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# THE EMPLOYMENT OF PERIODIC LYAPUNOV FUNCTIONS FOR ASYMPTOTIC ANALYSIS OF MULTIDIMENTIONAL PHASE CONTROL SYSTEMS

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**Abstract**

For continuous and discrete phase control systems the problem of gradient-like behavior and the problem of a number of slipped cycles are considered. By means of generalized periodic Lyapunov-type functions and Yakubovich-Kalman theorem new frequency-domain stability criteria as well as new frequency-domain estimates for the number of slipped cycles are obtained.

**Key words**

Phase control systems, gradient-like behavior, the number of slipped cycles, frequency-domain criteria, Lyapunov-type functions.

## 1 Introduction

This paper is devoted to asymptotic behavior of phase control systems, i.e. systems of indirect control with periodic nonlinearities. We consider here both systems described by differential equations and systems described by difference equations.

The phase system possesses a denumerable set of equilibria. Each equilibrium may be Lyapunov stable or not Lyapunov stable. And the main asymptotic characteristics of phase system is the convergence of any solution to a certain equilibrium. The systems which possess such a property are called gradient-like systems.

The problem of gradient-like behavior for phase systems has been examined in many published works. For detailed bibliography one can appeal to [Leonov, 2006].

More precise characteristics of gradient-like system is the "number of slipped cycles". They say that a phase system with a $\Delta$-periodic nonlinearity and an angular coordinate $\sigma(t)$ has slipped $k$ cycles if there exists a moment $\hat{t} > 0$ such, that $|\sigma(\hat{t}) - \sigma(0)| \geq k\Delta$ and $|\sigma(t) - \sigma(0)| < (k+1)\Delta$, for all $t \in \mathbf{R}^+$. This characteristics of a phase system was introduced by J. Stoker in [Stoker, 1950] for an equation of mathematical pendulum. It was then investigated for multivariable phase systems in [Yershova, Leonov, 1983]. The results of [Yershova, Leonov, 1983] were then extended to other classes of phase systems [Leonov, Smirnova, 2000], [Smirnova, Shepeljavyi and Utina, 2003], [Yang, Huang, 2007].

Both the problem of gradient-like behavior of a phase system and the problem of a number of slipped cycles have been fruitfully investigated by Lyapunov direct method. Efficient criteria of gradient-like behavior and efficient estimates for a number of slipped cycles has been obtained by means of two new types of Lyapunov functions.

One of them comprises Lyapunov functions which contain trajectories of gradient-like phase systems of low order (they are called reduction systems). So any stability assertion which is true for the reduction system can be extended to a phase system of high order [Leonov, 1984].

The functions which belong to the other type are often called periodic Lyapunov functions. They have the form of Lyr'e-Postnikov functions, i.e. they are constructed as "a quadratic form plus an integral of a nonlinear function". The nonlinear function which is under the integral sign is generated on the base of the periodic nonlinear function included in the phase system. This "new" nonlinear function is of the same period that the given one and has the same set of zeros.

Two kinds of such "new" nonlinear functions have been exploited in published works [Bakaev, Guzh, 1965], [Brockett, 1982], [ Leonov, Reitmann and Smirnova, 1992]. Consequently two varieties of periodic Lyapunov functions have been generated.

By means of these functions a number of theorems which give the opportunity to establish the fact of gradient-like behavior of a phase system and to get the estimations of the number of slipped cycles have been demonstrated. Usually the necessary and sufficient conditions for the existence of a periodic Lyapunov function are formulated with the help of Yakubovich-Kalman frequency-domain theorem [Yakubovich, 1973]. So stability theorems for phase systems usually contain a frequency-domain inequality with varying parameters.

In this paper a new periodic Lyapunov function is offered. It is a generalization of the two ones, used in above mentioned works. By means of this new Lyapunov function new multiparametric frequency-domain criteria for gradient-like behavior both for continuous and for discrete phase systems are generated. On the base of new multiparametric criteria improved frequency-domain estimates for the number of slipped cycles are obtained.

