# MOVEMENT OF A RIGID BODY THROUGH THE BOUNDARY OF A VISCOUS MEDIA

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**Abstract**
This article deals with mathematical model of a rigid body movement through the boundary of a viscous media. Movement is performed in a plane. Such movement of a body is described by system of the differential equations. The problem of attainability set construction of corresponding dynamic system in a class of the continuous limited controls is solved. Necessary conditions of optimality in the form of Euler–Lagrange equations in a problem of overcoming of media border with the minimum power consumption are received. Dependence of the trajectory of body movement on an occurrence in border between environments with different density is researched.

**Key words**
Control problem, Fluid dynamics.

## 1 Introduction

The control problem of objects motions in various media is actual. Various works which are produced in coastal shelves, such as a lining of pipelines, search of mineral deposits, service works are at the bottom of the given researches [Appazov, Lavrov and Mishin, 1966; Beletskii, 1973]. Construction adequate 3D models considering all physical nuances is a difficult task. Therefore we will be limited to flat consideration in absence of forces of a superficial tension. Models of this kind were considered in the book [Zavalishchin, 2002]. We also will consider only a movement through border of media. The created model can be used for design of perspective samples of new machines.

## 2 Mathematical model

In this section, we deal with construction of the model of moving through the boundary of a viscous media of solid body (see Fig.1) in plane $Oxy$. The medium of smaller density is located above axis $Ox$. More dense one is located below axis $Ox$. Body movement only through the border of viscous media is considered. In the initial state $l_t = l$ and in final state $l_t = 0$. Here $l$ – length of the body. Thus the length of a tail $l_t$ in the first medium changes from $l$ to $0$. Length of the body part being in the second one is equal $l - l_t$. The location of the body inertia center $l_c$ depends on size of body immersing in the second medium. It should be noted that the inertia center doesn't coincide with the center of mass.

### 2.1 Forces and moments

The state of the body is described by the generalized coordinates $x$, $y$ and $\varphi$. Let $V$ be the vector of centroid velocity $\mathbf{V} = (\dot{x}; \dot{y})^T$, $F$ be the force acting along a body axis $\mathbf{F} = (F\sin\varphi; F\cos\varphi)^T$, $U$ be the angular moment, $E$ be the unit vector $\mathbf{E} = (\sin\varphi; \cos\varphi)^T$, $D$ and $D^{\perp}$ are the drag force and lift force respectively

$$\begin{aligned}\mathbf{D} &= (-D\sin(\varphi-\alpha); -D\cos(\varphi-\alpha))^T,\\ \mathbf{D}^{\perp} &= (D^{\perp}\cos(\varphi-\alpha); -D^{\perp}\sin(\varphi-\alpha))^T.\end{aligned} \qquad (1)$$

It is necessary to note that because of presence of two viscous media drag forces and lift forces will be different. Let the body moves from the first medium 1 to the second medium 2. Forces acting in different media will create the moment. The drag force and lift force acting in $i$-th medium are equals

$$\begin{aligned}\mathbf{D}_i &= (-D_i\sin(\varphi-\alpha); -D_i\cos(\varphi-\alpha))^T,\\ \mathbf{D}_i^{\perp} &= (D_i^{\perp}\cos(\varphi-\alpha); -D_i^{\perp}\sin(\varphi-\alpha))^T.\end{aligned} \qquad (2)$$

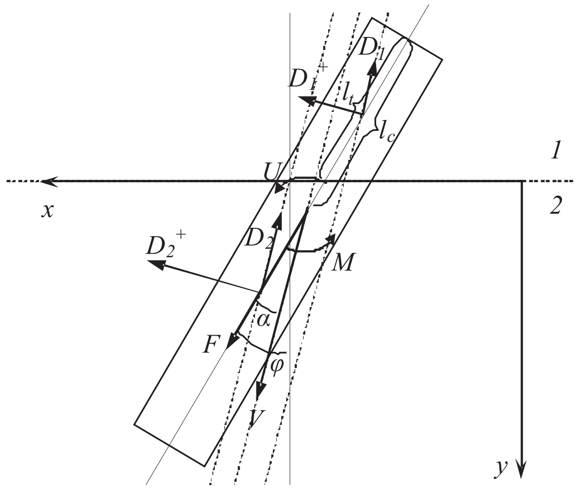


Figure 1. Forces and moments acting on a body.

