ADAPTIVE OUTPUT CONTROL OF LINEAR TIME-VARYING SYSTEMS[1]

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Abstract: The problem of output control of linear time-varying systems with unknown fast-changing bounded parameters is considered. It is assumed that the plant is affected by unknown bounded disturbance. The method assuring the solution of tracking problem with prescribed accuracy is represented.

Keywords: Adaptive control, Decision feedback, Output control, Single-input/single-output systems, Linear time-varying systems.

1. INTRODUCTION

This work deals with the relevant problem of analysis and synthesis of robust and adaptive output control of indeterminate linear time-varying systems. Among the works devoted to control of time-varying systems it is essential to distinguish the works of scientists Ioannou (Tsakalis and Ioannou, 1987; Tsakalis and Ioannou, 1993; Zhang, *et al.*, 2003), Marino and Tomei (Marino and Tomei, 2000; Marino and Tomei, 2003), Goodwin (Middleton and Goodwin, 1988), Bitmead (Zang and Bitmead, 1994; Mareels and Bitmead, 1986) and others. In spite of the fact that the problem of control of time-varying systems is not new and the set of publications is devoted to it, it is necessary to note, that a number of relevant problems still has no satisfactory decisions. For today a number of interesting results for linear systems touching a problem of control in conditions of slow change of parameters, periodic change of parameters and also for a case of special structures of description matrixes of time-varying plants in which the linear system consists of time-invariant and time-varying parts (it is known and coordinated with control input) (Fradkov, *et al.*, 1999) is received.

Among methods of control of time-varying systems with unknown parameters the algorithms providing set behaviour of system for a class of mathematical models of certain structure, as a rule, prevail. Adaptive and robust control algorithms, allowing solving problems of stabilization and tracking for time-varying plants in which uncertainty is coordinated with a control input (Tsykunov,1996).

In 80-90-th years a series of the publications devoted to development of adaptive controllers for linear time-varying systems has appeared (Tsakalis and Ioannou, 1987; Tsakalis and Ioannou, 1993; Middleton and Goodwin, 1988; Kreisselmeier, 1986). These results were based on an assumption that parameters of the plant vary slowly with time and affect on system as external disturbance. Using this assumption robust and adaptive control algorithms for linear time-varying systems providing small tracking error had been synthesized. Later the availability of some a priori information about the changing of parameters has resulted in development of new adaptive algorithms for systems with fast-changing parameters (Tsakalis and Ioannou, 1989; Tsakalis and Ioannou, 1990). However the given algorithms could not guarantee high quality of transients (Zang and Bitmead, 1994; Mareels and Bitmead, 1986), and generally cannot be expanded on nonlinear systems with variable parameters. The specified problems had been solved, with use of iterative procedure of control law synthesis (Zhang, *et al.*, 2003). However proposed controller has high order (Zhang, *et al.*, 2003). It is necessary to notice,

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that special complexity is represented with control problems in which the plant is affected by unknown disturbances (Marino and Tomei, 2000; Marino and Tomei, 2003).

In this paper a method of output control of linear time-varying systems with unknown bounded parameters is considered. An assumption, that the plant is affected by unknown bounded disturbance, was proposed. Furthermore proposed control scheme allows synthesizing adaptive controller of fixed order, which does not depend of unknown parameters as in work (Zhang, Fidan, Ioannou, 2003). The proposed method is based on the results published in (Bobtsov and Nikolaev, 2005), where the problem of stabilization of nonlinear system was considered. Thus the represented algorithms can be applied both for nonlinear and linear time-varying systems.

2. PROBLEM STATEMENT

We consider linear time-varying system

$$\begin{cases} \dot{z} = Fz + L(u+w) + \theta(t)y(t-\tau), \\ y = Sz, \end{cases} \tag{1}$$

where $z(t) \in R^n$ is vector of state variables; F, L and S are ($n \times n$), ($n \times 1$) and ($1 \times n$) unknown constant matrices; $\theta(t) \in R^n$ is vector of unknown time-varying parameters; $y(t) \in R$ is output variable; $w(t) \in R$ is bounded unknown disturbance.

