THE CHELOMEI PROBLEM: HIGH OR LOW FREQUENCY STABILIZATION?

Andrei A. Seyranian
MSTU n.a.Bauman
Moscow, Russia
aseiran@hotbox.ru

Alexander P. Seyranian
MSU n.a.Lomonosov
Moscow, Russia
seyran@imec.msu.ru

Abstract
V.N.Chelomei in his famous paper [Chelomei, 1956] recognized that statically unstable elastic systems can be stabilized by vibrations. In particular, he came to the conclusion that the elastic column compressed by an axial force, exceeding the critical Euler value, can be stabilized by high frequency axial vibration applied to the end of the column. In this paper we discuss contradictions between assumptions and results of the paper [Chelomei, 1956]. We derived and analyzed formulas for the higher and lower boundaries of the stabilization frequency. It is shown that unlike stabilization of an inverted pendulum by high frequency vibration of the support the column is stabilized by excitation frequencies of the order of the first frequency of transverse oscillations of the column belonging to a certain region.

Key words
Statically unstable systems, stabilization by vibration

1 Formulation of the problem

V.N. Chelomei in [Chelomei, 1956] considered a straight elastic rod of constant cross section, loaded by a periodic longitudinal force $P(t) = P_0 + P_t\phi(\omega\, t)$ applied to its end. The equation of transverse oscillations of the rod can be written as

$$EJ\frac{\partial^4 u}{\partial x^4} + P(t)\frac{\partial^2 u}{\partial x^2} + 2\gamma m\frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

where x is the coordinate along the rod axis; t is the time; $u(x,t)$ is the rod deflection function; m is the mass per unit length; EJ is the flexural rigidity; γ is the damping coefficient; P_t and ω are the excitation amplitude and frequency of the longitudinal vibration, respectively. Chelomei considered the case in which both ends of the rod were simply supported. Then solution of Eq. (1) can be found in the form of a series in eigenfunctions

$$u(x,t) = \sum_j \varphi_j(t)\sin(j\pi\, x/l)\,.$$

Substituting this series into Eq. (1), multiplying by $\sin(k\pi\, x/l)$, and integrating the result over the interval [0, 1], we arrive at an equation with respect to the functions $\varphi_k(t)$

$$\frac{d^2\varphi_k}{dt^2} + 2\gamma\frac{d\varphi_k}{dt} + \Omega_k^2\left\{1 - \frac{P_0}{P_k} - \frac{P_t\phi(\omega t)}{P_k}\right\}\varphi_k = 0$$

$$k = 1, 2, \ldots, \qquad (2)$$

where $\Omega_k = \pi^2 k^2\sqrt{EJ/m}\,/\,l^2$ is the k-th eigenfrequency of transverse free oscillations of the unloaded rod, and $P_k = \pi^2 k^2 EJ/l^2$ is the k-th critical Euler's force. Let us introduce notation $\beta_k = \gamma/\Omega_k$ and new time variable $\tau = \omega t$. Then, Eq. (2) can be rewritten as

$$\frac{d^2\varphi_k}{d\tau^2} + 2\beta_k\left(\frac{\Omega_k}{\omega}\right)\frac{d\varphi_k}{d\tau} + \left(\frac{\Omega_k}{\omega}\right)^2 \qquad (3)$$

$$\left\{1 - \frac{P_0}{P_k} - \frac{P_t\phi(\tau)}{P_k}\right\}\varphi_k = 0, \qquad k = 1, 2, \ldots$$

The trivial equilibrium of the rod $u(x,t) = 0$ is asymptotically stable if the functions asymptotically vanish $\varphi_k(t) \to 0$ as $t \to \infty$, $k = 1, 2, \ldots$., and it is unstable if at least one of the functions $\varphi_k(t)$ exhibits unlimited growth as $t \to \infty$.