## 2.2 About Hydrodynamic Forces and Coefficients

Let a body of bounded size with sufficiently smooth boundary $S$ move in fluid. One of the fluid mechanics axioms is the sticking condition: at the body surface points the velocity vector of fluid particle is equal to the velocity vector of the corresponding body point. This condition implies that in the case of translational motion of the body the following equality is fulfilled at its surface (see [Slezkin, 1955])

$$\Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^{*}\mathbf{n} = 0, \qquad (3)$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface $S$ at the point $x$.

The stress on an element $dS$ of the body surface is calculated by the formula $\mathbf{p}_n = P\mathbf{n}$, where $\mathbf{n}$ is the unit vector of the outward normal to $dS$. This equality and (3) yield the formula for the principal vector of the forces acting from fluid upon the body surface (hydrodynamic forces)

$$\mathbf{R} = \iint\limits_{S} \Big(-pE + \mu\frac{\partial \mathbf{v}}{\partial x}\Big)\mathbf{n}\,dS. \qquad (4)$$

We need further the so-called moving coordinate system $O_c y_1 y_2 y_3$ with the body inertia center as the origin and the axes rigidly connected with the body.

To find the principal vector and momentum, one has to calculate on the body surface the pressure and the Frechet derivative of the fluid velocity vector. To do this, one has to solve a certain boundary-value problem for the vector-valued Navier–Stokes equation. This equation is written out below in the moving system $O_c y_1 y_2 y_3$ with axes parallel to the corresponding axes of the system $Ox_1x_2x_3$ (the body is assumed to move translationally). Let $\mathbf{V}$ be the velocity vector of the body, and $x_c(t)$ be the radius vector of its inertia center. In the moving coordinate system, denote the absolute velocity vector of fluid and the pressure as follows: $\hat{\mathbf{v}}(t,y) = \mathbf{v}(t, x_c(t)+y)$, $\hat{p}(t,y) = p(t, x_c(t)+y)$. Then the Navier–Stokes equation is of the form

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = -\frac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V}) - \frac{1}{\rho}\Big(\frac{\partial \hat{p}}{\partial y}\Big)^{*} + \nu \mathrm{div}\frac{\partial \hat{\mathbf{v}}}{\partial y} + \mathbf{F}, \quad (5)$$

where $\mathbf{F}$ is the strength of the gravity field, $\rho$ is the fluid density, $\nu = \mu/\rho$ is the kinematic viscosity coefficient.

Now, the above-mentioned boundary-value problem is reduced to finding the solution of a system of partial differential equations, namely, equation (5) plus the equation of continuity $\mathrm{div}\hat{\mathbf{v}} = 0$. This solution must satisfy the sticking condition $\hat{\mathbf{v}}(t,y)\Big|_S = \mathbf{V}$ and the natural condition $\lim\limits_{y\to\infty} \hat{\mathbf{v}}(t,y) = 0$.

A flow is accepted to call established or stationary if the field of its absolute velocity vectors in the moving coordinate system does not change in time. Obviously, if the body moves translationally, the necessary condition for the flow to be stationary is $\mathbf{V} = \mathbf{V}_0 = \mathrm{const}$.

Suppose that the body has a symmetry axis. If the body moves in such a manner that this axis remains in a given plane (for example, in the plane $Oxy$), then, according to the statics theorems for an absolutely solid body, the totality of forces acting from fluid upon the body can be reduced to the resultant one called the hydrodynamic force. As usual the point of intersection of the symmetry axis and the line of the hydrodynamic force action is referred to as center of pressure. The hydrodynamic force is resolved into components parallel to the velocity vector $\mathbf{V}$ of the body inertia center and perpendicular to $\mathbf{V}$. The first component $\mathbf{D}$ is known as the drag force, and the second one $\mathbf{D}^{\perp}$ is called the lift force.

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors in the directions $Ox$ and $Oy$ respectively. We need further a mapping that puts a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ into correspondence to $\mathbf{a}^{\perp} = -a_2\mathbf{i} + a_1\mathbf{j}$. Let $V$ be the magnitude of $\mathbf{V}$, $D$ be that of the drag force, and $D^{\perp}$ be that of the lift force. For needs of forthcoming references, it is convenient to formulate the following assertion as lemma.

The drag and lift forces are calculated by the formulae

$$\begin{aligned} &\mathbf{D} = \mathrm{sign}(\mathbf{V},\mathbf{D})DV^{-1}\mathbf{V}, \\ &\mathbf{D}^{\perp} = \mathrm{sign}(\mathbf{V},\mathbf{D})sD^{\perp}V^{-1}\mathbf{V}^{\perp}, \\ &s = \mathrm{sign}((\mathbf{V},\mathbf{e})(\mathbf{V},\mathbf{e}^{\perp})), \end{aligned} \qquad (6)$$

where $\mathbf{e}$ is the directing vector of the body symmetry axis (see Fig. 2).

