

SYNCHRONIZATION ON GROWING DYNAMICAL NETWORKS: A DISCRETE EVENT APPROACH

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Abstract

We investigate the effects of growth process on the synchronization behavior of a dynamical network composed by coupled Logistic Maps. In particular, we assume that the growth process follows the wellknown Barabási-Albert model. In this context, we interpret the addition and preferential attachment of a new node as a discrete event where the network structure switches to a new configuration. We propose to add a new node iteratively slow in order to let the transitory effects die out. We examine then how the synchronization criterion for a dynamical network with fixed structure could be applied to the case in which this network grows. Our results show that the stability of the synchronized solution is preserved for the addition of only a limited number of nodes. Furthermore, the number of added nodes for which the stability is preserved directly depends on the structure and size of the initial structure of the network.

Key words

Dynamical networks; Discrete event systems; Switching systems; Synchronization.

1 Introduction

A Dynamical Network (DN) is composed by a set of coupled dynamical systems called nodes. The coupling between any pair on nodes is represented by an edge, and the pattern of the couplings is called the network structure. DN have attracted tremendous attention in many fields of science mainly due to their potential applications to model systems in nature such as the Internet, the World Wide Web, food webs, etc ([Newman (2010)]). One of the most significant phenomena in DN, is the synchronization behavior of dynamical nodes. Recent research have been focus on establish synchronization criteria in DN networks with a given fixed structure ([Boccaletti et al. (2008)]). However, in

order to model a more realistic situation, it is important to take into account that real-world networks actually evolves through different change processes like the addition or deletion of nodes and links.

Network evolution has been extensively addressed from the framework of graph theory. In this context, an evolution model consists of a set of structural change rules, which are repeated iteratively in order to emulate the network evolution. One of the first and more significant evolution model, designed to describe the way in which real-world networks evolves, is the so-called Barabási-Albert (BA) model ([Barabási and Albert (1999)]), which argued that there are two generic aspects of real-world networks, which are: 1) growth, at each iteration a new node is added to the network, and 2) preferential attachment, the new node is more likely to connect to an important node than to a less connected one. After repeat these rules a given number of iterations, we get a network where the vast majority of nodes will have few connections while some few nodes will have many connections. The network with this structural effect are called scale-free networks. However, BA model only focus on the structural features of the network and does not consider dynamical aspects of the collective behavior. In this sense, the authors in ([Jin Fan et al. (2004)]), have been proposed a synchronization-optimal growth model, where the BA preferential attachment rule is replaced by a rule where each new node connection is such that it optimize the synchronizability of the network. The model in ([Jin Fan et al. (2004)]) succeeds in construct a network with the scale-free feature that achieves synchronization; nevertheless, with this model we must to know the entire network dynamical state in order to select the new node connections.

On the other hand, when we model the structural growth in a DN, we can not use the synchronization criteria used for the case of DN with a fixed structure. This represent a challenging problem that has attracted the attention of current researchs, some of which, have been tackle this complication from the framework of

Siscrete Event Systems. This formalism allow us to interpret any change process in the network structure as a discrete event that causes a transition from one network structure to another. This discrete event occurs while the nodes are still evolving accordign to their dynamical nature. Discrete Event Systems permit us also to distinguish between to different dynamics: the time-driven and the event-driven dynamics. The first one is refered to systems whose dynamical states changes as time changes. In the context of DN, this kind of dynamics correspond to the evolution of nodes. On the contrary, for event-driven dynamics the system changes only at certain points in time through instantaneous transitions, which we can associate as an event. In the case of DN this corresponds to the structural changes on the network. Note that DN whit evolving structure combine time-driven and event-driven dynamics, so it can be seen as an hybrid dynamical system.