Let us assume that only output variable is measured, but not its derivatives, the state $z(t)$ and disturbance $w(t)$ are not measured and parameters of vector $\theta(t) \in R^n$ are smooth and bounded functions. We also assume that transfer function $H(p) = S(pI - F)^{-1}L = \dfrac{b(p)}{a(p)}$ is minimum-phase, i.e. $b(p)$ is a Hurwitz polynomial.

Together with the plant we consider the command signal y^* which is measured and satisfies the condition

$$\left| \frac{d^i y^*}{dt^i} \right| \le \overline{C} < \infty , \tag{2}$$

where $i = \overline{0, \rho}$ and number $\rho = n - m$ (where n and m dimensions of $a(p)$ and $b(p)$ polynomials accordingly) is a transfer function $H(p) = \dfrac{b(p)}{a(p)}$ relative degree.

We define the purpose of control as the solution of the problem of synthesizing the algorithm which at any initial conditions ensures the boundedness of all system signals as well as the execution of purpose condition

$$|e(t)| < \Delta , \tag{3}$$

for some $t \ge t_1$, where $e = y - y^*$ is a tracking error, Δ is a number which can be decreased by control law selection.

3. CONTROL DESIGN

We rewrite the system (1) in the following form

$$\begin{cases} \dot{z} = Fz + L(u+w) + \sum_{i=1}^{n} D_i \theta_i(t) y(t-\tau), \\ y = Sz, \end{cases} \tag{4}$$

where $D_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, …, $D_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ – ($n \times 1$) vectors; θ_1, θ_2, …, θ_n are components of the vector of unknown time-varying parameters $\theta(t) = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{n-1} \\ \theta_n \end{bmatrix}$.

The state-space model (4) can be represented in the input-output form

$$y = \frac{b(p)}{a(p)}(u+w) + \frac{c_1(p)}{a(p)}\theta_1(t)y(t-\tau) + \frac{c_2(p)}{a(p)}\theta_2(t)y(t-\tau) + \ldots + \frac{c_n(p)}{a(p)}\theta_n(t)y(t-\tau) = \frac{b(p)}{a(p)}(u+w) + \sum_{i=1}^{n} \frac{c_i(p)}{a(p)}\theta_i(t)y(t-\tau), \tag{5}$$

where $p = \dfrac{d}{dt}$ is differentiation operator, transfer function $\dfrac{c_i(p)}{a(p)} = S(pI - F)^{-1}D_i$.

Before beginning the synthesis of control law let us formulate the auxiliary result published in (Bobtsov and Nikolaev, 2005). Consider linear system time-invariant system

$$\begin{cases} \dot{x}' = A'x' + B'u', \\ y' = C'x', \end{cases} \tag{6}$$

where $x' \in R^n$, $y' \in R$, $u' \in R$, and matrixes A', B' and C' have appropriate dimensions. Transfer function of system (6) is determined by expression

$$\chi(p) = C'(pI - A')^{-1}B'.$$

Let the system (6) be closed

$$u' = -ky', \qquad (7)$$

in which number $k > 0$.

Let us put a question about existence of positively defined matrix $M = M^T$ and number k satisfying the correlations

$$M(A' + kB'C') + (A' + kB'C')^T M \le -G, \qquad (8)$$

$$MB' = (C')^T \qquad (9)$$

for some positively defined matrix $G = G^T$.

Lemma (Bobtsov, 2005; Bobtsov and Nikolaev, 2005). Let $\chi(p) = \frac{b'(p)}{a'(p)}$, where $b'(p) = b'_{n-1}p^{n-1} + ... + b'_0$ and $a'(p) = a'_n\ p^n + ... + a'_0$ are numerator and denominator of transfer function $\chi(p)$ accordingly. Let $b'(p)$ be a Hurwitz polynomial and $b'_{n-1} > 0$ then exists a number $k_0 > 0$ for which correlations (8), (9) are solvable for any $k \ge k_0$.