## 2 Frequency-domain conditions for gradient-like behavior.

Consider an autonomous phase system

$$\begin{array}{l}\dot{z} = Az + B\varphi(\sigma),\\ \dot{\sigma} = C^* z + R\varphi(\sigma),\end{array} \qquad (1)$$

where $A$ is a $(m \times m)$ - real matrix, $B$ and $C$ are real $m$ - vectors, $R$ is a number and $\varphi(\sigma)$ is a nonlinear function. The symbol $^*$ is used for Hermitian conjugation. We suppose that the pair $(A, B)$ is controllable, the pair $(A, C)$ is observable and matrix $A$ is a Hurwitz one.

We assume that $\varphi(\sigma)$ is $\Delta$-periodic, belongs to $C^1$ and has two simple zeros on $[0, \Delta]$. Assume also that

$$\int_0^\Delta \varphi(\sigma)d\sigma < 0. \qquad (2)$$

Let

$$\alpha_1 \leq \frac{d\varphi}{d\sigma} \leq \alpha_2 \qquad (3)$$

for all $\sigma \in \mathbf{R}$, where $\alpha_1 < 0 < \alpha_2$.

Let us determine the values

$$\nu = \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta |\varphi(\sigma)|\, d\sigma}, \qquad (4)$$

$$\nu_0 = \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta |\varphi(\sigma)| \sqrt{\left(1-\alpha_1^{-1}\varphi'(\sigma)\right)\left(1-\alpha_2^{-1}\varphi'(\sigma)\right)}d\sigma}. \qquad (5)$$

Let us introduce the transfer function for linear part of (1) from the input $\varphi$ to the output $(-\dot{\sigma})$

$$K(p) = -R + C^*(A - pE_m)^{-1}B \qquad (p \in \mathbf{C}), \quad (6)$$

where $E_m$ is a unit $m \times m$-matrix.

Let us also introduce the designation

$$\Re e H = \frac{1}{2}(H + H^*) \qquad (7)$$

for $l \times l$-matrix $H$.

**Theorem 1.** Suppose there exist such $æ \neq 0$, positive numbers $\varepsilon, \eta, \tau$ and nonnegative numbers $a, a_0$, that the following requirements are fulfilled:

1) for all $\omega \geq 0$ the inequality

$$\Re e \left\{ æK(i\omega) - \varepsilon K^*(i\omega)K(i\omega) - \tau \left(K(i\omega) + \alpha_1^{-1} i\omega\right)^* \cdot \left(K(i\omega) + \alpha_2^{-1} i\omega\right)\right\} - \eta \geq 0 \qquad (i^2 = -1) \qquad (8)$$

is valid;

2) $a + a_0 = 1$;

3) matrix

$$\left\| \begin{array}{ccc} \varepsilon, & \frac{æa\nu}{2}, & 0 \\ \frac{æa\nu}{2}, & \eta, & \frac{æa_0\nu_0}{2} \\ 0, & \frac{æa_0\nu_0}{2}, & \tau \end{array} \right\| \qquad (9)$$

is positive definite.

Then every solution of (1) converges to its equilibrium.

*Proof.* Theorem 1 is an extension of theorem 2.10.1 from monograph [Leonov, Ponomarenko and Smirnova, 1996]. So we shall borrow certain elements from the proof of the latter theorem. First of all we use here the transformation of system (1) to the system

$$\begin{array}{l}\frac{dy(t)}{dt} = Qy(t) + L\xi(t),\\ \frac{d\sigma(t)}{dt} = D^* y(t)\end{array} \qquad (10)$$

where

$$Q = \left\| \begin{array}{cc} A & B \\ 0 & 0 \end{array} \right\|, \quad L = \left\| \begin{array}{c} O \\ 1 \end{array} \right\|, \quad D = \left\| \begin{array}{c} C \\ R \end{array} \right\|,$$
$$y = \left\| \begin{array}{c} z(t) \\ \varphi(\sigma(t)) \end{array} \right\|, \; \xi = \frac{d}{dt}\varphi(\sigma(t)), \qquad (11)$$

and by $O$ a zero m-vector is designated. Next we borrow from [Leonov, Ponomarenko and Smirnova, 1996] the following quadratic form of $y \in \mathbf{R^{m+1}}$, $\xi \in \mathbf{R}$:

$$G(y,\xi) = 2y^* H(Qy + L\xi) + \varepsilon(D^*y)^2 + æy^*LD^*y- \\ \tau(D^*y - \alpha_1^{-1}\xi)(\alpha_2^{-1}\xi - D^*y) + \eta(L^*y)^2 \tag{12}$$

with a symmetric $(m+1)\times(m+1)$-matrix $H$ and numbers $\varepsilon$, æ, $\tau$ and $\eta$ which are introduced in the text of theorem 1.