In ([D.J Stilwell et al. (2006)]), the authors have tackle this problem from the framework of switching systems. They considers a set of DN whit different structure but whit the same type of dynamical nodes and the same number of nodes. Then the network switches its structure to a predefined one thought a switching signal. An important result is that if the switching is fast enough, then an average model can be used, and the synchronization can be achieve if the switching is fast enough. On the contrary, in ([David Hill et al. (2010)]) the authors have shown that if the average dwell time of the current structure is slow, then it is possible also to achieve synchronization. However, in these works the number of nodes are always fixed, that is, it is no allowed to the DN to growth, which, as was pointed out in BA model, is an important procees in the real-world network evolution.

In this contribution, we intepreted the BA model from the point of view of Discrete Event Systems. To do that, we define as an event the addition of a new node and its preferential attachment. In particular we consider the case of a DN composed by discrete-time systems called Logistic Maps, and we model the network evolution as in the BA model. We propose to add a new node iteratively slow in order to let the dynamical transitions of the nodes. We observe that if the dwell time of the new node is large enough, the synchronized behavior can be achieved when we add a few number of nodes.

This paper is dived as follows. On the second section we resume significant preliminaries for this work, in particular, we review the synchronization criterion for a discrete-time dynamical network with fixed structure and the BA model of network growth. On section tree we expose our interpretation of the BA model for dynamical network since the discrete event system approach, and in section four we analyze how to use the synchronization criterion for a dynamical network with fixed structure, to the case a growing dynamical networks. On section five we show our numerical results for the case of a DN where each node is a Logistic Map and where the structure evolve accordign to the

BA model. Finally we present the conclusions for this work.

2 Preliminaries

2.1 Discrete-time dynamical network

For a network of N identical discrete-time systems, lineally and bidirectionally coupled with unweighted edges, the dynamical evolution of each node is given by

$$x_i^{k+1} = f(x_i^k) + c \sum_{j=1}^N a_{ij} f(x_j^k), \quad i = 1, \dots, N \quad (1)$$

where x_i^k is the state variable of the i -th node at the discrete-time instant $k \in \mathbf{Z}$. The map $f(\cdot)$ describes the dynamics of a single node isolated from the network. For the remainder of this contribution, we consider that each node is a Logistic Map:

$$f(x^k) = rx^k(1 - x^k) \quad (2)$$

with $r = 3.9$ and $x_i^k \in \mathbf{R}$. The variable $c \in \mathbf{R}$ represents the uniform coupling strength, and the coupling matrix $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N \times N}$ describes the network structure as follows: if the i -th and j -th node are connected, the entries $a_{ij} = a_{ji}$ are set to one; if there is no connection between them, the entries are set to zero ($a_{ij} = a_{ji} = 0$). To complete the matrix, the diagonal entries are determine in the following manner:

$$a_{ii} = - \sum_{j=1}^N a_{ij} = - \sum_{i=1}^N a_{ij} = -d_i \quad (3)$$

where d_i is the node degree of the i -th node.

By construction, the connectivity in the network is difusive, that is, all sums by row or column of \mathcal{A} are zero. Further, if the network is connected in the sense that no node is isolated from the network, then the coupling matrix is symmetric, irreducible, and its eigenvalues (λ_i) can be ordered as:

$$0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_N \quad (4)$$

For a dynamical network, complete synchronization is defined as the phenomena in which the evolution of all its nodes moves at unison. In an other words, a dynamical network is said to (asymptotically) achieve complete synchronization if as $k \rightarrow \infty$ the states of each node in the network tend to the synchronized solution

$$x_1^k = x_2^k = \dots = x_N^k \quad (5)$$

From a stability analysis of the network dynamics linearized around the synchronized solution, diverse synchronization criteria can be derived (see for example [Arenas et al. (2008)]). In particular, for a dynamical network of discrete-time systems, a synchronization criterion was proposed in ([Li and Chen (2003)]). In that work, for a dynamical network of identical discrete-time systems with fixed structure and diffusive coupling, it was shown that the synchronized solution (5) is exponentially stable if the uniform coupling strength satisfies

$$\frac{1 - e^{-h_{max}}}{|\lambda_2|} < c < \frac{1 + e^{-h_{max}}}{|\lambda_N|} \quad (6)$$

where λ_2 and λ_N are the biggest and smallest nonzero eigenvalue of \mathcal{A} , respectively; while h_{max} is the largest Lyapunov exponent of an isolated node, which, for the case of a Logistic Map we have $h_{max} = \ln(2)$.