Choose the control law of the following form

$$u = -\phi(p)(k + \lambda)\hat{e}, \qquad (10)$$

where k is a positive number; the positive parameter λ is intended for compensation of the uncertainties $\sum_{i=1}^{n} D_i\theta_i(t)y(t-\tau)$ and $w(t)$; polynomial $\phi(p)$ is chosen for the polynomial $\beta(p) = \phi(p)b(p)$ to be Hurwitz and ($n-1$) order; function $\hat{e}(t)$ is the estimate of signal $e(t) = y(t) - y^*(t)$ which is calculated according to the following algorithm

$$\begin{cases} \dot{\xi}_1 = \sigma\xi_2, \\ \dot{\xi}_2 = \sigma\xi_3, \\ ... \\ \dot{\xi}_{\rho-1} = \sigma(-k_1\xi_1 - ... - k_{\rho-1}\xi_{\rho-1} + k_1 e), \end{cases} \qquad (11)$$

$$\hat{e} = \xi_1, \qquad (12)$$

where number $\sigma > k + \lambda$ (calculation procedure of σ is presented in Appendix, inequality (A.8)), and parameters k_i are calculated for the system (11) to be asymptotically stable for input $e = 0$.

Substituting (10) in equation $e(t) = y(t) - y^*(t)$ we obtain

$$e = \frac{b(p)}{a(p)}[-\phi(p)(k+\lambda)\hat{e} + w] + \sum_{i=1}^{n}\frac{c_i(p)}{a(p)}\theta_i(t)y - y^*$$

$$= \frac{b(p)}{a(p)}[-\phi(p)(k+\lambda)e + \phi(p)(k+\lambda)\varepsilon + w]$$

$$+ \sum_{i=1}^{n}\frac{c_i(p)}{a(p)}\theta_i(t)y(t-\tau) - y^*, \qquad (13)$$

where deviation function $\varepsilon(t)$ equals

$$\varepsilon = e - \hat{e}. \qquad (14)$$

Transform the equation (13) in the following way

$$a(p)e + k\phi(p)b(p)e = b(p)\phi(p)[(k+\lambda)\varepsilon - \lambda e]$$

$$+ b(p)w + \sum_{i=1}^{n} c_i(p)\theta_i(t)e(t-\tau) + \sum_{i=1}^{n} c_i(p)\theta_i(t)y^*(t-\tau)$$

$$-a(p)y^*,$$

where according to definition of error signal as $e(t) = y(t) - y^*(t)$ an equation $y(t-\tau) = e(t-\tau) + y^*(t-\tau)$ was used.

Let us introduce the following indication

$$f = w(t) - \frac{a(p)}{b(p)}y^* + \sum_{i=1}^{n}\frac{c_i(p)}{b(p)}\theta_i(t)y^*(t-\tau),$$

where according to the polynomial $b(p)$ is Hurwitz, parameters $\theta_i(t)$ are bounded and smooth, signal y^* and its derivatives up to order ρ including we obtain f is bounded.

Then for equation (13) obtain

$$e = \frac{\beta(p)}{a(p) + k\beta(p)}[-\lambda e + (k+\lambda)\varepsilon] + \frac{b(p)}{a(p) + k\beta(p)} f$$

$$+ \sum_{i=1}^{n}\frac{c_i(p)}{a(p) + k\beta(p)}\theta_i(t)e(t-\tau).$$

Let us denote

$$\gamma(p) = a(p) + k\beta(p),$$

$$\bar{f} = \frac{1}{\phi(p)} f,$$

then for (13) we have

$$e = \frac{\beta(p)}{\gamma(p)}[-\lambda e + (k+\lambda)\varepsilon + \bar{f}]$$

$$+ \sum_{i=1}^{n}\frac{c_i(p)}{\gamma(p)}\theta_i(t)e(t-\tau), \qquad (15)$$

where according to the polynomial $\phi(p)$ is Hurwitz and function $f(t)$ is bounded we obtain $\bar{f}(t)$ is also bounded.

Rewrite the input-output model (15) in state-space form

$$\begin{cases} \dot{x} = Ax + b(-\lambda e + (k+\lambda)\varepsilon + \bar{f}) + \sum_{i=1}^{n} q_i \theta_i(t) e(t-\tau), \\ e = c^T x, \end{cases} \tag{16}$$

where $x \in R^n$ is unmeasured state vector of system (16); A, b, q_i and c are appropriate matrixes of transition from input-output model to state-space model.

As $\beta(p)$ is ($n-1$) order Hurwitz polynomial then in view of lemma presented above number k_0 exists that it is possible to find number $k \ge k_0$ and symmetrical positively defined matrix P satisfying following matrix equations

$$A^T P + PA = -Q_1, \quad Pb = c, \tag{17}$$

where $Q_1 = Q_1^T$ is positively defined matrix.