In the framework of the listed constraints, the coefficient $C_D$ is a function of the body shape, Reynolds number and, probably, the angle of attack between the velocity vector of the body inertia center and the symmetry axis, i.e., $C_D = C_D(\mathrm{shape}, \mathrm{Re}, \alpha)$ [Daily and

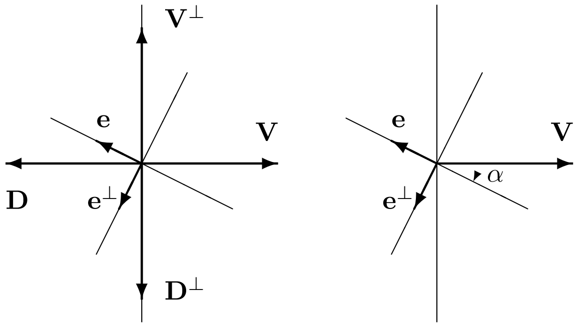


Figure 2. Drag and lift forces and the angle of attack

Harleman, 1966]. To determine the angle of attack (see Fig. 2), one can use the formula

$$\alpha = -s \arccos |(\mathbf{e}, \mathbf{V}/V)|. \tag{7}$$

The nonstationarity of the flow can be partially taken into account by means of introducing the apparent additional mass [Daily and Harleman, 1966].

### 2.3 Movement equations

Kinetic energy is equal to

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{ml^2}{3}\dot{\varphi}^2. \tag{8}$$

Using the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{9}$$

one can obtain body movement equations

$$\begin{aligned} m\ddot{x} &= Q_x \\ m\ddot{y} &= Q_y \\ \tfrac{1}{3} ml^2 \ddot{\varphi} &= Q_\varphi \end{aligned} \tag{10}$$

The generalized forces corresponding to the generalized coordinates will be the following

$$\begin{aligned} &Q_x = -D\sin(\varphi-\alpha) + D^{\perp}\cos(\varphi-\alpha) + F\sin(\varphi) \\ &Q_y = -D\cos(\varphi-\alpha) - D^{\perp}\sin(\varphi-\alpha) + F\cos(\varphi) - mg \\ &Q_\varphi = U + M \end{aligned} \tag{11}$$

The expression for the power of the control forces and momentums is of the form

$$\dot{W} = (\dot{x}\sin\varphi + \dot{y}\cos\varphi)F + \omega U\,. \tag{12}$$

The system of equations (10) and (11) describes body movement. Here $D$ and $D^{\perp}$ resultant forces acting in the point defined by $l_c$

$$l_c = \frac{(D_1 + D_2)l_t + D_2 l}{2(D_1 + D_2)} \tag{13}$$

## 3 Optimization problem

Now the optimization problem can be formulated.

Problem 1. It is required to find controls $F^0(t)$ $U^0(t)$, $0 \leqslant t \leqslant t_k$, moving with the minimum power expenses, $W(t_k) \to \min$, a body for given time $t_k$ for the set distance.

Such problem is nonregular. Euler–Lagrage equations do not contain controls and do not allow to define their optimum values in terms of the phase and interfaced variables.

The problem reduction is proved by that body movement occurs in a potential gravity field. And the changeable part of work of control forces is used on change of body kinetic energy. Therefore the varied part of work will be equivalent to power expenses for overcoming of hydrodynamic forces of resistance and will be equal to scalar product $(\mathbf{D}^T, \mathbf{V})$

$$N = -D\sin(\varphi-\alpha) - D^{\perp}\cos(\varphi-\alpha) - M\varphi \tag{14}$$

Power of hydrodynamic forces is equal to

$$\dot{N} = D(\dot{\varphi}-\dot{\alpha})(-\cos(\varphi-\alpha) + \sin(\varphi-\alpha)) - M\dot{\varphi}\,. \tag{15}$$

Now it is possible not to consider dynamics of a body, having assigned function of control to derivatives of the generalized coordinates. Thus the initial problem is to an equivalent following problem.