An important question related to the stability of the synchronized solution is whether or not there is a positive coupling function such that the criterion in (6) is satisfied. To this end, an alternative version of the criterion can be used. Consider the ratio $R = \frac{-\lambda_2}{\lambda_2 - \lambda_N}$, which measure the normalized distance of the eigen-spectrum of \mathcal{A} . Then a positive coupling strength exist if the ratio satisfies

$$\frac{1}{R} < \frac{2e^{-h_{max}}}{1 - e^{-h_{max}}} \quad (7)$$

For the dynamical network of Logistic Maps consider in this contribution, the condition becomes $\frac{1}{R} < 2$.

Notice that the synchronization criteria (6) and (7) are only valid for a static network structure, and in general, can not be consider a valid criteria while the structure of the network evolves. In particular, in the case of network growth, this is further complicated by the increment in the dimension of \mathcal{A} , and the change in its eigenvalues after each growth event. However, under the conditions that each growth event results in a network of identical nodes with diffusive structure, and that the time between growth events is large enough as to allow for the node dynamics to reach their steady-state behavior, one can argue that the synchronization conditions (6) and (7) can in fact be use to determine the stability of the synchronized solution on growing networks.

We assume that the growth events in the network are described by the scale-free model proposed by Barabasi and Albert ([Barabási and Albert (1999)]), which is described in the following subsection.

2.2 The BA model of network growth

One of the first and more significant network models designed to describe the way in which a real-world network grows is the so-called scale-free network model

proposed by Barabasi and Albert in 1999 ([Newman (2010)]), which states that as the network grows, it does so following a preferential attachment rule, that is, a new node in the network is more likely to connect to an important node that to a less connected one. In what follows we briefly describe the BA model network construction algorithm:

The BA model consists of two steps:

The first step is simply called *Growth*.

1. Starting with a small number (m_0) of nodes. Then, at every iteration of the model (or growth event) a new node is added to the network, and it is connected to m ($\leq m_0$) of the nodes already present in the network.

The second step tells us to which of the nodes already existing in the network our new node will be connected. The choice is made favoring connections to the nodes with the largest number of connections, for that reason is called *Preferential attachment*.

2. For each new node, say the q -th node, the m nodes to which it will be connected are selected from the nodes already in the network through a random process where the probability that the new node connects to the j -th node is given by

$$\Pi_{q \leftrightarrow j} = \frac{d_j}{\sum_{l=0}^N d_l} \quad (8)$$

These two steps are iterated until the network has grown to the desired number of nodes, say N , with $N = m_0 + \sigma$ where σ is the number of iteration of the construction algorithm, in other words, the number of growth events that lead to a network of size N .

A particularly significant aspect of the BA network model is that for a sufficiently large number of nodes, the statistical distribution of the node degrees of the resulting network is well approximated by a power-law distribution of the form $P(d) \sim d^{-3}$, which remains practically unchanged for larger number nodes, that is, this feature of the topology is independent of size; this is the scale-free feature that gives name to the model.

In this contribution, we investigate the effect of the growth events described by the BA model on the stability of the synchronized state of the resulting DN. In the following section, we propose an interpretation of the BA model as a Discrete Event System where the growth events result on changes in the dynamical description of the network. Then, we state the synchronization problem for the growing DN as the stability preservation of the synchronized solution for the resulting network after each growth event.