Notice matrix Q_1 parameters depend on parameter k and do not depend on λ.

Let us rewrite model (11), (12) in vector-matrix form

$$\begin{cases} \dot{\xi} = \sigma(\Gamma\xi + dk_1 e), \\ \hat{e} = h^T \xi, \end{cases} \tag{18}$$

where $\Gamma = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_1 & -k_2 & -k_3 & \dots & -k_{\rho-1} \end{bmatrix}$, $d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

and $h = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Consider new variable

$$\eta = he - \xi, \tag{19}$$

then according the matrix h structure, error ε will become

$$\varepsilon = e - \hat{e} = h^T he - h^T \xi = h^T(he - \xi) = h^T \eta .$$

For derivative of η we obtain

$$\begin{aligned} \dot{\eta} &= h\dot{e} - \sigma(\Gamma(he - \eta) + dk_1 e) \\ &= h\dot{e} + \sigma\Gamma\eta - \sigma(dk_1 + \Gamma h)e . \end{aligned} \tag{20}$$

As $dk_1 = -\Gamma h$ (can be checked by substitution) then

$$\begin{cases} \dot{\eta} = h\dot{e} + \sigma\Gamma\eta, \\ \varepsilon = h^T \eta, \end{cases} \tag{21}$$

where matrix Γ according the calculation of parameters k_i of model (11) has proper numbers with negative real component and satisfies the Lyapunov equation:

$$\Gamma^T N + N\Gamma = -Q_2, \tag{22}$$

where $N = N^T$ and $Q_2 = Q_2^T$ are positively defined matrixes.

Theorem. There exist numbers $\sigma > k + \lambda$ and $\lambda > 0$ such that all trajectories of system (16), (21) are bounded and control purpose (3) is executed.

The proof of the theorem is presented in Appendix.

4. ADAPTIVE TUNING OF PARAMETERS

In this part we consider the problem of choosing the controller (10) – (12) parameters k, λ, σ satisfying the theorem conditions (see expressions (A.4), (A.7) and (A.8)). Possible variant of tuning the coefficients k, λ, σ is to increase them as long as the purpose condition (3) is executed.

For realization of this idea we use the following algorithm

$$\tilde{k}(t) = \int_{t_0}^{t} \mu(\tau) d\tau , \tag{23}$$

where $\tilde{k} = k + \lambda$ and function $\mu(t)$ is calculated in the following way

$$\mu(t) = \begin{cases} \mu_0 & for \ |e(t)| \ge \Delta, \\ 0 & for \ |e(t)| < \Delta, \end{cases}$$

where number $\mu_0 > 0$.
Choose σ in the following way

$$\sigma = \sigma_0 \tilde{k}^2, \tag{24}$$

where number $\sigma_0 > 0$.
It is obvious that with such calculation of k, λ, σ exists a point of time $t_1 > t_0$ for which, condition (17) and inequalities (A.4), (A.7), (A.8) are executed.

5. SIMULATION RESULTS

Let us consider the *Two-Stage Chemical Reactor with Recycle Streams* (Nguang, 2000):

$$\dot{z}_1(t) = -\frac{1}{\varsigma_1(t)} z_1(t) - \upsilon_1(t) z_1(t) + \frac{1 - R_2(t)}{V_1} z_2(t), \quad (25)$$

$$\dot{z}_2(t) = -\frac{1}{\varsigma_2(t)} z_2(t) - \upsilon_2(t) z_2(t) + \frac{R_1(t)}{V_2} z_1(t-\tau) + \frac{R_2(t)}{V_2} z_2(t-\tau) + \frac{F_2}{V_2} u, \quad (26)$$

$$y(t) = z_1(t) \quad (27)$$

where $z_1(t)$ and $z_2(t)$ are the compositions, R_1 and R_2 are the recycle flow rates, ς_i are the reactor residence times, υ_i are the reaction constants, F_2 is the feed rate and V_i are the reactor volumes.