It follows from condition 1) of theorem 1 that there exists a real symmetric matrix $H$, such that the inequality

$$G(y,\xi) \le 0 \qquad (\forall y \in \mathbf{R^{1+m}}, \forall \xi \in \mathbf{R}) \tag{13}$$

is true [Leonov, Ponomarenko and Smirnova, 1996].

We are going to use here periodic functions

$$F(\sigma) = \varphi(\sigma) - \nu\,|\varphi(\sigma)|, \\ \Phi(\sigma) = \sqrt{\left(1-\alpha_1^{-1}\varphi'(\sigma)\right)\left(1-\alpha_2^{-1}\varphi'(\sigma)\right)}, \tag{14}$$

$$\Psi(\sigma) = \varphi(\sigma) - \nu_0\Phi(\sigma)\,|\varphi(\sigma)|, \tag{15}$$

which have been introduced in [Leonov, Ponomarenko and Smirnova, 1996] after [Bakaev, Guzh, 1965] and [Brockett, 1982]. It is clear that

$$\int_0^\Delta F(\sigma)d\sigma = 0, \quad \int_0^\Delta \Psi(\sigma)d\sigma = 0. \tag{16}$$

With the help of $F(\sigma)$ and $\Psi(\sigma)$ we construct a new Lyapunov-type function

$$v(t) = y^*(t)Hy(t) + æ\left(a\int_0^{\sigma(t)} F(\sigma)d\sigma + \right. \\ \left. +a_0\int_0^{\sigma(t)}\Psi(\sigma)d\sigma\right). \tag{17}$$

Let $\frac{dv}{dt}$ be the derivative of $v(t)$ in virtue of system (10).

We have

$$\frac{dv(t)}{dt} = 2y^*(t)H(Qy(t) + L\xi(t)) + \\ +æ\left[aF(\sigma(t)) + a_0\Psi(\sigma(t))\right]\dot\sigma(t). \tag{18}$$

It follows from (13) that

$$\frac{dv(t)}{dt} \le \left(-\varepsilon\dot\sigma^2(t) - æ\varphi(\sigma(t))\dot\sigma(t) - \right. \\ -\eta\varphi^2(\sigma(t)) - \tau\Phi^2(\sigma(t))\dot\sigma^2(t) + \\ \left. +(æaF(\sigma(t)) + æa_0\Psi(\sigma(t)))\,\dot\sigma(t)\right). \tag{19}$$

Using formulas (14) and (15) we conclude from (19) that

$$\frac{dv(t)}{dt} \le \left(-\varepsilon\dot\sigma^2(t) - \eta\varphi^2(\sigma(t)) - \right. \\ -\tau\left(\Phi(\sigma(t))\dot\sigma(t)\right)^2 - æa\nu\,|\varphi(\sigma)|\,\dot\sigma(t) - \\ \left. -æa_0\nu_0\Phi(\sigma(t))\,|\varphi(\sigma)|\,\dot\sigma(t)\right). \tag{20}$$

The right part of inequality (20) is a quadratic form with regard to $\dot\sigma(t)$, $|\varphi(\sigma)|$, $\Phi(\sigma(t))\dot\sigma(t)$. According to condition 3) of theorem 1 it is negative definite. So we have

$$\frac{dv(t)}{dt} \le -\delta\varphi^2(\sigma(t)) \tag{21}$$

with $\delta > 0$. Then

$$v(t) - v(0) \le -\int_0^t \delta\varphi^2(\sigma(t))\,dt,\ \forall t \ge 0. \tag{22}$$