In his paper [Chelomei, 1956], that became famous, Chelomei stated the problem of stabilization of an elastic rod loaded by a periodic longitudinal force exceeding in average the Euler's critical value $P_0 > P_1$, i.e., of a statically unstable rod, by means of longitudinal vibration bringing it to the straight position. Assuming that the excitation frequency is high as compared with the natural frequency of the transverse oscillations of the uncompressed rod $(\omega >> \Omega_1)$ and applying the perturbation method and the averaging method to Eq. (3) $(k = 1)$, Chelomei [Chelomei, 1956] obtained an inequality describing the rod stabilization domain. In the case of $\phi(\tau) = \cos\tau$, this inequality can be written as

$$\left(\frac{\omega}{\Omega_1}\right)^2 < \frac{\varepsilon^2}{2\alpha} - 4\beta_1^2 , \qquad (4)$$

where $\varepsilon = P_t / P_1$, $\alpha = P_0 / P_1 - 1$, and $\beta_1 = \gamma / \Omega_1$. There is an obvious discrepancy: in deriving this relation, it was assumed that $\omega / \Omega_1 >> 1$, while the critical stabilization frequency turns out to be of the order of the natural frequency. Indeed, setting, for example $\varepsilon = 0.1$, $\alpha = 0.05$, $\beta_1 = 0$ we obtain from Eq. (4) that $\omega / \Omega_1 < 1/\sqrt{10} = 0.316$. As it follows from Eq. (4), taking damping into account one only lowers the upper boundary of the stabilization frequency. Besides, Chelomei [Chelomei, 1956] did not impose a limitation on the stabilization frequency from below. The absence of this lower boundary leads to the paradoxical conclusion that the rod can be stabilized by applying an arbitrarily low longitudinal vibration!

The Chelomei problem [Chelomei, 1956] of stabilizing the elastic rod was included with slight corrections into the well-known monograph by Bogolyubov and Mitropol'skii [Bogolyubov and Mitropol'skii, 1974]. Bolotin in [Bolotin, 1999] without referring to [Chelomei, 1956] also analyzed a possibility of the longitudinal vibrations use to stabilize an elastic rod compressed by a periodic longitudinal force exceeding in average Euler's critical value. Based on numerical results, he arrived at the conclusion that the analogy with the problem of stabilization of the inverted pendulum is not correct due to the presence of intermittent resonance zones of instability caused by higher harmonics, which decrease the domains of rod stabilization. In addition, Chelomei [Chelomei, 1956; Chelomei, 1983] always mentioned the high-frequency stabilization of a rod loaded by a periodic force exceeding in average Euler's critical value, but he did not report the particular values of the stabilization frequency achieved in the experiments [Chelomei, 1983]. Recently, new attempts to investigate the stabilization of a statically unstable rod by means of high-frequency longitudinal vibrations were undertaken in [Jensen , 2000; Jensen, Tcherniak and Thomsen, 2000]. However, the conclusions made in these papers were rather ambiguous. It was stated [Jensen, 2000], in particular, that a rod subjected to high-frequency excitation had both a curved stable configuration and a straight stable equilibrium shape; the experiments [Jensen, Tcherniak and Thomsen, 2000] confirmed the stiffening effect (an increase of the natural frequencies of transverse oscillations) by longitudinal high-frequency excitation, but the critical forces or stabilization frequencies were not found.

2 Stability analysis

For analysis of the rod stabilization, let us use the results of investigation of the stability domains for the Hill equation with damping [Seyranian, 2001; Seyranian and Seyranian, 2006]. Applying these results to Eq. (3) with $k = 1$ for the case in which the constant component of the longitudinal force is slightly greater than Euler's critical value, $0 < \alpha = P_0 / P_1 - 1 << 1$, assuming a small excitation amplitude $\varepsilon = P_t / P_1 << 1$ and $\phi(\tau) = \cos\tau$, we arrive at an inequality determining the stabilization domain:

$$\left(\frac{\omega}{\Omega_1}\right)^2 < \frac{\varepsilon^2}{2\alpha} - 4\beta_1^2 - \frac{7\alpha}{8} . \qquad (5)$$

This formula differs from the Chelomei formula (4) only by the last small term.