Choose control law according to equations (10) – (12)

$$u = -\phi(p)(k+\lambda)\hat{e} = -(p+1)(k+\lambda)\hat{e} = -(p+1)\tilde{k}\xi_1 = -(\dot{\tilde{k}}\xi_1 + \tilde{k}\dot{\xi}_1) - \tilde{k}\xi_1, \quad (28)$$

$$\dot{\xi}_1 = \sigma(-k_1\xi_1 + k_1 e) = \sigma(-\xi_1 + e), \quad (29)$$

where polynomial $\phi(p) = p+1$ and coefficient $k_1 = 1$.

To tune the parameters $\tilde{k}$ and σ we use the method proposed in the previous part. Assigning the precision $\Delta = 0{,}05$ and command signal $y^*(t) = \sin t + 0{,}5\cos 0{,}3t$ we simulate the system for $\mu_0 = 5$ and $\sigma_0 = 0{,}3$. Results of computer simulation for unknown time-varying parameters $\varsigma_1 = 2\cos 3t$, $\varsigma_2 = 2$, $R_1 = 0{,}25\cos t \cdot e^{-t}$, $R_2 = 0{,}5$, $\upsilon_1 = 2 + 0{,}3\sin 0{,}1t + 0{,}7\sin 2t$, $\upsilon_2 = 0{,}3$, $V_1 = V_2 = F_2 = 0{,}5$ and $\tau = 2$ on variables $e(t)$, $u(t)$ and $\tilde{k}(t)$ are presented in the Fig. 1 – 3 accordingly. Computer simulation graphics for $y(0) = 0$ and $\hat{e}(0) = 0$ illustrate the achievement of proposed control purpose.

Let us consider one more time-varying plant:

$$\begin{cases} \dot{z}_1 = z_2 + \theta_1(t) z_1(t-\tau), \\ \dot{z}_2 = u + w + \theta_2(t) z_1(t-\tau), \end{cases} \quad (30)$$

$$y = z_1, \quad (31)$$

where $\theta_1(t)$ and $\theta_2(t)$ are unknown time-varying parameters, $w(t)$ is unknown disturbance, τ is delay.

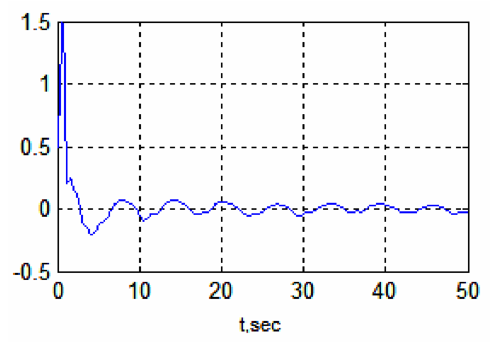

Fig. 1. Transients in control system for variable $e(t)$.

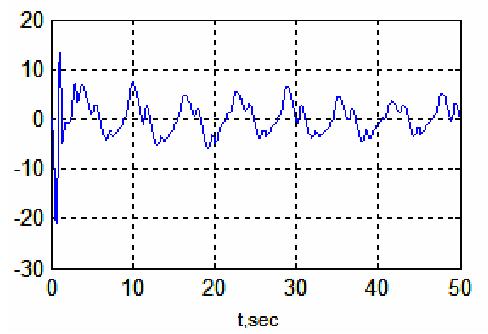

Fig. 2 Transients in control system for variable $u(t)$.

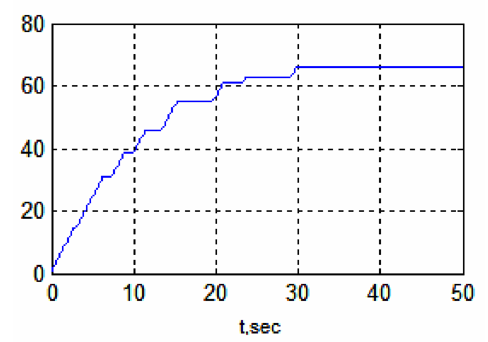

Fig. 3. Transients in control system for variable $\tilde{k}(t)$.

Choose control law according to equations (10) – (12)

$$u = -\phi(p)(k+\lambda)\hat{e} = -(p+1)(k+\lambda)\hat{e} = -(p+1)\tilde{k}\xi_1 = -(\dot{\tilde{k}}\xi_1 + \tilde{k}\dot{\xi}_1) - \tilde{k}\xi_1 \quad (32)$$

$$\dot{\xi}_1 = \sigma(-k_1\xi_1 + k_1 e) = \sigma(-\xi_1 + e), \quad (33)$$

where polynomial $\phi(p) = p+1$ and coefficient $k_1 = 1$.