It follows from (16) that function $v(t)$ is bounded from bellow. Then from (22) we have that

$$\int_0^\infty \varphi^2(\sigma(t))dt \le +\infty. \tag{23}$$

Since matrix $A$ is Hurwitzian, function $\varphi(\sigma(t))$ is uniformly continuous on $[0,+\infty)$. Then it follows from (23) according to Barbalat lemma that

$$\varphi(\sigma(t)) \to 0 \quad as \quad t \to +\infty, \tag{24}$$

This limit relation and lemma 2.5.1 [Leonov, Ponomarenko and Smirnova, 1996] imply that

$$\sigma(t) \to \hat\sigma \quad as \quad t \to +\infty, \tag{25}$$

where $\varphi(\hat\sigma) = 0$, The first equation of system (1) can be rewritten in the form

$$z(t) = e^{At}z(0) + \int_0^t e^{A(t-\tau)}B\varphi(\sigma(\tau))d\tau. \tag{26}$$

From (23) and the fact that the convolution of two functions from $L_2[0,+\infty)$ tends to 0 as $t \to +\infty$ we deduce that

$$z(t) \to 0 \quad as \quad t \to +\infty. \tag{27}$$

Theorem 1 is proved.

Theorem 1 was applied to an autonomous second-order phase-locked loop with proportional-integrating filter. Its transfer function has the form

$$K(p) = T\frac{1+\beta Tp}{1+Tp}, \tag{28}$$

where $T > 0$ and $\beta \in (0,1)$ are constants. The case of $\beta = 0.2$ and $\varphi(\sigma) = \sin(\sigma) - \gamma$, where $\gamma \in (0,1)$, was considered. Theorem 1 gave the opportunity to improve the estimate for lock-in range in the space of parameters $(T^{-1}, \gamma)$, obtained in [Leonov, Ponomarenko and Smirnova, 1996] by theorem 2.10.1. With the help of theorem 1 the gap between the genuine boundary of lock-in range and its frequency-domain estimate from [Leonov, Ponomarenko and Smirnova, 1996] was diminished for $T \le 1$ by 15% at least.

Consider a discrete phase system

$$\begin{array}{l} z(n+1) = Az(n) + B\varphi(\sigma(n)), \\ \sigma(n+1) = \sigma(n) + C^* z(n) + R\varphi(\sigma(n)) \\ (n = 0, 1, 2, \ldots), \end{array} \qquad (29)$$

where $A$, $B$, $C$, $R$ are described in the beginning of the section. We suppose that the pair $(A, B)$ is controllable, the pair $(A, C)$ is observable and all eigenvalues of matrix $A$ are situated inside the open unit circle. All the properties of $\varphi(\sigma)$ are just the same as in the beginning of the section. We shall need numbers $k_1 = 2\alpha_1 - \alpha_2$ and $k_2 = 2\alpha_2 - \alpha_1$. The transfer function for the linear part of system (29) is the same as that of system (1).

**Theorem 2.** Suppose there exist such $æ \ne 0$, positive numbers $\varepsilon, \eta, \tau$ and nonnegative numbers $a, a_0$, that the following requirements are fulfilled:

1) for all $p \in \mathbf{C}$, $|p| = 1$ the inequality

$$\begin{array}{l} \Re e \left\{ æK(p) - \tau \left( K(p) + (p-1)k_1^{-1} \right)^* \cdot \right. \\ \left. \cdot \left( K(p) + (p-1)k_2^{-1} \right) \right\} - \varepsilon K^*(p)K(p) - \eta \ge 0 \end{array} \qquad (30)$$

is valid;

2) $a + a_0 = 1$;

3) matrix

$$\left\| \begin{array}{ccc} \varepsilon - \frac{æ\alpha_0}{2} \left( a(1+\nu) + a_0 \left( 1 - \frac{\alpha_2 - \alpha_1}{\sqrt{|\alpha_1|\alpha_2}} \right) \right), & \frac{æ\nu a}{2}, & 0 \\ \frac{æ\nu a}{2}, & \eta, & \frac{æ a_0 \nu_0}{2} \\ 0, & \frac{æ a_0 \nu_0}{2}, & \tau \frac{\alpha_1 \alpha_2}{k_1 k_2} \end{array} \right\|, \qquad (31)$$

where $\alpha_0 = \alpha_2$ if $æ > 0$ and $\alpha_0 = \alpha_1$ if $æ < 0$, is positive definite.