3 Interpretation of the BA model for dynamical network growth

We start with a network composed by a small number (m_0) of Logistic Maps connected in a fully coupled structure. Rewriting the dynamical description (1) of our initial network in vector form we have

$$\mathbf{X}^{k+1,0} = F^0(\mathbf{X}^{k,0}) + c\mathcal{A}^0 F^0(\mathbf{X}^{k,0}) \quad (9)$$

where $\mathbf{X}^{k,0} = [x_1^{k,0}, \dots, x_{m_0}^{k,0}]^\top \in \mathbf{R}^{m_0}$; $F^0(\mathbf{X}^{k,0}) = [f(x_1^{k,0}), \dots, f(x_{m_0}^{k,0})]^\top \in \mathbf{R}^{m_0}$; and the initial coupling matrix has the form

$$\mathcal{A}_0 = \begin{pmatrix} -m_0 + 1 & 1 & \dots & 1 \\ 1 & -m_0 + 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -m_0 + 1 \end{pmatrix} \quad (10)$$

By construction, the eigenspectrum of \mathcal{A}_0 is $\lambda_1 = 0$, and $\lambda_j = -m_0$ for $j = 2, \dots, m_0$. In order to ensure that our initial network synchronizes, we select a coupling strength c such that the criterion (6) is satisfied.

We assume that the first growth event ($\tau(k) = 1$) occur after sufficient time has passed such that the transitory behaviors have die out ($k = k_0$). Further we assume that all subsequent growth events occur periodically such that

$$\tau(k) = \begin{cases} 0, & \text{if } 0 \leq k < k_0 \\ n, & \text{if } k_0 + (n-1)T \leq k < k_0 + nT \end{cases} \quad (11)$$

for $n = 1, 2, \dots, N$; where $\tau(k)$ is the growth event index and T is the time period between growth events.

The initial conditions for our initial network ($\mathbf{X}^{0,0}$) are randomly selected from $[0, 1]$. As discrete-time moves along the k index, the event index $\tau(k)$ moves according to (11). Then, at the time instant in which the first growth event occurs, the dynamical description of network changes to:

$$\mathbf{X}^{k+1,1} = F^1(\mathbf{X}^{k,1}) + c\mathcal{A}^1 F^1(\mathbf{X}^{k,1})$$

where, according to the BA network growth model, a new node is added into the network, this means that the vector of state variables becomes

$$\mathbf{X}^{k,1} = [\mathbf{X}^{k,0}, x_{m_0+1}^{k,1}]^\top \in \mathbf{R}^{m_0+1}$$

The initial condition of the added Logistic Map is a value randomly selected from $[0, 1]$. In a similar manner, $F^1(\mathbf{X}^{k,1})$ is the previous vector function appended with the dynamics of the added node

$$F^1(\mathbf{X}^{k,1}) = [F^1(\mathbf{X}^{k,0}), f(x_{m_0+1}^{k,1})]^\top \in \mathbf{R}^{m_0+1}$$

The coupling matrix of the network with an added node becomes

$$\tilde{\mathcal{A}}_0 = \phi_1(\mathcal{A}_0) = \begin{pmatrix} \mathcal{A}_0 & v_1 \\ v_1^\top & 0 \end{pmatrix}$$

with $v_1 \in \mathbf{R}^{m_0}$ a zero vector. The preferential attachment step of the BA network model, becomes the random selection of which m entries of v_1 to change from 0 to 1, then, we have

$$\hat{\mathcal{A}}_0 = \phi_2(\tilde{\mathcal{A}}_0) = \begin{pmatrix} \mathcal{A}_0 & \hat{v}_1 \\ \hat{v}_1^\top & 0 \end{pmatrix}$$

where \hat{v}_1 is the zero vector v_1 with m randomly selected entries as ones.

In order to have a diffusive connection in the resulting network, the diagonal entries of \mathcal{A}_1 are calculated from $\hat{\mathcal{A}}_0$ as $a_{ii} = -\sum_{j=1}^N a_{ij}$. Then, finally we have

$$\mathcal{A}_1 = \phi_3(\hat{\mathcal{A}}_0)$$

Summarizing in our interpretation of the BA model, a growth event signifies a three part process: first the previous coupling matrix is appended with zero vector ($\phi_1(\mathcal{A}_0)$); then, the new node is randomly coupled to m nodes nodes ($\phi_2 \circ \phi_1(\mathcal{A}_0)$), and finally, the diagonal entries are recalculated ($\phi_3 \circ \phi_2 \circ \phi_1(\mathcal{A}_0)$).