According to the Strutt-Ince diagram for the Mathieu-Hill equation [Panovko, 1987; Merkin, 1987], there exists a lower boundary for the frequency ω. In order to obtain a formula describing this boundary, it is necessary to analyze the stability domain occurring near the first critical frequency. Let us first consider the case where damping is absent, $\beta_1 = 0$. Retaining terms of the first and second order of smallness in ε , we obtain

$$\left(\frac{\omega}{\Omega_1}\right)^2 > H, \qquad (6)$$

$$H = \varepsilon - 2\alpha + \sqrt{(\varepsilon - 2\alpha)^2 + \frac{\varepsilon^2}{2}}$$

Formula (6) can be extended to the case of small damping. Assuming the critical value of the frequency to be $(\omega / \Omega_1)^2 = H + \Delta\beta_1^2$, where Δ is a coefficient, and using the formula for the first stability domain of the Mathieu-Hill equation with small damping [Thomsen, 2003], we obtain

$$\Delta = \frac{2H^2}{\varepsilon\sqrt{(\varepsilon - 2\alpha)^2 + \frac{\varepsilon^2}{2}}} \qquad (7)$$

Finally, combining Eqs. (5)-(7) we find the rod stabilization conditions in the form of a two-sided inequality for the stabilization frequency

$$\varepsilon - 2\alpha + \sqrt{(\varepsilon - 2\alpha)^2 + \frac{\varepsilon^2}{2}} - \Delta\beta_1^2 < \left(\frac{\omega}{\Omega_1}\right)^2 < \frac{\varepsilon^2}{2\alpha} - \frac{7\alpha}{8} - 4\beta_1^2 \qquad (8)$$

Thus, the stabilization domain boundaries depend on three small parameters, namely, ε, α and β_1. Inequality (8) indicates that taking damping into account lowers both the upper and lower boundaries of the stabilization frequency. Note that the stabilization domain exists only provided that the right-hand side of inequality (8) is greater than zero, or

$$\varepsilon^2 > \frac{7\alpha^2}{4} + 8\alpha\beta_1^2 \qquad (9)$$

which implies that the excitation amplitude must be fairly large.

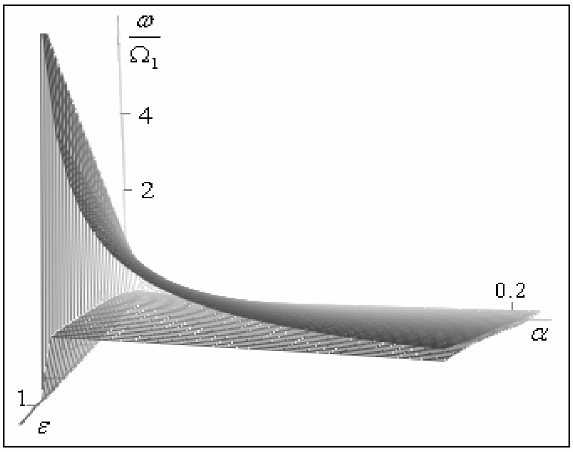

Figure 1. Stabilization domain of the rod.

Figure 1 shows the dependence of the lower and upper boundaries of the stabilization frequency on the parameters ε and α calculated from Eq. (8) at the damping coefficient $\beta_1 = 0.05$. As can be seen, the stabilization domain is located between two surfaces. The two-dimensional domains of stability (projections) of the trivial equilibrium of the rod in the presence of longitudinal excitation obtained for a damping coefficient of $\beta_1 = 0.05$ and the values $\alpha = 0.05$ and $\alpha = 0.1$, are presented in Figs. 2 and 3, respectively. The instability domains (shaded) were obtained numerically by determining the monodromy matrix followed by the calculation of the system multipliers and the estimation of their moduli (Floquet theory). Bold curves represent the analytical dependence of the frequency on the excitation amplitude in accordance with Eq. (8). Figures 2 and 3 show a good agreement between the analytical and numerical results.