Then

$$\begin{array}{l} \lim\limits_{n \to \infty} \varphi(\sigma(n)) = 0, \\ \lim\limits_{n \to \infty} z(n) = 0, \\ \lim\limits_{n \to \infty} (\sigma(n+1) - \sigma(n)) = 0, \\ \lim\limits_{n \to \infty} (\sigma(n)) = \hat{\sigma}, \end{array} \qquad (32)$$

where $\varphi(\hat{\sigma}) = 0$.

Theorem 2 is a generalization of theorem 1 of [Smirnova, Shepeljavyi, 2007]

## 3 Frequency-domain estimates for the number of slipped cycles.

We shall show in this section how a frequency-domain criterion for gradient-like behavior can be transformed into a frequency-domain estimate for the number of slipped cycles. For the purpose we shall bring the number of slipped cycles into algebraic conditions on varying parameters. Instead of constants $\nu_i$ and $\nu_{0i}$ we shall use the functions

$$r_j(k, æ, x) = \frac{\int_0^\Delta \varphi(\sigma)\, d\sigma + (-1)^j \frac{x}{æk}}{\int_0^\Delta |\varphi(\sigma)|\, d\sigma}, \qquad (33)$$

$$r_{0j}(k, æ, x) = \frac{\int_0^\Delta \varphi(\sigma)\, d\sigma + (-1)^j \frac{x}{æk}}{\int_0^\Delta \varphi_1(\sigma)|\varphi(\sigma)|\, d\sigma} (j = 1, 2), \qquad (34)$$

where

$$\varphi_1(\sigma) = \sqrt{(1 - \alpha_1^{-1}\varphi'(\sigma))(1 - \alpha_2^{-1}\varphi'(\sigma))}. \qquad (35)$$

Further investigation is based on a Lyapunov–type lemma and on Yakubovich–Kalman frequency–domain theorem.

**Lemma.** Let $a_0$, $a \in \mathbf{R}^+$, $\varepsilon, \eta, \tau \in \mathbf{R}^+/\{0\}$, $æ \ne 0$, $k \in \mathbf{N}$. Let $\varphi(\sigma)$ be a $\Delta$-periodic continuously differentiable function, satisfying conditions alike (2), (3), and functions $\sigma(t)$, $W(t)$ be continuously differentiable on $\mathbf{R}^+$. Suppose the following requirements are fulfilled:

1)

$$\begin{array}{l} \frac{dW(t)}{dt} + æ\varphi(\sigma(t))\frac{d\sigma(t)}{dt} + \varepsilon(\frac{d\sigma(t)}{dt})^2 + \\ + \eta\varphi^2(\sigma(t)) + \tau\varphi_1^2(\sigma(t))(\frac{d\sigma(t)}{dt})^2 \le 0, \end{array} \qquad (36)$$

2) $a + a_0 = 1$;

3) the matrices $T(W(0))$, where

$$T_j(x) = \left\| \begin{array}{ccc} \varepsilon, & \frac{aær_j(k,æ,x)}{2}, & 0 \\ \frac{aær_j(k,æ,x)}{2}, & \eta, & \frac{a_0ær_{0j}(k,æ,x)}{2} \\ 0, & \frac{a_0ær_{0j}(k,æ,x)}{2}, & \tau \end{array} \right\|, \qquad (37)$$

are positive definite $(j = 1, 2)$.

4) a value $\bar{t} > 0$ is such that $W(\bar{t}) \ge 0$.

Then

$$|\sigma(\bar{t}) - \sigma(0)| \ne k\Delta. \qquad (38)$$

*Proof.* If follows from the requirement 3) of the lemma that matrices $T_j(W(0) + \varepsilon_0)$ $(j = 1, 2)$, where $\varepsilon_0$ is a small positive number, are positive definite. Let us define the functions

$$\begin{aligned} F_j(\sigma) &= \varphi(\sigma) - r_j|\varphi(\sigma)|, \\ \Psi_j(\sigma) &= \varphi(\sigma) - r_{0j}|\varphi(\sigma)|\varphi_1(\sigma), \end{aligned} \tag{39}$$

where

$$\begin{aligned} r_j &= r_j(k, æ, W(0)+\varepsilon_0), \\ r_{0j} &= r_{0j}(k, æ, W(0)+\varepsilon_0) \ \ (j=1,2), \end{aligned} \tag{40}$$

and Lyapunov–type functions

$$V_j(t) = W(t) + æ(a \int\limits_{\sigma(0)}^{\sigma(t)} F_j(\sigma)\, d\sigma + a_0 \int\limits_{\sigma(0)}^{\sigma(t)} \Psi_j(\sigma)\, d\sigma). \tag{41}$$