Problem 2. It is required to find functions $\mathbf{V}(t) = (V_x(t), V_y(t))^T$ $\omega(t)$, minimizing terminal functional $N(t_k)$ at dynamical relations (12) and restrictions

$$\begin{aligned} &x(t_k) = x_k\,, \quad y(t_k) = y_k\,, \quad \varphi(t_k) = \varphi_k\,, \\ &\cos\alpha = \dot{x}\cos\varphi + \dot{y}\cos\varphi\,. \end{aligned} \tag{16}$$

According to classical Euler–Lagrange procedure it is necessary to write out Hamiltonian

$$H = \lambda_0\dot{Y} + \lambda_1\dot{x} + \lambda_2\dot{y} + \lambda_3\dot{\varphi}$$

and conjugated system with boundary conditions

$$\begin{aligned} -\dot{\lambda}_0 &= \frac{\partial H}{\partial N} = 0\,, \; \lambda_0(t_k) = \frac{\partial \Phi}{\partial N(t_k)} \\ -\dot{\lambda}_1 &= \frac{\partial H}{\partial x}\,, \qquad \lambda_1(t_k) = \frac{\partial \Phi}{\partial x(t_k)} \\ -\dot{\lambda}_2 &= \frac{\partial H}{\partial y}\,, \qquad \lambda_2(t_k) = \frac{\partial \Phi}{\partial y(t_k)} \\ -\dot{\lambda}_3 &= \frac{\partial H}{\partial \varphi}\,, \qquad \lambda_3(t_k) = \frac{\partial \Phi}{\partial \varphi(t_k)} \end{aligned} \tag{17}$$

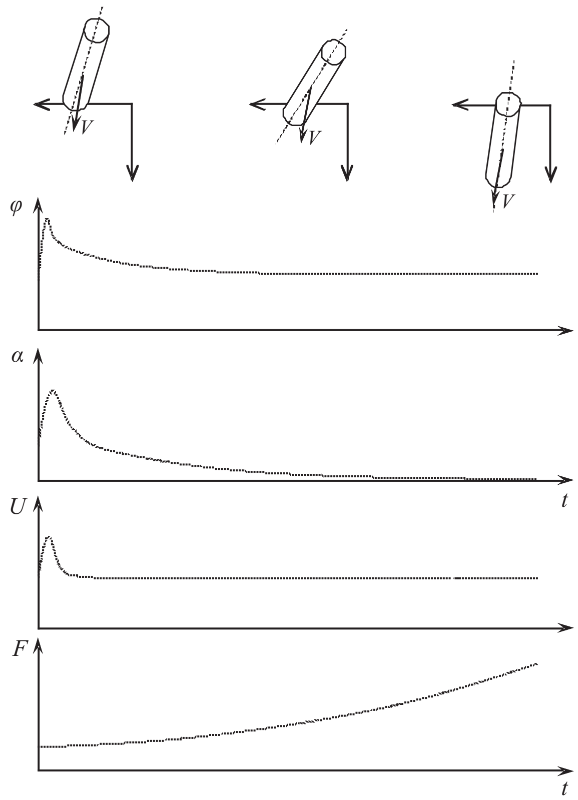


Figure 3. The angles of orientation and attack and control forces

Here $\Phi = N(t_k) + \nu_1(x(t_k) - x_k) + \nu_2(y(t_k) - y_k) + \nu_3(\varphi(t_k) - \varphi_k)$ is functional describing boundary conditions.

Euler–Lagrange equations

$$\begin{aligned} \frac{\partial H}{\partial \dot{x}} &= \lambda_1 + \frac{\partial \dot{N}}{\partial \dot{x}} = 0 \\ \frac{\partial H}{\partial \dot{y}} &= \lambda_2 + \frac{\partial \dot{N}}{\partial \dot{y}} = 0 \\ \frac{\partial H}{\partial \dot{\varphi}} &= \lambda_3 + \frac{\partial \dot{N}}{\partial \dot{\varphi}} = 0 \end{aligned} \quad (18)$$

allow to calculate Lagrange multipliers and at having substituted them in the conjugated system (17) to write out the equations of optimal movement

$$\begin{aligned} \dot{x} = V_x\,, &\quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial V_x}\right) = \frac{\partial \dot{N}}{\partial x} \\ \dot{y} = V_y\,, &\quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial V_y}\right) = \frac{\partial \dot{N}}{\partial y} \\ \dot{\varphi} = \omega\,, &\quad \frac{d}{dt}\left(\frac{\partial \dot{N}}{\partial \omega}\right) = \frac{\partial \dot{N}}{\partial \varphi} \end{aligned} \quad (19)$$

On Fig. 3 the basic modes of border overcoming are presented.

## 4 Conclusion

Thus the system of the differential equations describing a rigid body movement through the boundary of a viscous media is obtained. It allows to model various modes of such movement.

**Acknowledgements**

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