The dynamical description of the network including growth events is

$$\mathbf{X}^{k+1,\tau(k)} = F^{\tau(k)}(\mathbf{X}^{k,\tau(k)}) + c\mathcal{A}^{\tau(k)} F^{\tau(k)}(\mathbf{X}^{k,\tau(k)}) \quad (12)$$

where $\mathbf{X}^{k,\tau(k)} = [\mathbf{X}^{k,\tau(k)-1}, x_{m_0+\tau(k)}^{k,\tau(k)}]^\top \in \mathbf{R}^{m_0+\tau(k)}$; $F^{\tau(k)}(\mathbf{X}^{k,\tau(k)}) = [F^{\tau(k)-1}(\mathbf{X}^{k,\tau(k)-1}), f(x_{m_0+\tau(k)}^{k,\tau(k)})]^\top \in \mathbf{R}^{m_0+\tau(k)}$; and $\mathcal{A}^{\tau(k)} = \phi_3 \circ \phi_2 \circ \phi_1(\mathcal{A}_{\tau(k)-1})$; with $\tau(k)$ given by equation (11).

Notice that when a growth event occurs, lets say at $k = \bar{k}$, $\tau(k)$ increases by one and the structure of the network changes with the inclusion of the new node as described above. However, the dynamical evolution of the nodes continues along the discrete-time index k without change. This means at the following discrete-time instant after the growth event ($k = \bar{k} + 1$), the dynamical network continues its evolution with the corresponding new structure until a new growth event occurs ($k = \bar{k} + T$), then the structure changes again, and the growth process continues in that way until the network has grown to the desired N nodes.

4 A synchronization criterion for growing dynamical networks

Following the same basic ideas presented in subsection 2.1, we define synchronization on a growing

dynamical network as the phenomenon in which the nodes existing in the network move at unison. That is, a growing dynamical network is said to be synchronize if the solution

$$x_1^{k,\tau(k)} = x_2^{k,\tau(k)} = \dots = x_{m_0+\tau(k)}^{k,\tau(k)} \quad (13)$$

is asymptotically stable. Moreover, if after a growth event the synchronized solution (13) remains asymptotically stable, we say that the network preserves its synchronization. On the other hand, if for the resulting network, the synchronized solution becomes unstable, we say that the event desynchronizes the network.

As mention before, we consider a growing network where the growth events occur only after a sufficiently large time has passed, such that all transient behaviors have died out. This dwell time restriction allows us to determine the stability of the resulting synchronized solution (13) using the criteria (6)-(7) at each growth event. That is, the corresponding synchronized solution will be stable if the uniform coupling strength satisfies

$$\frac{1 - e^{-h_{max}}}{|\lambda_2^{\tau(k)}|} < c < \frac{1 + e^{-h_{max}}}{|\lambda_N^{\tau(k)}|}$$

where $\lambda_i^{\tau(k)}$ is the i -th eigenvalue of the coupling matrix $\mathcal{A}^{\tau(k)}$. In particular, for our Logistic map network, stability of the synchronized solution is determine by the criteria

$$\frac{0.5}{|\lambda_2^{\tau(k)}|} < c < \frac{1.5}{|\lambda_N^{\tau(k)}|} \quad (14)$$

5 Numerical Results

According to criterion (6), for a network connected in a fully coupled structure, as larger the number of initial nodes (m_0) is, the smaller the value range of the coupling strength (c) to synchronize the nodes. On the other side, if we chose to start with a small number of nodes, the synchronized behavior will be lost at the arrive and attachment of few new nodes. So, in this work we take $m_0 = 5$ initial nodes, for which $c \in [0.1, 0.3]$. As we can see on figure (1), for $c = 0.11$ we can keep the synchronization behavior even when six or seven new nodes arrive and are attached with $m = m_0$ nodes already present in the network.