Then

$$\frac{dV_j}{dt} = \frac{dW(t)}{dt} + æ\left(aF_j(\sigma(t)) + a_0\Psi_j(\sigma(t))\dot\sigma(t)\right). \tag{42}$$

In virtue of requirement 1) of the lemma we have

$$\begin{aligned} \tfrac{dV_j}{dt} &\le -\varepsilon\dot\sigma^2 - \tau(\varphi_1(\sigma)\dot\sigma)^2 - \eta\varphi^2(\sigma(t)) - \\ &- æar_j|\varphi(\sigma)|\dot\sigma - æa_0r_{0j}|\varphi(\sigma)|\varphi_1(\sigma)\dot\sigma. \end{aligned} \tag{43}$$

Since matrices $T_j(W(0)+\varepsilon_0)$ $(j=1,2)$ are positive definite it follows that

$$\frac{dV_j(t)}{dt} \le 0 \ \ (j=1,2), \tag{44}$$

and consequently for all $t \in \mathbf{R}^+$

$$V_j(t) \le V_j(0) = W(0). \tag{45}$$

Suppose now that $\sigma(\bar t) = \sigma(0) + k\Delta$. Then

$$V_1(\bar t) = W(\bar t) + kæ\int\limits_0^\Delta (aF_1(\sigma) + a_1\Psi_1(\sigma))d\sigma. \tag{46}$$

But

$$\int\limits_0^\Delta F_1(\sigma)\, d\sigma = \int\limits_0^\Delta \Psi_1(\sigma)\, d\sigma = \frac{1}{æk}(W(0)+\varepsilon_0). \tag{47}$$

Thus

$$V_1(\bar t) = W(\bar t) + W(0) + \varepsilon_0 \tag{48}$$

and

$$V_1(\bar t) > W(0), \tag{49}$$

which contradicts (45).

If we suppose that $\sigma(\bar t) = \sigma(0) - k\Delta$ then by analogous reasoning we shall come to the conclusion that

$$V_2(\bar t) > W(0), \tag{50}$$

which also contradicts (45). So lemma is proved.

Further we shall use the proof of theorem 1. So consider the quadratic form $G_0(y,\xi)$ of $y \in \mathbf{R}^{m+1}$ and $\xi \in \mathbf{R}$, such that

$$G(y,\xi) = 2y^*H(Py + L\xi) + G_0(y,\xi). \tag{51}$$

Here $H = H^*$ is a $((m+1)\times(m+1))$ - matrix.

Let the values of parameters $\varepsilon$, $\eta$, æ and $\tau$ in $G(y,\xi)$ be such that condition 1) of the theorem 1 is true. Then there exists a real symmetric matrix $H$ such that the inequality

$$G(y,\xi) \le 0 \ \ (\forall\, y \in \mathbf{R}^{m+1}, \forall\, \xi \in \mathbf{R}) \tag{52}$$

is true.

**Theorem 3.** Let $\sigma(0) \in (\sigma_1, \sigma_2)$ with $\varphi(\sigma_1) = \varphi(\sigma_2) = 0$ and $|\sigma_1 - \sigma_2| < \Delta$. Suppose there exist such $æ \ne 0$, positive numbers $\varepsilon, \eta, \tau$, nonnegative numbers $a, a_0$ and natural $k$ that the following conditions are fulfilled:

1) the condition 1) of the theorem 1 is true;

2) $a + a_0 = 1$;

3) matrices

$$T_1 \cdot (sign(æ)(y^*(0)Hy(0) - æ\int_{\sigma(0)}^{\Delta_j} \varphi(\sigma)d\sigma)), \tag{53}$$

where j=1,2, $\Delta_1 = \sigma_1, \Delta_2 = \sigma_1 + \Delta$, if $æ\varphi(\sigma(0)) < 0$, and $\Delta_1 = \sigma_2, \Delta_2 = \sigma_2 - \Delta$, if $æ\varphi(\sigma(0)) > 0$, are positive definite for a certain matrix $H = H^*$ satisfying (52)

($T_1(x)$ is defined in the text of the lemma).