It is interesting to investigate the possibility of stabilizing the rod by means of longitudinal vibrations at a given frequency with the variable parameters ε and α. Figure 4 shows the stabilization domain (unshaded) obtained numerically for the values of $\omega/\Omega_1 = 1$ and $\beta_1 = 0.05$. The corresponding analytical dependences can be obtained from the two-sided

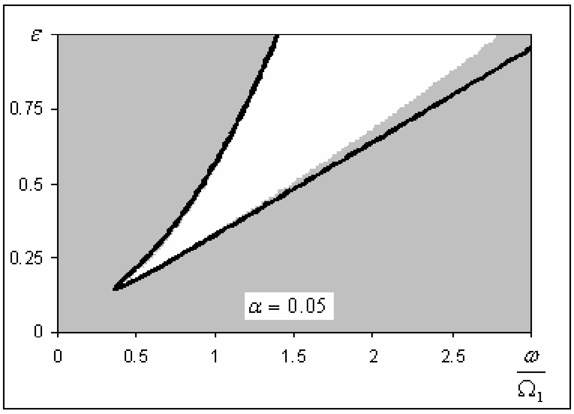

Figure 2. Projection of the stabilization domain at $\alpha = 0.05$.

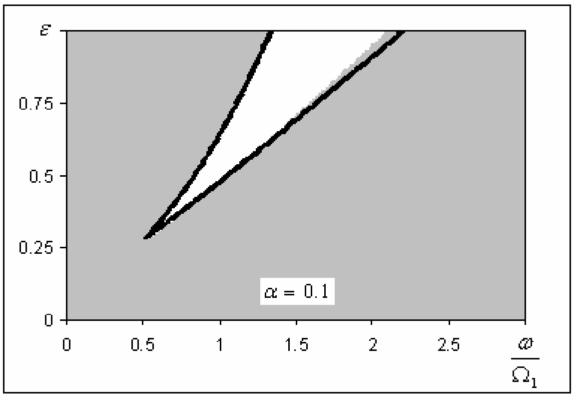

Figure 3. Projection of the stabilization domain at $\alpha = 0.1$.

inequality (8). In particular, for small ε and β_1, the approximate formula for the stabilization domain $\alpha < \varepsilon^2/2$ is valid. From Fig. 4 it follows that, at a moderate excitation amplitude ε, stabilization of the rod is possible only for forces slightly exceeding the critical value: $\alpha = P_0/P_1 - 1 << 1$.

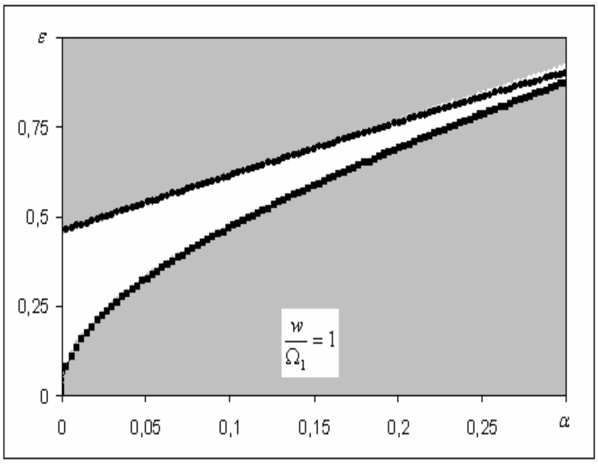

Figure 4. Stabilization domain (unshaded) at given excitation frequency $\omega/\Omega_1 = 1$.

3 Influence of higher harmonics

Let us estimate the influence of the instability (parametric resonance) domains for Eq. (3) at $k = 2,3,\ldots$ on the stabilization domain found above. It is possible that the instability domains for Eq. (3) at $k = 2,3,\ldots$ intersect with the stability domain, thus narrowing the stabilization region. It is known [Merkin, 1987; Seyranian, 2001] that the parametric resonance for the Mathieu-Hill equations (3) takes place at frequencies close to

$$\left(\frac{\Omega_k}{\omega}\right)^2\left(1-\frac{P_0}{P_k}\right)=\frac{n^2}{4}, \qquad n=1,2,\ldots \quad (10)$$

Using this expression and taking into account Eqs. (2) and (3), we obtain the following critical values of the excitation frequencies:

$$\omega=\frac{2\Omega_1 k^2}{n}\sqrt{1-\frac{P_0}{P_1 k^2}}, \qquad n=1,2,\ldots \quad (11)$$