The time series of the nodes dynamics as the network grows is shown in figure (2), which is plotted in terms of the error between the nodes dynamical states. In order to avoid the transitory behavior of the first nodes, we let the nodes to evolve until $k_0 = 100$ time steps, which, as we can seen in figure (2.a), is enough to allow the error between the dynamical states of the m_0 nodes reach to zero, i.e, be synchronized. After this k_0

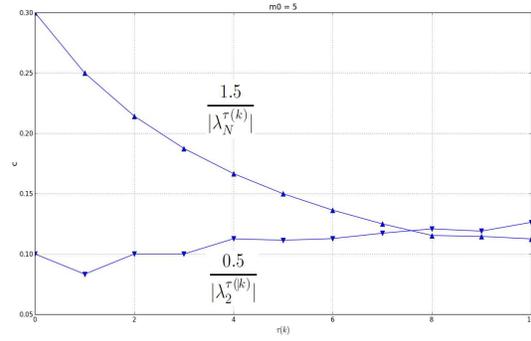


Figure 1. Synchronization area for a dynamical network of $m_0 = 5$ Logistic Maps which structure evolves according to the BA model.

time steps, the network start to growth according to the BA model, and we assume that each new node es added after $T = k_0$ iterations. In figure (2.a) we can see that at $k = 100$, the first node arrives and begins to evolve until be synchronized with the first nodes. Note that this first event does not affect the dynamical evolution of the m_0 nodes. Again, in figure (2.b), we observe that at $k = 200$ and $k = 300$ two new events occurs with the addition of two the nodes , which alter very little the synchronized behavior. The same is seen again in figure (2.c) with two next events, where the synchronized behavior is significantly altered. Finally, on figure, we observe that when the last node is added, the nodes are no longer synchronized, that is, this event desynchronizes the network.

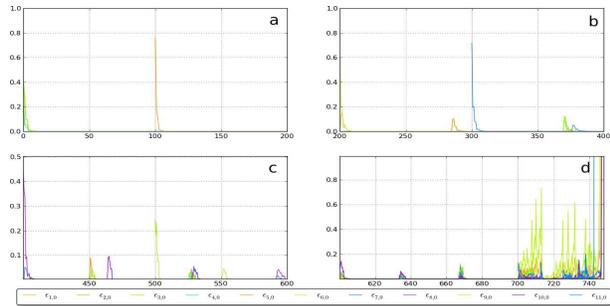


Figure 2. Errors between the nodes dynamical states as the network grows

6 Conclusions

In this contribution we analyze how the synchronization criterion for a dynamical network with fixed structure and composed by discrete-time systems could be applied to the case in which this network grows according to the BA model. In order to perform such analysis, we describe mathematically the growth process from

the perspective of Discrete Event Systems, which let us to define as an event, the addition and attachment of a new node to the network, and also let us to introduce another factor on the growing process, the time in which a new event occurs, which, for this particular work, we consider that it is present at equal time intervals. We examine then how this event affects the collective dynamics of the nodes. In particular, each event changes the dynamical equation in two important forms: a new dimension is added to the equation (1), and the structure of the network changes with the inclusion of the new node. Given that the synchronization criterion depends hardly on the eigenvalue spectrum of the coupling matrix, the growing process is limited to the addition of a few nodes if we want to achieve the synchronization. However, we consider that the Discrete Event approach is convenient to develop a mathematically formalism to analyze the synchronization criterion of a dynamical network with growing structure.

Our next step in the study of growing processes in a dynamical network is to analyze how synchronization can be improved through extra processes defined in the BA model, like the deletion of nodes or links, rewiring the connections of some nodes, etc. These extra processes must be designed based on the way in which the eigenvalue spectrum of the coupling matrix changes.

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