Assigning the precision $\Delta = 0{,}1$ and command signal $y^*(t) = \sin(t)$ we simulate the system for $\mu_0 = 2$ and $\sigma_0 = 0{,}2$. Results of computer simulation for unknown time-varying parameters $\theta_1(t) = 2 + \sin 0{,}1t + \sin 10t$, $\theta_2(t) = 2\cos t$, disturbance $w(t) = 2 + \cos 3t$ and $\tau = 3$ on variables $e(t)$, $u(t)$ and $\tilde{k}(t)$ are presented in the Fig. 4 – 6 accordingly.

Computer simulation graphics for $y(0) = 0$ and $\hat{e}(0) = 0$ illustrate the achievement of proposed control purpose.

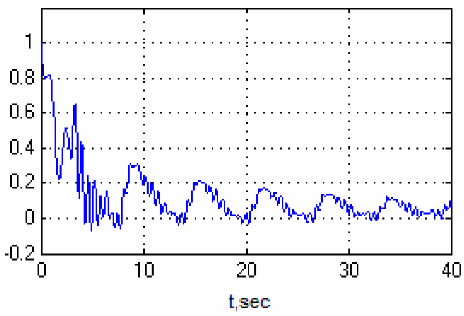

Fig. 4. Transients in control system for variable $e(t)$.

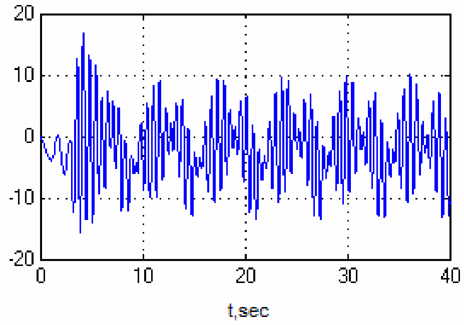

Fig. 5 Transients in control system for variable $u(t)$.

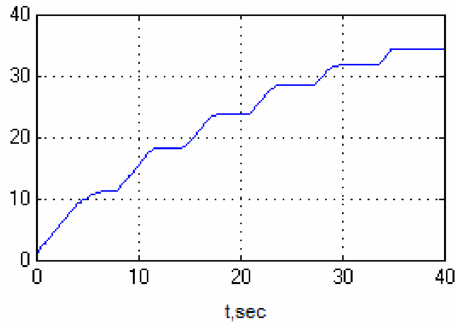

Fig. 6. Transients in control system for variable $\tilde{k}(t)$.

6. CONCLUSION

In the work the problem of synthesis of the output control for time-varying system (1) is considered. The control law (10) – (12), (23), (24) providing the execution of the control purpose (3), was designed. Adaptation algorithm (10) – (12), (23), (24) has dimension equal to ρ, where ρ is a relative degree of transfer function $H(p) = S(pI - F)^{-1} L = \frac{b(p)}{a(p)}$ of system (1).

The advantages of the proposed approach consist in the following:

- in comparison with (Tsykunov,1996) more general form of time-varying system is considered, but not a special case of structures of description matrixes of time-varying plants when the time-varying part is coordinated with control input as in (Tsykunov,1996);
- in contrast to work (Tsykunov,1996), the proposed controller is an output controller so the output is the only measured variable;
- intensifying the result represented in work (Zhang, *et al.*, 2003), an assumption, that the plant is affected by unknown bounded disturbance, is proposed;
- in contrast to algorithm from work (Zhang, *et al.*, 2003), proposed control scheme is easier in realization, does not require $5n - m$ additional filters and allows synthesizing adaptive controller of fixed order ρ, which depends only on relative degree of transfer function $H(p) = S(pI - F)^{-1} L = \frac{b(p)}{a(p)}$, but not on number of unknown parameters as in work (Zhang, *et al.*, 2003);

The disadvantage of the proposed controller in comparison with works (Zhang, *et al.*, 2003; Marino and Tomei, 2000; Marino and Tomei, 2003) is the following:

- matrix L was considered to be time-invariant.

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APPENDIX

Proof of the theorem.

