Springer-Verlag.

Kats, I. Ja. and N. N. Krasovskii (1960). The stability of systems with random parameters. *Prikl. Mat. Mech*, 24, 809–23.

Khasminskii, R. Z. (1980). *Stochastic stability of differential equations.* Sijthoff & Noordhoff, Alpen aan den Rijn.

Krasnosel'skii, M. A., E. A. Lifshits and A. V. Sobolev (1985). *Positive Linear Systems. The method of positive operators.* Nauka, Moscow.

Krasovskii, N. N. (1963) *Stability of Motion.* CA:Stanford University Press, Stanford.

Kushner, H. J. (1967). *Stochastic stability and control.* Academic Press, New York.

Mao, X. (1994). *Exponential stability of stochastic differential equations.* Marcel Dekker.

Ryashko, L. B. (1979). Stabilization of linear stochastic systems with state and control-dependent perturbations. *Prikl. Mat. Mekh.*, 43, 612–620.

Ryashko, L. B. (1981). Stabilization of linear systems with multiplicative noise under incomplete information. *Prikl. Mat. Mekh.*, 45, 778–786.

Ryashko, L. B. (1996). The stability of stochastically perturbed orbital motions. *J.Appl.Math.Mech.*, 60, 579–90.

Ryashko, L. B. (1999). Stability and stabilization of SDEs with periodic coefficients. *Dynamic Systems and Applications*, 8, 21–34.

Ryashko, L. B. (2001). Lyapunov function technique for stability analysis and stabilization of invariant 2-torus. *Appl.Nonlinear Dynamics*, 4-5, 140–154.

Ryashko, L. B. and E. E. Shnol (2003). On exponentially attracting invariant manifolds of ODEs. *Nonlinearity*, 16, 147–160.

Wiggins, S. (1994). *Normally Hyperbolic Invariant Manifolds in Dynamical Systems.* Springer-Verlag, Berlin.