Then for any solution of (1) with initial data $(z(0), \sigma(0))$ the estimate

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta \tag{54}$$

is true for all $t \in \mathbf{R}^+$.

*Proof.* Consider the solution with the initial data $(z(0), \sigma(0))$. We shall use the lemma here. Let

$$W(t) = y^*(t)Hy(t). \tag{55}$$

Then

$$\frac{dW(t)}{dt} = 2y^*(t)H(Qy(t) + L\xi(t)), \tag{56}$$

where the derivative is calculated in virtue of system (10). We have from (52)

$$\frac{dW(t)}{dt} \le -G_0(y(t), \xi(t)) = \\ = -æ\varphi(\sigma(t))\dot{\sigma}(t) - \varepsilon\dot{\sigma}^2(t) - \eta\varphi^2(\sigma(t)) - \\ -\tau\dot{\sigma}^2(t)(1 - \alpha_1^{-1}\varphi'(\sigma(t))) \cdot (1 - \alpha_2^{-1}\varphi'(\sigma(t))). \tag{57}$$

So condition 1) of the lemma is fulfilled for any solution of (1). Suppose that for a certain moment $t = \hat{t}$ we have either $\sigma(\hat{t}) = \Delta_1$ or $\sigma(\hat{t}) = \Delta_2$. Let us consider then the solution with $z(0) = z(\hat{t})$ and $\sigma(0) = \sigma(\hat{t})$ and denote it by $(\hat{z}(t), \hat{\sigma}(t))$. Let us also use the notations

$$\hat{y} = \left\| \begin{matrix} \hat{z} \\ \varphi(\hat{\sigma}) \end{matrix} \right\|, \tag{58}$$

$$\hat{W}(t) = \hat{y}^*(t) H \hat{y}(t). \tag{59}$$

Note that $\varphi(\hat{\sigma}(0)) = 0$. Suppose that $\hat{\sigma}(\bar{t}) = \hat{\sigma}(0) \pm k\Delta$. Then $\varphi(\hat{\sigma}(\bar{t})) = 0$.

Let

$$H = \left\| \begin{matrix} H_0 & h \\ h^T & \alpha \end{matrix} \right\|, \tag{60}$$

where $H_0 = H_0^*$ is a $m \times m$-matrix. Then $\hat{W}(\bar{t}) = \hat{z}^*(\bar{t}) H_0 \hat{z}(\bar{t})$. It is easy to demonstrate that $\hat{W}(\bar{t}) \ge 0$ [Smirnova, Shepeljavyi and Utina, 2003]. Note that

$$0 \le \hat{W}(0) = W(\hat{t}) \le W(0) - æ \int_{\sigma(0)}^{\Delta_j} \varphi(\sigma) d\sigma. \tag{61}$$

Condition 3) of theorem 3 guarantees that condition 3) of the lemma is true. It follows from the fact that $W(\hat{t}) \ge 0$. So all the conditions of the lemma are fulfilled for the solution $(\hat{z}(t), \hat{\sigma}(t))$ and

$$\hat{\sigma}(0) - k\Delta < \hat{\sigma}(\bar{t}) < \hat{\sigma}(0) + k\Delta. \tag{62}$$

It follows then that for the solution $(z(t), \sigma(t))$ of (1) the estimate

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta \tag{63}$$

is true for all $t \in \mathbf{R}^+$. Theorem 3 is proved.

## 4 Conclusions

In this paper a certain generalization of periodic Lyapunov-type functions and sequences, traditionally used for phase control systems, is offered. With the help of this generalization and Yakubovich-Kalman frequency-domain theorem new frequency-domain criteria with many varying parameters are obtained. The latter give the opportunity to improve the estimates of stability regions for concrete phase systems.


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