Consider the following Lyapunov function

$$V = x^T Px + \eta^T N\eta + \gamma \int_{t-\tau}^{t} e^2(\omega)d\omega . \qquad \text{(A.1)}$$

Differentiating (A.1) on time in view of the equations (16) and (21) we obtain

$$\begin{aligned}\dot V &= x^T(A^T P + PA)x + 2(k+\lambda)x^T Pbh^T\eta \\ &+ 2x^T P \sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau) - 2\lambda x^T Pbe + 2x^T Pb\bar f \\ &+ \eta^T\sigma(\Gamma^T N + N\Gamma)\eta + 2\eta^T Nhc^T Ax \\ &+ 2(k+\lambda)\eta^T Nhc^T bh^T\eta + 2\eta^T Nhc^T \sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau) \\ &- 2\lambda\eta^T Nhc^T be + 2\eta^T Nhc^T b\bar f + \gamma e^2 - \gamma e^2(t-\tau) ,\end{aligned} \qquad \text{(A.2)}$$

where instead of the function $\dot e$ of system (21) and (A.2) the following item was used

$$\dot e = c^T(Ax - \lambda be + b\bar f + (k+\lambda)bh^T\eta + \sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)) .$$

Substituting in (A.2) the equations (17) and (22) and taking into account correlations

$$2(k+\lambda)x^T Pbh^T\eta \le (k+\lambda)(\delta x^T Pbb^T Px + \delta^{-1}\eta^T hh^T\eta) ,$$

$$2x^T P \sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau) \le \delta x^T P^2 x + \delta^{-1}\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2 ,$$

$$2x^T Pb\bar f \le \lambda x^T Pbb^T Px + \lambda^{-1}\bar f^2 ,$$

$$2\eta^T Nhc^T Ax \le \delta^{-1}\eta^T Nhc^T AA^T ch^T N\eta + \delta x^T x ,$$

$$2(k+\lambda)\eta^T Nhc^T bh^T\eta \le (k+\lambda)(\eta^T Nhc^T bb^T ch^T N\eta + \eta^T hh^T\eta) ,$$

$$2\eta^T Nhc^T \sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau) \le \lambda\eta^T Nhc^T ch^T N\eta + \lambda^{-1}\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2 ,$$

$$-2\lambda\eta^T Nhc^T be \le \delta^{-1}\lambda\eta^T Nhc^T ch^T N\eta + \delta\lambda x^T Pbb^T Px ,$$

$$2\eta^T Nhc^T b\bar f \le 2\lambda\eta^T Nhc^T bb^T ch^T N\eta + \frac{1}{2}\lambda^{-1}\bar f^2$$

for the derivative of Lyapunov function (A.1) we obtain

$$\begin{aligned}\dot V &\le -x^T Q_1 x - \sigma\eta^T Q_2\eta - \frac{3}{2}\lambda x^T Pbb^T Px - \frac{1}{2}\lambda e^2 \\ &+ \delta(k+\lambda)x^T Pbb^T Px + \delta^{-1}(k+\lambda)\eta^T hh^T\eta + \delta x^T P^2 x \\ &+ \delta^{-1}\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2 + \lambda x^T Pbb^T Px \\ &+ (k+\lambda)\eta^T Nhc^T bb^T ch^T N\eta + (k+\lambda)\eta^T hh^T\eta \\ &+ \delta^{-1}\eta^T Nhc^T AA^T ch^T N\eta + \delta x^T x + \lambda\eta^T Nhc^T ch^T N\eta \\ &+ \delta\lambda x^T Pbb^T Px + \lambda^{-1}\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2 \\ &+ \delta^{-1}\lambda\eta^T Nhc^T ch^T N\eta + 2\lambda\eta^T Nhc^T bb^T ch^T N\eta \\ &+ \frac{3}{2}\lambda^{-1}\bar f^2 + \gamma e^2 - \gamma e^2(t-\tau) ,\end{aligned} \qquad \text{(A.3)}$$

where the number $0 < \delta \le \frac{1}{2}$ satisfies the following condition

$$-Q_1 + \delta I + (\delta k + 2\delta\lambda - \frac{1}{2}\lambda)Pbb^T P + \delta P^2 \le -Q < 0 . \qquad \text{(A.4)}$$