Taking into account that P_0, in the problem under consideration is close to P_1, we determine the first four resonance frequencies. For $k = 2$, Eq. (11) yields the following approximate values: $4\sqrt{3}\,\Omega_1$, $2\sqrt{3}\,\Omega_1$, $4\sqrt{3}/3\,\Omega_1$, $\sqrt{3}\,\Omega_1,\ldots$; while for $k = 3$ we have: $12\sqrt{2}\,\Omega_1$, $6\sqrt{2}\,\Omega_1$, $4\sqrt{2}\,\Omega_1$, $3\sqrt{2}\,\Omega_1,\ldots$. It is known that, for the Mathieu-Hill equations (3), only the first instability domains near the frequencies $4\sqrt{3}\,\Omega_1$ and $12\sqrt{2}\,\Omega_1$ are wide; in the presence of even a small damping, the instability corresponding to large values of n vanishes at moderate excitation amplitudes. Hence, it follows that, in the presence of damping, the instability domains for Eq. (3) at $k = 2,3,\ldots$ do not influence the stabilization domain (8). Numerical calculations confirm this conclusion. However, this conclusion contradicts the results of [Bolotin, 1999], according to which the stabilization domain alternates with the instability domains for higher harmonics. Therefore, the stabilization domain obtained numerically in [Bolotin, 1999] agrees with the results presented in Fig. 2 only for small ε and ω.

4 Conclusion

The analogy between the Chelomei problem and the problem of stabilization of an inverted pendulum with a vibrating suspension point, noted in [Chelomei, 1956], seems quite natural. Indeed, in both cases statically unstable systems are stabilized by means of vibrations. Both problems reduce to an analysis of the stability domain for the Mathieu-Hill equation at negative frequencies close to zero. The difference is that, at small excitation amplitude, the pendulum in the upper vertical position is stabilized by a frequency which is greater than the critical value and high as compared with the natural frequency of the pendulum, whereas an elastic rod is stabilized by excitation frequencies of the order of the natural frequency of transverse oscillations belonging to a certain interval.

Acknowledgment

This work was supported by INTAS, grant no. 06-1000013-9019.

References

Bolotin, V.V. (1999). *Vibrations in Engineering. Handbook.* V.1. *Oscillations of Linear Systems*. Mashinostroenie. Moscow.

Bogolyubov, N.N. and Mitropol'skii, Yu.A. (1974). *Asymptotic Methods in the Theory of Nonlinear Oscillations.* Nauka. Moscow.

Chelomei, V.N. (1956). On possibility to increase stability of elastic systems by vibration. *Doklady Akad. Nauk SSSR*, **110** (3), pp. 345-347.

Chelomei, V.N. (1983). Paradoxes in mechanics caused by vibration. *Doklady Akad. Nauk SSSR*, **270** (1), pp. 62-67.

Chelomei, V.N. (1989). *Selected Works.* Mashinostroenie. Moscow.

Jensen, J.S. (2000). Buckling of an elastic beam with added high-frequency excitation. *International Journal of Non-Linear Mechanics*, **35**, pp. 217-227.

Jensen, J.S., Tcherniak, D.M. and Thomsen, J.J. (2000). Stiffening effects of high-frequency excitation: experiments for an axially loaded beam. *ASME J. Applied Mechanics*, **67**, pp. 397-402.

Merkin, D. R. (1987). *An Introduction to the Theory of Stability of Motion*. Nauka. Moscow.

Panovko, Ya.G. and Gubanova, I.I. (1987). *Stability and Vibrations of Elastic Systems. Modern Concepts, Paradoxes and Mistakes.* Nauka. Moscow.

Seyranian, A.P. (2001). Resonance domains for the Hill equation with allowance for damping. *Doklady Physics*, **46** (1), pp. 41-44.

Seyranian, A.A. and Seyranian, A.P. (2006). The stability of an inverted pendulum with a vibrating suspension point. *J. Appl. Math. Mech.*, **70**, pp. 754-761.

Thomsen, J.J. (2003). *Vibrations and Stability. Advanced Theory, Analysis and Tools*. Springer. Berlin.