Substituting expression (A.4) in an inequality (A.3) we obtain

$$\dot{V} \le -x^T Qx - \sigma\eta^T Q_2\eta - \frac{1}{2}\lambda e^2 + \delta^{-1}(k+\lambda)\eta^T hh^T\eta$$

$$+\delta^{-1}\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2 + \delta^{-1}\eta^T Nhc^T AA^T ch^T N\eta$$

$$+(k+\lambda)\eta^T Nhc^T bb^T ch^T N\eta + (k+\lambda)\eta^T hh^T\eta$$

$$+\lambda\eta^T Nhc^T ch^T N\eta + \lambda^{-1}\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2$$

$$+\delta^{-1}\lambda\eta^T Nhc^T ch^T N\eta + 2\lambda\eta^T Nhc^T bb^T ch^T N\eta$$

$$+\frac{3}{2}\lambda^{-1}\bar{f}^2 + \gamma e^2 - \gamma e^2(t-\tau). \qquad \text{(A.5)}$$

As the function $\theta_i(t)$ and each element of vector q_i are bounded, for the norm $\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2$ is possible to find such positive number C_0 that the following condition is executed

$$C_0 e(t-\tau)^2 \ge \left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2. \qquad \text{(A.6)}$$

Let number λ satisfies the following inequality

$$\lambda \ge 2C_0\left(\frac{1}{\lambda}+\frac{1}{\delta}\right). \qquad \text{(A.7)}$$

Let number σ such, that the following correlation is executed

$$-\sigma Q_2 + \delta^{-1}(k+\lambda)hh^T + (k+\lambda)Nhc^T bb^T ch^T N$$

$$+(k+\lambda)hh^T + \delta^{-1}Nhc^T AA^T ch^T N + \lambda Nhc^T ch^T N$$

$$+2\lambda Nhc^T bb^T ch^T N + \delta^{-1}\lambda Nhc^T ch^T N \le -Q. \qquad \text{(A.8)}$$

Substituting expression (A.8) in the inequality (A.5) we obtain

$$\dot{V} \le -x^T Qx - \eta^T Q\eta - \frac{1}{2}\lambda e^2 + \frac{3}{2}\lambda^{-1}\bar{f}^2$$

$$+(\delta^{-1}+\lambda^{-1})\left|\sum_{i=1}^{n} q_i\theta_i(t)e(t-\tau)\right|^2. \qquad \text{(A.9)}$$

Taking into account condition (A.6), for the inequality (A.8) we obtain

$$\dot{V} \le -x^T Qx - \eta^T Q\eta - \frac{1}{2}\lambda e^2 + \frac{3}{2}\lambda^{-1}\bar{f}^2$$

$$+(\delta^{-1}+\lambda^{-1})C_0 e^2(t-\tau) + \gamma e^2 - \gamma e^2(t-\tau). \qquad \text{(A.10)}$$

Choose γ in the following way:

$$\gamma = (\delta^{-1}+\lambda^{-1})C_0. \qquad \text{(A.11)}$$

Substituting expression (A.11) into (A.10) obtain

$$\dot{V} \le -x^T Qx - \eta^T Q\eta - \frac{1}{2}\lambda e^2 + \frac{3}{2}\lambda^{-1}\bar{f}^2$$

$$+(\delta^{-1}+\lambda^{-1})C_0 e^2 = -x^T Qx - \eta^T Q\eta - \left(\frac{1}{2}\lambda - \frac{C_0}{\delta} - \frac{C_0}{\lambda}\right)e^2$$

$$+\frac{3}{2}\lambda^{-1}\bar{f}^2. \qquad \text{(A.12)}$$

Substituting the inequality (A.7) into expression (A.12), we obtain

$$\dot{V} \le -x^T Qx - \eta^T Q\eta + \frac{3}{2}\lambda^{-1}\bar{f}^2$$

$$\le -x^T Qx - \eta^T Q\eta + \frac{3}{2}\lambda^{-1}C_1, \qquad \text{(A.13)}$$

where numbers $\nu > 0$ and $C_1 = \max\{\bar{f}^2\}$.

From expression (A.13) we obtain that all system (16), (21) trajectories are bounded and there exists a number $\lambda > 0$, that the control purpose (3) is executed.